Navigation of a Non-Holonomic Vehicle for Gradient Climbing and Source Seeking without Gradient Estimation

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Abstract—We consider a single Dubins-like mobile vehicle traveling with a constant longitudinal speed in a planar region supporting an unknown field distribution. A single sensor provides the distribution value at the current vehicle location. We present a new sliding mode navigation strategy that drives the vehicle to the location where the field distribution attains its maximum. The proposed control algorithm does not employ gradient estimation and is non-demanding with respect to both computation and motion. Its mathematically rigorous analysis and justification are provided. Simulation results confirm the applicability and performance of the proposed guidance approach.

I. INTRODUCTION

In this paper, we consider the problem of steering a single non-holonomic vehicle to the extremum of an unknown scalar environmental field. This field may represent the strength of a spatially distributed signal, e.g., electromagnetic or acoustic one, concentration of a chemical, physical or biological agent, a distribution of a physical quantity, like thermal, magnetic, electric, or optical field distributions, etc. Some examples of missions where steering to extrema is of interest include detecting and localizing the sources of hazardous chemicals leakage or vapor emission, sources of pollutants and plumes, locating hydrothermal vents, environmental studies, etc. In the literature, the problems of the kind to be examined are basically referred to as those of source seeking/localizing [10], [21], [26] or gradient climbing/descent/ascent [2], [5], [6], [20]. In the first case, it is assumed that the field distribution is caused by emission of some substance from a local (pointwise) source and at the location of the source, the distribution attains its maximum. The second reference underscores that the maximum search is arranged via ensuring constant increase of the distribution value at the vehicle location, with the best option being to move in the direction of the gradient.

In the last decades, an extensive body of research was devoted to sensor-based motion control for source seeking/gradient climbing; see e.g., [5], [10], [20] and the references therein. Most of the proposed approaches are based on gradient estimation. In the context of mobile sensor networks, which enjoyed much attention recently [2], [5], [15], [20], [21], a team of vehicles has extended capabilities for gradient estimation thanks to collaborative sensing the field distribution at various points and sensor data exchange. The single vehicle scenario is more challenging in this respect, unless the vehicle is large and equipped with numerous sensors providing the distribution values at various locations from its vicinity. In the case where the multiple sensor information is unavailable, a typical method to compensate for the lack of data is to get extra information via extra maneuvers by e.g., ‘dithering’ the sensor position [6], [10], [26]. For example, in accordance with a general approach to the wider problem of extremum-seeking design [1], the vehicle is excited with probing high-frequency sinusoidal inputs [10], [26]. The disadvantage of this method is that systematic costly and superfluous maneuvers may be required to collect rich enough data. At the same time, the complementary multiple vehicle/sensor scenario means more complicated and costly hardware.

In this paper, we propose a navigation strategy that steers a single autonomous vehicle to the location where an unknown field distribution attains its maximum. Modulo natural and partly unavoidable assumptions, the distribution is arbitrary. The vehicle is modeled as unicycle, travels with a constant longitudinal speed, and is controlled by the time-varying angular velocity limited by a given constant. The kinematics of the vehicle is given by the standard model of the Dubins-like car, i.e. a unicycle moving along planar paths of bounded curvature without reversing direction [11]. The model is applicable to many mechanical systems such as wheeled robots, unmanned aerial vehicles (UAVs), missiles and autonomous underwater vehicles (AUVs); see e.g. [14], [18], [19] and the references therein. A single sensor provides the current distribution value, and the vehicle is also capable to access the rate at which this measurement evolves as time progresses, but no further sensing capabilities are assumed. The proposed controller is motivated by the equiangular navigation guidance (ENG) law, which navigates a wheeled robot towards an unknown target using the range and range-rate measurements [23].

Unlike the mainstream in the area, the proposed control law does not employ gradient estimates and related systematic exploration maneuvers. Instead, we propose a ‘direct action’ discontinuous controller, non-demanding with respect to both computation and motion. Mathematically rigorous analysis and justification of the proposed strategy are offered. Unlike many papers in the area, we provide explicit description of the design parameters that ensure the control objective: the vehicle inevitably reaches the desired neighborhood of the maximum point for a finite time and...
remains there afterwards. The applicability of the proposed navigation law is confirmed by computer simulations.

The proposed strategy belongs to the class of sliding mode control laws [24]. Due to the well-known benefits such as high insensitivity to disturbances and noises, robustness against system uncertainties, good dynamic response, simple implementation, etc. [24], the sliding mode approach attracts an increasing interest in the area of motion control. However, the major obstacle to implementation of sliding mode controllers is a harmful phenomenon called “chattering”, i.e., undesirable finite frequency oscillations around the ideal trajectory due to un-modeled system dynamics and restrictions on the control switching frequency and delays. An extensive body of research was devoted to the problem of chattering elimination and reduction; we refer the reader to [12], [17] for a survey. This discipline offers a variety of rather effective approaches, including smooth approximation of the discontinuity [7], inserting low-pass filters/observers into the control loop [3], [4], [9], [25], combining sliding mode and adaptive control techniques [22], employing fuzzy logic technique [8], etc. The controller proposed in this paper typically exhibits a combination of ordinary and sliding motions. Whether undesirable chattering be encountered in its practical applications, this problem is supposed to be handled within the framework of the above general discipline.

The body of the paper is organized as follows. Section II offers system description and problem setup, whereas Section III introduces and discusses assumptions. The main result is stated in Section IV. In Section V, this result and related controller design are illustrated for the simple yet instructive case of the isotropic distribution. The results of computer simulation are discussed in Section VI.

II. SYSTEM DESCRIPTION AND PROBLEM SETUP

We consider a planar mobile vehicle modeled as unicycle. It travels with a constant speed \( v \) and is controlled by the time-varying angular velocity \( u \) limited by a given constant \( \pi \). The vehicle moves in the area supporting an unknown field distribution \( D(r) \) of a certain quantity. Here \( r := \text{col}(x, y) \) is the vector of the Cartesian coordinates \( x, y \) in the plane. The objective is to drive the vehicle into a vicinity of the point \( r^0 \) where the distribution \( D(r) \) attains its maximum and then to keep the vehicle in this vicinity, thus displaying the approximate location of \( r^0 \). To accomplish the mission, the on-board control system has access only to the concentration \( d := D(x, y) \) at the robot current position \( x = x(t), y = y(t) \) and the rate \( d \) at which this measurement evolves as time \( t \) progresses. This means violation of the key assumption adopted in many works on general optimization techniques, i.e., that the entire gradient \( \nabla D(x, y) \) is accessible.

The kinematic model of the vehicle is as follows:

\[
\begin{align*}
\dot{x} &= v \cos \theta, \\
\dot{y} &= v \sin \theta, \\
\dot{\theta} &= u, \quad |u| \leq \pi,
\end{align*}
\]

where \( \theta \) gives the vehicle orientation.

The problem is to design a controller that for a finite time \( t_0 \) drives the vehicle into the \( R_* \)-neighborhood

\[
V_* := \{ r : \| r - r^0 \| \leq R_* \}
\]

of the point \( r^0 \) where the concentration attains the maximum \( D(r^0) = \max_{r \in \mathbb{R}^2} D(r) \) and afterwards \( t \geq t_0 \), keeps the vehicle within this neighborhood.

In this paper, we examine the following control algorithm:

\[
u(t) = \pi \text{sgn} \{ \dot{d}(t) - v_* \}.
\]

The control parameter \( v_* > 0 \) is to be designed to achieve the control objective.

III. ASSUMPTIONS

In the above setting, the problem comprises the general problem of global optimization: find the absolutely best point to maximize the objective function. As is well known, global optimization problems are typically quite difficult to solve. In the presence of local extrema, NP-hardness, this mathematical seal for intractability, was established for even the simplest classes of such problems [16]. We do not mean to deal with NP-hard problems and impose the following assumption, which is realistic for many applications.

Assumption 3.1: The function \( D(\cdot) \) is twice continuously differentiable. The maximum \( \gamma_{\text{max}} := \max_{r \in \mathbb{R}^2} D(r) \) is attained at some point \( r^0 \), which is the only critical point, i.e., \( \nabla D(r) \neq 0 \forall r \neq r^0 \). The set \( \{ r \in \mathbb{R}^2 : D(r) \geq \gamma_r \} \) is bounded whenever inf \( \text{r}_e \in \mathbb{R}^2 \) \( D(r) < \gamma_r \leq \gamma_{\text{max}} \).

In many practical situations, the concentration \( D(r) \geq 0 \forall r \) and \( D(r) \to 0 \) as \( x^2 + y^2 \to \infty \). Then the last requirement from the assumption necessarily holds.

A good option in looking for the maximum would be constant improvement of the vehicle position \( r(t) \) due to the monotone growth of the related concentration \( d(t) := D[r(t)] \).

The parameter \( v_* \) from (3) can be informally interpreted as the desired growth rate.

Definition 3.1: A point \( r \in \mathbb{R}^2 \) is said to be growth controllable if whenever the growth \( \dot{d} \geq 0 \) of the concentration \( d \) was achieved for the vehicle at the location \( r \), it can be kept any longer. In other words, \( \dot{d} = 0 \Rightarrow \exists u \in [-\pi, \pi] : \dot{d} \geq 0 \).

The points outside a vicinity of the maximum one typically are growth controllable, as is implied by following criterion.

Proposition 3.1: A point \( r \) is growth controllable if and only if the (unsigned) curvature of the isoline

\[
I(\gamma) := \{ \tau : D(\tau) = \gamma \}
\]

passing through this point \( \gamma := D(r) \) is no greater than the maximal turning curvature \( \kappa_{\text{veh}} = \frac{\pi}{\gamma} \) of the vehicle whenever the isoline signed curvature \( \kappa(r) \) is negative; or in brief,

\[
\kappa(r) = \frac{D''(r) \Phi_\pi \nabla D(r); \Phi_\pi \nabla D(r)}{\| \nabla D(r) \|^3} \geq -\kappa_{\text{veh}}.
\]

The curvature sign corresponds to the case where the isoline is run in the direction for which grater values of \( D \).
are to the right, i.e., the pair of the vectors $\nabla D, \dot{r}$ is normally oriented. The curvature $\kappa_{veh}$ is the reciprocal $R^{-1}$ of the minimal turning radius $R$ of the vehicle, $\| \cdot \|$ and $\langle \cdot ; \cdot \rangle$ stand for the standard norm and inner product in the Euclidian plane, whereas $D''(r)$ and $\Phi_\alpha$ are the matrices of the second derivatives and rotation at angle $\alpha$, respectively:

$$
D''(r) := \begin{pmatrix}
D''_{xx}(r) & D''_{xy}(r) \\
D''_{yx}(r) & D''_{yy}(r)
\end{pmatrix}, \quad \Phi_\alpha := \begin{pmatrix}
\cos \alpha & -\sin \alpha \\
\sin \alpha & \cos \alpha
\end{pmatrix}.
$$

The proof of Proposition 3.1 will be given in the full version of this paper.

Isolines (4) with $\gamma \leq \max, D(r)$ typically are not much contorted. So growth uncontrollable points are usually confined to those with $\gamma \approx \max, D(r)$, which, in total, form a small region around the maximum point $r^0$.

To make the objective of driving the vehicle into the desired vicinity (2) of $r^0$ realistic, it is natural to require that all points outside this vicinity be growth controllable. The given information constraints, under which $\nabla D(r)$ and $\theta$ are not measured, along with the need to keep the vehicle in the above vicinity, motivate enhancement of this requirement. To state it, we note that thanks to Assumption 3.1,

$$
\gamma_* := \max_{r: \|r-r^0\|=R_*} D(r) < \gamma_{max},
$$

$$
R_* := \min_{r \in \Gamma(\gamma_*)} \|r-r^0\| > 0,
$$

(6)

where the isoline $I(\gamma_*)$ is given by (4).

Assumption 3.2: The minimal turning radius $R = v/\pi$ of the vehicle is less than $1/3R_*$; all points $r$ outside the disk $\{r: \|r-r^0\| < R_* - 2R\}$ are growth controllable. Moreover, (5) holds with $> \ $ put in place of $\geq \ $ for them.

IV. MAIN RESULT

The circle $C_{in}$ of the radius $R$ passing the vehicle initial location tangentially to its initial velocity so that the circle center is to the right with respect to the velocity vector is said to be minus-initial. In other words, this is the trajectory of the vehicle driven by $u \equiv -\pi$. We put

$$
\gamma_{in} := \min_{r \in C_{in}} D(r),
$$

$$
W := \left\{ r \in \mathbb{R}^2 : D(r) \geq \min_{r \in \mathbb{R}^2} \gamma_{in}, \gamma_* \right\},
$$

$$
\|r-r^0\| \geq R_* - 2R \right\},
$$

(7)

where $\gamma_*$ is defined in (6). Due to the last claim from Assumption 3.1, the set $W$ is compact since $\gamma_{in}, \gamma_* \geq \inf_{r \in \mathbb{R}^2} D(r)$. (Otherwise, the point furnishing the minimum $\gamma_{in}$ or $\gamma_*$ would furnish $\min_{r \in \mathbb{R}^2} D(r)$ and so would be critical, in violation of Assumption 3.1.)

Now we are in a position to state the main result of the paper.

Theorem 4.1: Suppose that Assumptions 3.1 and 3.2 hold and the parameter $v_*$ of the controller (3) is chosen so that

$$
0 < v_* < \min_{r \in W} \|\nabla D(r)\|,
$$

$$
\gamma(r) \cos \alpha + \frac{\sin^2 \alpha \langle D''(r) \nabla D(r); \nabla D(r) \rangle}{\cos \alpha \|\nabla D(r)\|^3} - 2\sin \alpha \frac{\langle D''(r) \Phi_\alpha \nabla D(r); \nabla D(r) \rangle}{\|\nabla D(r)\|^3} > -\kappa_{veh}
$$

for all $r \in W$, where

$$
\alpha = \alpha(v_*, r) := \arcsin \frac{v_*}{v/\|\nabla D(r)\|}
$$

(9)

and the signed curvature $\gamma(r)$ and the set $W$ are given by (5) and (7), respectively. Then the vehicle driven by the controller (3) reaches the desired vicinity (2) of the maximum point for a finite time $t_0$ and remains there afterwards: $r(t) \in V_*, t \geq t_0$.

The proof of this theorem will be given in the full version of this paper.

Remark 4.1: i) As $v_*$ → 0, the left hand side of (9) converges to that of (5) uniformly over the compact set $W$. So due to Assumption 3.2, the key requirements (8) and (9) are necessarily met whenever $v_*$ is small enough. At the same time, small $v_*$ may have a detrimental effect on the transient performance: it is expected that $t_0 \uparrow$ as $v_* \downarrow$.

ii) Practically, the choice of $v_*$ is based on putting an upper estimate $\hat{W} \supset W$ in place of $W$ in (8) and (9) and estimates of the quantities encountered there.

iii) A way to accomplish the choice of $v_*$ along the above lines is to first estimate the discrepancy in (5), i.e., to find $\Delta > 0$ such that

$$
\gamma(r) \geq -\kappa_{veh} + \Delta \quad \forall \ r \in \hat{W}.
$$

(11)

Then we observe that for $r \in \hat{W}$, (9) holds whenever

$$
a \tan^2 \alpha - 2b \tan \alpha + \Delta > 0,
$$

(12)

where

$$
a := \frac{\langle D''(r) \nabla D(r); \nabla D(r) \rangle}{\|\nabla D(r)\|^3}
$$

and

$$
b := \frac{\langle D''(r) \Phi_\alpha \nabla D(r); \nabla D(r) \rangle}{\|\nabla D(r)\|^3},
$$

(13)

which is equivalent to

$$
\tan \alpha < \chi(r) := \begin{cases}
\frac{1}{a} \left( 1 \leq \sqrt{b^2 - a\Delta} \right) & \text{if } a\Delta \leq b^2 \\
+\infty & \text{otherwise}
\end{cases}
$$

(14)

If $\chi(r) = \infty \forall r \in \hat{W}$, this is necessarily true and so (9) does hold. Otherwise, this can be re-written in the following form with regard to (10).

$$
0 < v_* < \min_{r \in W : \chi(r) < \infty} \frac{\chi(r) \|\nabla D(r)\|}{\sqrt{\chi(r)^2 + 1}},
$$

(14)

iv) The discontinuous control law (3) may exhibit a sliding motion. Theorem 4.1 addresses the equivalent dynamics [13].
V. THE ISOTROPIC DISTRIBUTION

In this section, we illustrate the controller design and Theorem 4.1 in the simple yet instructive case of the isotropic distribution:

\[ D(r) = \varphi(||r - r^0||), \quad \text{where} \]
\[ \varphi(p) \geq 0, \, \varphi(p) \to 0 \text{ as } p \to \infty, \]
\[ \varphi'(p) < 0 \, \forall p > 0, \, \varphi'(0) = 0. \]

The function \( \varphi(\cdot) \) is twice continuously differentiable. The maximum point \( r^0 \) is unknown. At the same time, we suppose that a rough estimate of the distance from the vehicle initial location \( r_{\text{in}} \) to \( r^0 \) is available:

\[ \| r_{\text{in}} - r^0 \| \leq R_{\text{est}}. \tag{16} \]

To simplify the matters, we assume that \( R_{\text{est}} > R_* + 4R \).

Assumption 3.1 is clearly fulfilled. The isolines are circles centered at \( r^0 \). So the growth controllable points are those outside the disk of the radius \( R \) centered at \( r^0 \). Hence \( \gamma_* = \varphi(R_*), R_* = R \), and Assumption 3.2 reduces to \( R_* > 3R \).

**Corollary 5.1:** Suppose that (15) and (16) hold, and the required distance to the maximum point \( R_* > 3R \). Then the vehicle driven by the controller (3) reaches the desired vicinity (2) of the maximum point for a finite time and remains there afterwards provided that the controller parameter \( v_* \) is chosen from the intersection of the following two intervals:

\[ 0 < v_* < v \min_{p \in [R_* - 2R]} |\varphi'(p)|, \quad R_- := R_* - 2R \]

\[ 0 < v_* < v \min_{p \in [R_* - 2R]} A(p), \quad R_+ := R_* + 2R, \]

\[ A(p) := \frac{|\varphi'(p)|}{\sqrt{1 + \frac{R(R_* - 2R)|\varphi'(p)|}{(R_* - 3R)|\varphi'(p)|}}}, \tag{17} \]

and the minimum over the empty set is assumed to be \( +\infty \).

\textit{a) Proof:} We employ iii) of Remark 4.1. By (7),

\[ W \subset \tilde{W} := \{ r : R_* - 2R \leq ||r - r^0|| \leq R_{\text{est}} + 2R \}. \tag{18} \]

The isoline passing through \( r \) is the circle of the radius \( ||r - r^0|| \). Its signed curvature \( \kappa(r) = -||r - r^0||^{-1} \). So in (11),

\[ \Delta = \min_{r \in \tilde{W}} \left[ \frac{1}{R} - \frac{1}{||r - r^0||} \right] = \frac{R_* - 3R}{R(R_* - 2R)}. \tag{19} \]

Elementary computation shows that now \( ||\nabla D(r)|| = |\varphi'(||r - r^0||)| \) and in (12) and (13),

\[ a = \frac{\varphi''(||r - r^0||)}{|\varphi'(||r - r^0||)|}, \quad b = 0. \]

It follows that (14) takes the form

\[ 0 < v_* < v \min_{r \in \tilde{W}, \varphi'(p) < 0} \frac{|\varphi'(p)|}{\sqrt{1 + \frac{R_* - 3R}{R(R_* - 2R)||\varphi'(p)||}}}, \]

where \( p := ||r - r^0||. \) By invoking (18) and (19), we arrive at the conclusion of the corollary.

If a certain substance undergoes diffusion from an instantaneous point-wise source in an isotropic unbounded environment, its distribution profile at a given time is often Gaussian:

\[ \varphi(p) = qe^{-\frac{p^2}{\sigma^2}}, \quad q \geq q_\text{min} > 0, \sigma > 0. \]

Then conditions (17) are implied by:

\[ 0 < v_* < \frac{\sqrt{q_\text{min}}}{\sigma \sqrt{2}} \min_{r \in [R_* - 2R]} e^{-\frac{1}{\sigma^2} \frac{r^2}{2R^2}} \]

\[ \text{if } \sigma < R_*; \quad 0 < v_* < \frac{q - q_\text{min}}{\sigma^2} \left\{ \begin{array}{ll} +\infty & \text{if } \sigma < R_*; \\ \sqrt{\frac{R_* - r^0}{R}} & \text{otherwise} \end{array} \right. \tag{20} \]

If the second moment \( \sigma \) is unknown but its estimate \( \sigma \in [\sigma_-, \sigma_+] \) is available, replacement of the right hand sides in the first and second lines of (20) by their minima over \( \sigma \in [\sigma_-, \sigma_+] \) provides constructive constraints on the parameter \( v_* \) under which the control objective is achieved.

VI. SIMULATION RESULTS

To illustrate the performance of the navigation law (3), we simulate a unicycle robot governed by kinematic equations (1) moving in a planar region which supports an unknown field distribution. The robot moves with constant linear velocity \( v = 0.5 \text{ m/s} \) and its maximum angular velocity is limited to \( \pi = 1 \text{ rad/s} \). The vehicle is supposed to travel to the \( R_* \)-neighborhood of the location at which the field distribution attains its maximum.

We consider the Gaussian distribution \( D(x,y) = 10e^{-((x-8)^2+(y-5)^2)/600} \) and choose \( R_* = 3m > 3R \) where \( R = v/\pi = 0.5m \). The controller parameter \( v_* = 0.05 \) is chosen to meet the conditions (20). The filed distribution is shown in Fig. 1. Under the navigation law (3), the robot moves along an equiangular spiral and approaches the \( R_* \)-neighborhood of the maximum point, shown in Fig. 2. In this figure, the dotted circles are the level sets of unknown field and the desired neighborhood of the maximum point with \( R_* = 3m \) is shown with the centered circle. Fig. 3 depicts the distance between the robot and the maximum point. As can be seen, the distance first monotonically decays and then stays within the pre-defined margin.

\textit{Remark 6.1:} On the sliding surface, the proposed controller generates high-frequency switching control signal.
In practice, this may lead to undesirable chattering due to constraints on available switching frequency and delay. A way to overcome this problem is to replace the discontinuous sign function in the control law (3) by its continuous approximation. Our computer simulation employs this method with approximation by a linear function with saturation.

### References


