Periodic Observer Design for Networked Embedded Control Systems

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Abstract—In this paper periodic observer design for linear continuous-time systems with periodically time-varying sampling intervals and time delays is considered. Such systems arise e.g. in networked embedded control applications where limited computation and communication resources have to be scheduled. A periodically time-varying discrete-time plant model and cost function regarding the timing effects are derived for the observer design. The design problem is formulated as a periodic linear quadratic (LQ) optimization problem. By applying the lifting procedure, this periodically time-varying LQ problem is transformed into a standard time-invariant LQ problem. The observer gain then follows from solving a periodic algebraic Riccati equation. Both a prediction and a current state observer are derived. Moreover, the separation principle for periodic systems is proved and observability is analyzed. The periodic observer design is illustrated for a benchmark system.

I. INTRODUCTION

The computation and communication resources of modern control systems are increasingly constrained. Commonly, several controllers have to run on a single processor and have to communicate with sensors and actuators via a shared communication medium. To utilize these resources efficiently, scheduling is necessary. Scheduling, however, leads to time-varying sampling intervals. Additionally, computation and transmission times cause time-varying time delays. Thus, conventional sampled-data control theory assuming constant sampling intervals and negligible computation and transmission times is no longer applicable. Instead, a control and scheduling codesign is crucial to ensure stability and performance of such networked embedded control systems [1].

Control and scheduling codesign has received considerable attention in recent years [2]–[9]. The controller and scheduler are optimized jointly. Standard real-time scheduling algorithms like RM and EDF and medium access control protocols like TDMA rely on periodic schedules. Therefore, for control and scheduling codesign usually also periodic schedules are considered [2]–[7]. Besides, optimization and implementation are relieved considerably thereby.

Periodic schedules lead to periodically time-varying sampling intervals and time delays. Usually, the sampling intervals and time delays are assumed as multiples of a basic interval [2]–[6]. Sampling intervals and time delays may, however, differ considerably among control algorithms and data transmissions. Therefore, in [7] a control and scheduling codesign allowing for arbitrary periodically time-varying sampling intervals and time delays has been proposed. The method is based on a continuous-time plant model and cost function which is crucial to account for arbitrary periodically time-varying sampling intervals and time delays. During discretization, the periodically time-varying sampling intervals and time delays are regarded. In this way, these timing effects are included in the resulting discrete-time cost function. The resulting periodically time-varying discrete-time system is controlled by a periodic linear quadratic regulator (PLQR). The optimal schedule is finally determined by combinatorial optimization.

Most publications on control and scheduling codesign focus on full state feedback. Only few works address output feedback, e.g. [5] where LQG is considered and [6] where a stabilizing controller and observer are proposed. These results are rather specific since time delays and zero order hold have not been regarded. On the other hand, there exists a rich literature on output feedback for general periodic systems, see [10] and the references therein. These methods are clearly not tailored to control and scheduling codesign.

In this paper a periodic observer for control and scheduling codesign is presented. The observer is based on a continuous-time plant model and cost function weighting the observer error and input. Using a continuous-time plant model and cost function is crucial to account for irregular sampling and varying computation and transmission times. This has not been considered before for observer design. The plant model and the cost function are discretized for periodically time-varying sampling intervals and time delays. The timing variations are thus quantized in the cost function. The observer design then relies on a periodic linear quadratic (LQ) optimization problem. Using the lifting procedure, a standard LQ optimization problem is obtained. The observer gain then results from solving algebraic Riccati equations. Both a prediction and a current state observer are considered. Furthermore, the separation principle and the observability are addressed. Finally, the periodic observer is evaluated for control and scheduling codesign of an active suspension system. The periodic observer complements the control and scheduling codesign proposed in [7], but applies to other approaches as well.

II. PERIODIC OBSERVER DESIGN

A. Problem formulation

It is assumed that a plant is controlled by a discrete-time full state feedback controller utilizing a state estimation provided by a discrete-time observer. Both controller and observer are executed on a shared processor and the control signal is transmitted via a shared network according to a
given periodic scheduling policy. The periodic scheduling leads to irregular execution and transmission patterns for the controller and the observer within the period of the schedule as depicted in Fig. 1. Here, the computation and transmission time can be split into segment $S_1$ where the control signal is computed and transmitted and segment $S_2$ where the internal states of the observer are updated. The periodic schedule can be mapped onto a $P$-periodic sequence of sampling intervals $h(k)$ and input delays $\tau(k) < h(k)$ denoted as

$$ (h(k), \tau(k)) = (h(k + P), \tau(k + P)) \quad \forall k \in \mathbb{Z}. $$ (1)

The linear continuous-time plant with input delay is described by

$$ \dot{x}(t) = Ax(t) + Bu(t - \tau(k)) , \quad x(0) = x_0 $$ (2a)

$$ \hat{y}(t) = C\hat{x}(t) $$ (2b)

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the control input vector and $y \in \mathbb{R}^p$ is the measured output vector. The triple $(A, B, C)$ denotes the system, the input and the output matrix of appropriate dimension. Note that the input delay $\tau(k)$ caused by the computation time of the controller and observer algorithm and the transmission time of the control signal is modeled in the plant. In order to be able to design a discrete-time observer while considering the time-varying sampling intervals and input delays properly, a continuous-time observer model is considered in the first step

$$ \hat{x}(t) = A\hat{x}(t) + Bu(t - \tau(k)) - u_{OB}(t), $$ (3a)

$$ \hat{x}(0) = \hat{x}_0 $$

$$ \hat{y}(t) = C\hat{x}(t). $$ (3b)

The estimated value of $x(t)$ is denoted as $\hat{x}(t)$. The observer input vector is $u_{OB} \in \mathbb{R}^n$. The computation and transmission times have no influence on the estimated value $\hat{x}(t)$ since $\hat{x}(t)$ will be later on evaluated for discrete time instants $t_k = \sum_{i=0}^{k-1} h(i)$ only using a discrete-time equivalence. Therefore, the observer input vector $u_{OB}$ is not delayed unlike the control input vector $u$. The observer error $e(t) = x(t) - \hat{x}(t)$ obeys the continuous-time dynamics

$$ \dot{e}(t) = Ae(t) + u_{OB}(t), \quad e(0) = x_0 - \hat{x}_0 $$ (4a)

$$ e_y(t) = Ce(t) $$ (4b)

where $e_y(t) \in \mathbb{R}^p$ is the error output signal. Since a discrete-time observer is to be designed, the discrete dynamics (4) is influenced by a piecewise constant error output feedback

$$ u_{OB}(t) = -L(t_k)e_y(t_k) \quad \text{for} \quad t_k \leq t < t_{k+1}. $$ (5)

We assume that the observer gain matrix $L(t_k) \in \mathbb{R}^{n \times p}$ varies periodically according to the sampling interval $h(k)$ at the discrete time step $t_k$. Because (5) is rather an output feedback law than a state feedback law, the PLQR design procedure proposed in [7] cannot be adopted directly to the observer design problem. Therefore, the observer gain is determined on the dual error dynamics

$$ \dot{e}^d(t) = A^T e^d(t) + C^T u_{OB}^d(t), $$ (6a)

$$ e^d(0) = e_0^d = x_0^d - \hat{x}_0^d $$

$$ e_y^d(t) = e^d(t) $$ (6b)

where the dual signals are $e^d, e_y^d \in \mathbb{R}^n$ and $u_{OB}^d \in \mathbb{R}^p$. The triple $(A^T, C^T, I)$ denotes the system, the input and the output matrix of the dual system. It is worth noting that due to (6b) the dual output vector is identical to the dual state vector. Thus, the dual output feedback is actually a dual full state feedback

$$ u_{OB}^d(t) = -L(t_k)^T e^d(t_k) \quad \text{for} \quad t_k \leq t < t_{k+1}. $$ (7)

The primal system (4) and the corresponding dual system (6) have the same dynamic properties [11, Section 7.7.1].

For the control performance the observer error $e(t)$ at time instants $t_k$ together with its duration $h(k)$ are relevant. Since the observer error is modeled in the continuous-time domain, both $e(t_k)$ and $h(k)$ can exactly be treated by a continuous-time cost function. Hence, the continuous-time LQ optimization problem

$$ \min_{u_{OB}(t)} J = \min_{u_{OB}(t)} \int_0^\infty \left( e^d(T) \right)^T Q e^d(T) + \left( e^d(t) - e_y^d(t) \right)^T R \left( e^d(t) - e_y^d(t) \right) dt $$ (8a)

s.t. (6) (8b)

can be formulated, where $Q \in \mathbb{R}^{n \times n}$ and $R \in \mathbb{R}^{p \times p}$ are symmetric and positive definite weighting matrices. The continuous-time cost function (8a) as well as the dual continuous-time observer error dynamics (6) are discretized [12, Section 9.3.4] to determine the observer gain $L(t_k)$ for the piecewise constant error output feedback (5). Thereby the periodically time-varying sampling intervals (1) are regarded. The resulting periodically time-varying discrete-time LQ optimization problem is

$$ \min_{u_{OB}^d(k)} \sum_{k=0} \left( e^d(k) \right)^T Q_{12}(k) Q_{12}(k) \left( e^d(k) \right) $$ (9a)

s.t. \left\{ \begin{align*}
 e^d(k+1) &= \Phi^d(k)e^d(k) + \Gamma_{OB}^d(k)u_{OB}^d(k) \\
 e^d(0) &= e_0^d
 \end{align*} \right. $$ (9b)

where the discretized periodically time-varying weighting matrices are given by

$$ Q_{1}(k) = \int_0^{h(k)} \Phi^d(t)Q\Phi^d(t) dt $$ (10a)

$$ Q_{2}(k) = \int_0^{h(k)} \Gamma_{OB}^d(t)Q\Gamma_{OB}^d(t) + R dt $$ (10b)

$$ Q_{12}(k) = \int_0^{h(k)} \Phi^d(t)Q\Gamma_{OB}^d(t) dt $$ (10c)
with
\[ \Phi^d(t) = e^{A^T t}, \quad \Gamma^d_{DB}(t) = \int_0^t e^{A^T s} ds C^T \] (11)
and \( \Phi^d(k) = \Phi^d(h(k)) \) and \( \Gamma^d_{OB}(k) = \Gamma^d_{OB}(h(k)) \).

**B. Problem transformation and solution**

The periodically time-varying LQ optimization problem (9) can further be transformed into a standard time-invariant LQ optimization problem using the lifting procedure [10] to facilitate its solution. However, before lifting a cross-term elimination and factorization of the cost function (9a) are recommendable. These transformations simplify the lifting procedure for the cost function. Hence, only the dual error dynamics has to be lifted afterwards.

In the first step, the cross terms \( Q_{12}(k) \) of the cost function (9a) are eliminated using the input transformation

\[
\bar{u}^d_{OB}(k) = u^d_{OB}(k) + M^T(k) e^d(k) \quad \text{(12a)}
\]

\[
M(k) = Q_{12}(k) Q^{-1}(k) \quad \text{(12b)}
\]

\[
\tilde{\Phi}^d(k) = \Phi^d(k) - \Gamma^d_{OB}(k) M^T(k) \quad \text{(12c)}
\]

\[
\tilde{Q}_1(k) = Q_1(k) - M(k) Q^{-1}_{12}(k). \quad \text{(12d)}
\]

In the second step, the weighting matrices of the cost function are eliminated using the Cholesky decomposition, which leads to \( \tilde{Q}_1(k) = \tilde{U}^T(k) \tilde{U}(k), \quad Q_2(k) = \tilde{V}^T(k) \tilde{V}(k) \) where \( \tilde{U}(k) \) and \( \tilde{V}(k) \) are full rank lower triangular matrices. Since the matrix \( Q_2(k) \) is positive definite, the corresponding matrix \( \tilde{V}(k) \) is regular and the substitution

\[
\tilde{\Gamma}^d_{OB}(k) = \Gamma^d_{OB}(k) \tilde{V}^{-1}(k), \quad \bar{u}^d_{OB}(k) = \tilde{V}(k) \bar{u}^d_{OB}(k) \quad \text{(13)}
\]

leads to the factorized optimization problem with an unweighted and time-invariant cost function

\[
\min_{\bar{u}^d_{OB}(k)} J = \min_{\bar{u}^d_{OB}(k)} \sum_{k=0}^{\infty} ||\bar{u}^d_{OB}(k)||^2 \quad \text{(14a)}
\]

\[
\text{s.t. } \begin{align*}
    \bar{u}^d_{OB}(k+1) &= \tilde{\Phi}^d(k) \bar{u}^d_{OB}(k) + \tilde{\Gamma}^d_{OB}(k) \bar{u}^d_{OB}(k) \\
    \bar{y}^d_{OB}(k) &= \tilde{U}(k) \bar{u}^d_{OB}(k) \\
    \bar{e}^d_{0}(0) &= e^d_{0}.
\end{align*} \quad \text{(14b)}
\]

The lifting procedure converts this optimization problem with periodically time-varying constraints into a time-invariant optimization problem. The difference equation (14b) is solved for every starting index \( k_s \) within one period \( P \). The discretization index of the lifted system is \( \kappa \) with

\[
\kappa = \frac{k - k_s}{P} = \left\lfloor \frac{k}{P} \right\rfloor \quad \text{where} \quad k_s = k \mod P. \quad \text{(15)}
\]

The lifted optimization problem is

\[
\min_{\bar{u}^d_{OBk},(\kappa)} J = \min_{\bar{u}^d_{OBk},(\kappa)} \sum_{\kappa=0}^{\infty} ||\bar{y}^d_{OBk}(\kappa)||^2 + ||\bar{u}^d_{OBk}(\kappa)||^2 \quad \text{(16a)}
\]

\[
\text{s.t. } \begin{align*}
    \bar{y}^d_{OBk}(\kappa+1) &= \tilde{\Phi}^d_{k_s}(\kappa) \bar{y}^d_{OBk}(\kappa) + \tilde{\Gamma}^d_{OBk}(\kappa) \bar{u}^d_{OBk}(\kappa) \\
    \bar{y}^d_{OBk}(\kappa) &= \tilde{U}_{k_s} \bar{u}^d_{OBk}(\kappa) + \tilde{D}_{k_s} \bar{y}^d_{OBk}(\kappa) \\
    \bar{y}^d_{OBk}(0) &= e^d_{0}.
\end{align*} \quad \text{(16b)}
\]

where

\[
\bar{y}^d_{OBk}(\kappa) = \bar{y}^d_{OBk}(\kappa) + \tilde{\Gamma}^d_{OBk}(\kappa) \bar{u}^d_{OBk}(\kappa) \quad \text{(17a)}
\]

\[
\bar{u}^d_{OBk}(\kappa) = \left( \bar{u}^d_{OBk}(\kappa) + \tilde{\Gamma}^d_{OBk}(\kappa) \bar{u}^d_{OBk}(\kappa) \right) \quad \text{(17b)}
\]

\[
\bar{y}^d_{OBk}(\kappa) = \left( \bar{y}^d_{OBk}(\kappa) + \tilde{\Gamma}^d_{OBk}(\kappa) \bar{u}^d_{OBk}(\kappa) \right) \quad \text{(17c)}
\]

\[
\bar{y}^d_{OBk}(\kappa) = \left( \bar{y}^d_{OBk}(\kappa) + \tilde{\Gamma}^d_{OBk}(\kappa) \bar{u}^d_{OBk}(\kappa) \right) \quad \text{(17d)}
\]

with \( \tilde{\Phi}^d_{k_s} \in \mathbb{R}^{n \times n}, \tilde{\Gamma}^d_{OBk_s} \in \mathbb{R}^{n \times P}, \tilde{U}_{k_s} \in \mathbb{R}^{P \times n} \) and \( \tilde{D}_{k_s} \in \mathbb{R}^{P \times n \times n} \). The matrix \( \tilde{\Psi}_{k_s} \in \mathbb{R}^{n \times n} \) is the transition matrix of the error dynamics (14b) and defined as

\[
\tilde{\Psi}_{k_s} = \left\{ \begin{array}{c}
    I, \\
    \bar{\Phi}^d((k-1)) \bar{\Phi}^d((k-2)) \cdots \bar{\Phi}^d((k_s)),
\end{array} \right. \quad \text{(18)}
\]

The special case \( \tilde{\Psi}_{k_s} \) is called monodromy matrix with time-invariant eigenvalues, which are denoted as characteristic multipliers. The corresponding periodic system is stable iff all characteristic multipliers lie within the unit disk of the complex plane [10, Section 3.3]. The lifted signal vectors are

\[
\bar{\Psi}^d_{k_s} = \bar{\Psi}_{k_s} + P \bar{\kappa} \quad \text{(19a)}
\]

\[
\bar{y}^d_{OBk_s}(\kappa) = \left( \bar{y}^d_{OBk_s}(\kappa) + \tilde{\Gamma}^d_{OBk_s}(\kappa) \bar{y}^d_{OBk_s}(\kappa) \right) \quad \text{(19b)}
\]

\[
\bar{y}^d_{OBk_s}(\kappa) = \left( \bar{y}^d_{OBk_s}(\kappa) + \tilde{\Gamma}^d_{OBk_s}(\kappa) \bar{y}^d_{OBk_s}(\kappa) \right) \quad \text{(19c)}
\]

with \( \bar{\Psi}^d_{k_s} \in \mathbb{R}^{n \times n}, \bar{\Gamma}^d_{OBk_s}(\kappa) \in \mathbb{R}^{n \times P}, \bar{y}^d_{k_s} \in \mathbb{R}^{P \times n} \) and \( \bar{y}^d_{OBk_s}(\kappa) \in \mathbb{R}^{n \times n} \). To convert the time-invariant LQ optimization problem (16) into a standard LQ optimization problem, the constraint for \( \bar{y}^d_{k_s}(\kappa) \) is inserted into the cost function (16a). The resulting standard problem is

\[
\min_{\bar{y}^d_{OBk_s}(\kappa)} \sum_{\kappa=0}^{\infty} \left( \bar{\Psi}^d_{k_s}(\kappa) \right)^T \bar{D}_{k_s} \bar{N}_{k_s} \bar{D}_{k_s} \bar{N}_{k_s} \left( \bar{\Psi}^d_{k_s}(\kappa) \right) \quad \text{(20a)}
\]

\[
\text{s.t. } \begin{align*}
    \bar{y}^d_{OBk_s}(\kappa+1) &= \bar{\Psi}^d_{k_s}(\kappa) \bar{y}^d_{OBk_s}(\kappa) + \tilde{\Gamma}^d_{OBk_s}(\kappa) \bar{u}^d_{OBk_s}(\kappa) \\
    \bar{y}^d_{OBk_s}(0) &= e^d_{0}.
\end{align*} \quad \text{(20b)}
\]

The solution of the optimization problem (20) is given by the solution of the periodic algebraic Riccati equation

\[
\bar{D}_{k_s} \bar{P}_{k_s} \bar{D}_{k_s} + \bar{Q}_{k_s} \bar{P}_{k_s} + \bar{Q}_{k_s} \bar{D}_{k_s} \bar{P}_{k_s} + \bar{Q}_{k_s} = \bar{D}_{k_s} \bar{D}_{k_s} \bar{N}_{k_s} \quad \text{(21)}
\]
with
\[ T_k^d = \left[ T_{k_s}^T + T^T_{OBk_s} T_{k_s} + T^T_{OBk_s} \right]^{-1} \left[ T^T_{OBk_s} T_{k_s} + T^T_{OBk_s} \right]. \]

To obtain \( T_k^d \), the Riccati equation must be solved for every starting index \( k_s = 0, 1, \ldots, (P - 1) \). The solution of the original problem (9) is then given by the inverse transformation of \( T_k^d \). The dual periodic observer gain matrix results as
\[
L^T(k) = \hat{V}(k)^{-1} \left\{ (T^T_{k_s})_{zz} \right\} + Q_2(k)^{-1}Q_1z(k)^T
\]
\[ 1 \leq z \leq p, \quad 1 \leq s \leq (n + p). \] (22)

This leads to the primal observer input vector
\[ u_{OB}(k) = -L(k)e_y(k). \] (23)

C. Determination of the observer algorithm

To compute the estimated value \( \hat{x}(t) \) at the discrete time instants \( t_k \), the continuos-time observer model (3) is discretized. The solution of the state equation (3a) is
\[
\hat{x}(t) = e^{A(t-t_0)} \hat{x}(t_0) + \int_{t_0}^{t} e^{A(t-s)} B u(s) - \tau (k) ds
\]
\[ - \int_{t_0}^{t} e^{A(t-s)} u_{OB}(s) ds. \] (24)

For the discretization one sampling interval with \( t_0 = t_k \) and \( t = t_k+1 \) is considered where the delayed control input vector \( u(t - \tau (k)) \) is held piecewise constant by a ZOH. Because of the delay the change of \( u(t) \) occurs within the sampling interval. Hence, the control input vector is
\[ u(t) = \begin{cases} u(t_{k-1}) & \text{for} \quad t_k \leq t < t_k + \tau(k) \\ u(t_k) & \text{for} \quad t_k + \tau(k) \leq t < t_{k+1}. \end{cases} \] (25)

The observer input vector \( u_{OB}(t) \) is not delayed. Thus, its change occurs only at the discrete time instants \( t_k \). The difference equation of the prediction observer is then
\[
\hat{x}(k+1) = \Phi(k)\hat{x}(k) + \Gamma_0(k)u(k) + \Gamma_1(k)u(k-1)
\]
\[ + L_p(k)(y(k) - C\hat{x}(k)) \] (26)
with
\[ L_p(k) = \Gamma OB(k)L(k) \] (27)

where the discretized periodically time-varying system and input matrices are
\[ \Phi(k) = e^{A_h(k)} \] (28a)
\[ \Gamma_0(k) = \int_{0}^{h(k)-\tau(k)} e^{A_s} ds B \] (28b)
\[ \Gamma_1(k) = e^{A(h(k)-\tau(k))} \int_{0}^{\tau(k)} e^{A_s} ds B \] (28c)
\[ \Gamma OB(k) = \int_{0}^{h(k)} e^{A_s} ds. \] (28d)

The discrete-time model of the plant (2a) is determined analogously as
\[
x(k+1) = \Phi(k)x(k) + \Gamma_0(k)u(k) + \Gamma_1(k)u(k-1).
\] (29)

The discrete-time primal error dynamics
\[ e(k+1) = (\Phi(k) - L_p(k)C) e(k) = \Phi_{cil}(k)e(k) \] (30)
then results by subtracting (26) from (29). The structure of the prediction observer is shown in Fig. 2. One drawback of a prediction observer is that the estimated value \( \hat{x}(k) \) is based on the previous measurement \( y(k-1) \). Hence, the response of the observer is delayed. Especially if the observed system is influenced by disturbances, the delay degrades the performance. To overcome this drawback, a current state observer can be designed. The estimated value of a current state observer is based on the current measurement \( y(k) \). The computation of the estimated value is split into two equations
\[
\hat{x}(k) = \Phi(k-1)\hat{x}(k-1) + \Gamma_0(k-1)u(k-1)
\]
\[ + \Gamma_1(k-1)u(k-2) \] (31a)
\[ \hat{x}(k) = \hat{x}(k) + L_c(k) (y(k) - C\hat{x}(k)). \] (31b)

The estimation error of the current state observer is defined as \( e(k) = x(k) - \hat{x}(k) \). Inserting (29) and (31) delivers
\[ e(k+1) = (\Phi(k) - \Phi(k) L_c(k)C) e(k). \] (32)

A comparison between \( \hat{x}(k) \) of the current state observer and the estimated value \( \hat{x}(k) \) of the prediction observer shows that both observers are represented by the same dynamic system. Only the output signal \( \hat{x}(k) \) is chosen differently as shown in Figs. 2 and 3. Furthermore, a comparison of (30) and (32) delivers the current state observer gain matrix
\[ L_c(k) = \Phi(k)^{-1}L_p(k). \] (33)

III. ANALYSIS

A. Separation principle for periodic systems

In the following the separability of the characteristic multipliers is shown. Particularly, a full state feedback controller
designed with the PLQR method given in [7] using the state estimation provided by a prediction observer (26) is considered. The full state feedback law according to [7] is
\[ u(k) = -K_x(k)\hat{x}(k) - K_u(k)u(k-1). \] (34)

To obtain a standard state-space model with only one input vector, the delayed input vector \( u(k-1) \) is considered as an augmented state. For the closed-loop system the following state-space model holds
\[ x_a(k+1) = (A_a(k) - B_a(k)K_a(k))x_a(k) \]
\[ = A_{cl}(k)x_a(k) \] (35)
where \( x_a(k) \in \mathbb{R}^{(n+m)} \) is the augmented state vector, \( A_a(k) \in \mathbb{R}^{(n+m)\times(n+m)} \) the augmented system matrix, \( B_a(k) \in \mathbb{R}^{(n+m)\times m} \) the augmented input matrix and \( K_a(k) \in \mathbb{R}^m \) the augmented controller with
\[ x_a(k) = \begin{pmatrix} x(k) \\ u(k-1) \end{pmatrix}, \quad K_a(k) = \begin{pmatrix} K_x(k) & K_u(k) \end{pmatrix} \]
\[ A_a(k) = \begin{pmatrix} \Phi(k) & \Gamma_0(k) \\ 0 & 0 \end{pmatrix}, \quad B_a(k) = \begin{pmatrix} \Gamma_0(k) \\ I \end{pmatrix}. \]
Now this controller is used with the prediction observer. The control signal based on the estimated value \( \hat{x}(k) \) is
\[ u(k) = -K_x(k)\hat{x}(k) - K_u(k)u(k-1) \]
\[ = -K_x(k)x(k) - K_u(k)u(k-1) + K_x(e(k)). \] (36)
This leads to the overall model
\[ \begin{pmatrix} x_a(k+1) \\ e(k+1) \end{pmatrix} = \begin{pmatrix} A_{cl}(k) & B_a(k)K_x(k) \\ 0 & \Phi_{cl}(k) \end{pmatrix} \begin{pmatrix} x_a(k) \\ e(k) \end{pmatrix} \] (37)
of the observer-based control system. The monodromy matrix of the lifted overall model (37) is
\[ \Psi_{k_a+P,k_a} = \prod_{k=k_a+P-1}^{k_a} \begin{pmatrix} A_{cl}(k) & B_a(k)K_x(k) \\ 0 & \Phi_{cl}(k) \end{pmatrix}. \] (38)
The monodromy matrix \( \Psi_{k_a+P,k_a} \) is the product of upper triangular matrices. Hence,
\[ \Psi_{k_a+P,k_a} = \left( \prod_{k=k_a+P-1}^{k_a} A_{cl}(k) \right) \prod_{k=k_a+P-1}^{k_a} \Phi_{cl}(k) \] (39)
where \( \Delta \) denotes a cross term which is not relevant for the determination of the characteristic multipliers. The elements of the main diagonal of (39) are the monodromy matrices of the closed-loop control system (35) and the observer error dynamics (30). Hence, the characteristic multipliers of the overall system are composed of the controller and observer characteristic multipliers, i.e. the use of the observer does not change the characteristic multipliers of (35).

For the current state observer the separation principle can be derived analogously.

### B. Observability of periodic systems

The observability of a general periodic system
\[ x(k+1) = \Phi(k)x(k) \] (40a)
\[ y(k) = C(k)x(k) \] (40b)
is analyzed on the lifted system description
\[ \begin{pmatrix} \tilde{x}_k(k+1) \\ \tilde{y}_k(k) \end{pmatrix} = \begin{pmatrix} \Phi_k & \Phi_k \end{pmatrix} \begin{pmatrix} x_k(k) \\ e_k(k) \end{pmatrix}. \] (41a)
\[ \begin{pmatrix} \tilde{x}_k(k) \\ \tilde{y}_k(k) \end{pmatrix} = \begin{pmatrix} C_k & C_k \end{pmatrix} \begin{pmatrix} x_k(k) \\ e_k(k) \end{pmatrix}. \] (41b)

**Proposition 1:** The system (40) is observable if the observability matrix
\[ O_{k_a} = \begin{pmatrix} C_k & C_k \\ \vdots & \vdots \end{pmatrix} \] (42)
is regular \( \forall k_a \in \{0, \ldots, P-1\} \).

**Proof:** Follows by utilizing Definition 4.3 and Remark 6.2 in [10].

### IV. Application Example

An active suspension system of a passenger car is studied. For simplicity reasons the car is modeled by quarter cars (see Fig. 4) and a single axle consisting of two quarter cars is considered. The chassis mass \( m_c = 315 \text{kg} \) is connected to the wheel by a linear spring \( k_s = 29.5 \text{kN/m} \) and a linear damper \( b_s = 1.5 \text{kN}\text{s/m} \). The wheel is modeled as an unsprung mass \( m_u = 37.5 \text{kg} \), the tire as a linear spring \( k_u = 208 \text{kN/m} \). Two identical linear state-space models with the state vector \( x = (z_s - z_u \ z_s - z_r \ z_u)^T \) are built. The input is the force \( F \) which is generated by an actuator, the measured output is \( y = (\ddot{z}_s \ \ddot{z}_u)^T \) and the controlled output is \( \tilde{y} = (\ddot{z}_s \ \ddot{z}_u)^T \).

Two periodic controllers and a periodic scheduling policy are to be codesigned to control the two quarter cars with a single processor. Therefor the control and scheduling codesign proposed in [7] is used minimizing the cost function
\[ J_{PLQR} = \sum_{i=1}^{P} \int \left( \begin{pmatrix} F_1(t - T_i(k)) \\ 0 \end{pmatrix} \begin{pmatrix} Q_{PLQR} & 0 \\ 0 & R_{PLQR} \end{pmatrix} \begin{pmatrix} \ddot{y}_1(t) \\ \ddot{y}_2(t) \end{pmatrix} \right) dt \]
with $\tilde{Q}_{\text{PLQR}} = \text{diag}(20, 10^4)$, $R_{\text{PLQR}} = 2 \cdot 10^{-5}$ and $i$ denoting the $i^{th}$ quarter car. An initial suspension deflection is considered, leading to the initial conditions $x_{0,1} = (0.005 \text{ m } 0 \ 0 \ 0)^T$ and $x_{0,2} = (0.05 \text{ m } 0 \ 0 \ 0)^T$. For the control algorithm and the observer algorithm a total computation time of 2 ms is assumed. Particularly, the evaluation of (36), (31a) and (31b) takes 0.67 ms and the evaluation of (26) takes 1.33 ms. The total computation time is segmented as follows: If a prediction observer is used, in segment $S_1$ the control input (36) is computed, leading to the input delay $\tau(k) = 0.67$ ms, and in segment $S_2$ the estimated state (26) for the next time instant is predicted. If a current state observer is used, in segment $S_1$ the estimated state (31b) for the current time instant is updated and the control input (36) is computed, leading to the input delay $\tau(k) = 1.33$ ms, and in segment $S_2$ the internal state $\hat{x}$ is computed from (31a).

From the control and scheduling codesign two PLQRs and the periodic schedule (2221) which determines the execution order of the control algorithms are obtained. The resulting periodic sequences of sampling intervals and input delays are

$$\begin{align*}
(h_1(k), \tau_1(k)) &= ((8, \tau(k))) \text{ ms} \\
(h_2(k), \tau_2(k)) &= ((2, \tau(k)), (2, \tau(k)), (4, \tau(k))) \text{ ms}.
\end{align*}$$

For these sequences a prediction observer and a current state observer are designed with the weighting matrices $Q = \text{diag}(5000, 150, 10, 50)$ and $R = I$.

The designed PLQRs are simulated with full state measurement and with state estimation. In Fig. 5 a comparison between the measurement-based control system and the observer-based control system using the prediction observer is shown for the second quarter car. The measurement-based controller ($J_{\text{meas}}^{\text{PLQR}} = 18.64$) reacts faster to the initial condition than the prediction observer-based controller ($J_{\text{pred}}^{\text{PLQR}} = 20.36$) due the non-zero observer error. Furthermore, the prediction observer needs one time step to produce the first estimate. However, the settling time of both control systems is almost identical. After 0.7 s the deviation from zero is negligible. The current state observer generally reacts faster than the prediction observer, however, due to the bigger input delay the overall costs are slightly higher ($J_{\text{sum}}^{\text{PLQR}} = 20.42$).

V. CONCLUSIONS AND FUTURE WORK

In this paper a periodic observer for control and scheduling codesign has been presented. The PLQR proposed in [7] has been extended to an observer-based PLQR. Timing effects have been fully considered in the observer design. Furthermore, the separation principle has been proved and an observability condition has been given. The functionality of the periodic observer algorithm has been verified on a benchmark system. The use of the observer degrades the performance only marginally if an unknown initial condition is imposed. If also a disturbance is present, the performance loss is higher. In this case, the variance of the observer error can be minimized instead, leading to a periodic Kalman filter.

Future work will focus on extending the controller-scheduler codesign proposed in [7] to a controller-observer-scheduler codesign. Furthermore, uncertain time-varying sampling intervals and time delays will be addressed following the ideas in [13].

REFERENCES