A convex framework for the design of dynamic anti-windup for state-delayed systems

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Abstract—This work considers the design of dynamic anti-windup compensators for state-delayed systems subject to saturating actuators. Based on the use of a Lyapunov-Krasovskii approach, a generalized sector condition and some congruence transformations, an unified LMI-based framework for the synthesis of both rational and non-rational dynamic anti-windup compensators is proposed. Theoretical results ensure to guarantee the asymptotic and the input-to-state stability of the closed-loop system are presented both in local and in a global context. The proposed conditions are cast in convex optimization problems to compute anti-windup compensators aiming at maximizing the bound of admissible $L_2$ disturbances or maximizing the $L_2$-gain from the disturbance to the regulated output. A numerical example illustrates the application of the methodology.

I. INTRODUCTION

The anti-windup compensation is a well-known and efficient technique to cope with undesirable effects (on performances and stability) produced by actuator saturation in control loops. The first results regarding the design of anti-windup compensators were motivated by the degradation of the transient performance induced by saturation in feedback control systems containing integral actions (see for instance [1], [2]). More recently, the study of the anti-windup problem has been considered in a formal context and a large amount of systematic synthesis methods have been proposed (see for instance [3], [4] and the survey [5] for a large overview). In particular, some of these works are based on LMI (or “almost” LMI) conditions (see among others [6], [7], [8], [9]). The advantage of the LMI-based methods lies on the fact that the anti-windup design can be carried out through convex optimization problems. In this case, different optimal synthesis criteria (such as $L_2$-gain attenuation or enlargement of the basin of attraction) can be directly addressed in an optimal way.

Besides the actuator saturation, it is well-known that time-delays are present in many control applications and are also source of performance degradation and even instability (see for instance [10], [11] and references therein). However, it appears that most of the anti-windup design methods (as the ones mentioned in the previous paragraph) regard only undelayed systems. The anti-windup compensation for time-delay systems, was addressed, for instance, in [12], [13], [14] and [15]. In [12] and [13] plants subject to input and/or output delays are considered. For this case, it is considered the synthesis of a dynamic anti-windup compensator aiming at minimizing a cost function, that measures the absolute difference between the controller state considering saturation free actuators and the controller state when the plant input saturation is considered. It should however be pointed out that the results apply only to stable open-loop systems and that the approach does not consider systems presenting state delays. In [14] and [15], an LMI approach to synthesize stabilizing static anti-windup has been proposed. Differently from the classical objective of recovering performance, in these works the anti-windup compensation have been used to enlarge the region of attraction of the closed-loop system. In particular, the action of disturbances and closed-loop performance issues were not considered in these works. The dynamic anti-windup synthesis for state-delayed systems has recently addressed in [16] and [17]. The approach followed in [16] was based on congruence transformations, similar to the ones proposed in [18], allowing only the synthesis of non-rational compensators (i.e. presenting delayed terms in the dynamics). From a projection Lemma approach, in [17] it is shown that the synthesis of rational compensator can be carried out by true LMI conditions.

In this work, we also address the problem of synthesizing dynamic anti-windup compensators for state-delayed linear systems. Based on the use of a Lyapunov-Krasovskii approach, a generalized sector condition and the application of congruence transformations, differently from the ones considered in [16], true LMI conditions for the synthesis of both rational and non-rational anti-windup compensators are proposed. Results concerning the guarantee of local (regional) input-to-state as well as asymptotic stability are obtained and from them, the global case is derived as a particular case. The computation of the anti-windup compensator aiming at ensuring both input-to-state stability (considering $L_2$ bounded disturbances) and internal stability of the closed-loop system are therefore carried out from the solution of convex optimization problems. Some optimization criteria are considered for the synthesis; maximization of the $L_2$-norm upper bound on the admissible disturbances for which the trajectories are assured bounded and minimization of the $L_2$-gain of the disturbance to the system regulated output; The main contribution with respect to [16] and [17] is that the present approach unifies, in the same framework, both the synthesis of rational as well as non-rational anti-windup...
compensators, allowing different structural configurations for
the compensators. It should be noticed that this could not be
done in the previous cited approaches.

Notation - For two symmetric matrices, A and B, A > B means that A - B is positive definite. A' denotes the transpose of
A. A_{(i)} denotes the i^{th} line of matrix A. \ast stands for symmetric
blocks; \Lambda(P) and \bar{\Lambda}(P) denote the minimal and maximal eigenvalues of matrix
P, respectively. \Phi \Sigma = \mathcal{C}([-\tau, 0], \mathbb{R}^n) is the Banach Space of
continuous vector functions mapping the interval \([-\tau, 0]\) into \mathbb{R}^n
with the norm \| \phi \|_c = \sup_{\tau < t < 0} \| \phi(t) \|. \| \cdot \| refers to the
Euclidean vector norm. \Phi \Sigma^* is the set defined by \Phi \Sigma^* = \{ \phi \in \Phi \Sigma; \| \phi \|_c < v, v > 0 \}. For v \in \mathbb{R}^m, sat(v) : \mathbb{R}^m \to \mathbb{R}^m denotes
the classical symmetric saturation function defined as (sat(v))(i) =
\text{sign}(v(i)) \min(u_{\text{sat}}(i), |v(i)|), \forall i = 1, ..., m, where
u_{\text{sat}}(i) > 0 denotes the i^{th} magnitude bound. blockdiag(\cdots) is
a block diagonal matrix whose diagonal blocks are the ordered
arguments. H_{\epsilon}(A) = A' + A.

II. PROBLEM STATEMENT

Consider the following plant model:
\begin{align*}
\dot{x}(t) &= Ax(t) + A_d x(t - \tau) + Bu(t) + B_w w(t) \\
y(t) &= C_y x(t) \\
z(t) &= C_z x(t) + D_z u(t)
\end{align*}
(1)
where vectors \( x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m, w(t) \in \mathbb{R}^q, y(t) \in \mathbb{R}^p, z(t) \in \mathbb{R}^d \) are the plant state, input, disturbance, measured output and regulated output, respectively. The time-delay \( \tau \) is assumed to be constant. \( A, A_d, B, B_w, C_y, C_z, D_z \) are matrices of appropriate dimensions.

The plant inputs are supposed to be bounded as follows:
\begin{equation}
-u_{\text{sat}}(i) \leq u(i), u_{\text{sat}}(i) > 0, i = 1, \cdots, m
\end{equation}
(2)
The disturbance vector \( w(t) \) is assumed to be limited in energy, i.e. \( w(t) \in \mathcal{L}_2 \), and for some scalar \( \delta \), \( 0 \leq \frac{1}{\delta} < \infty \), the disturbance \( w(t) \) is bounded as follows:
\begin{equation}
\| w \|_2^2 = \int_{0}^{\infty} w(t)'w(t)dt \leq \frac{1}{\delta}
\end{equation}
(3)

In order to control the plant (1), we assume that the following controller has been designed for stabilizing the system disregarding the control bounds given in (2):
\begin{align*}
\dot{x}_c(t) &= A_c x_c(t) + A_{c,d} x_c(t - \tau) + B_c u_c(t) \\
y_c(t) &= C_y x_c(t) + C_{c,d} x_c(t - \tau) + D_c u_c(t)
\end{align*}
(4)
where \( x_c(t) \in \mathbb{R}^{n_c}, u_c(t) \in \mathbb{R}^p \) and \( y_c(t) \in \mathbb{R}^{m_c} \). Matrices \( A_c, A_{c,d}, B_c, C_y, C_{c,d}, D_c \) are matrices of appropriate dimensions. Note that due to the delayed terms in (4), the transfer function of this controller is non-rational and then this controller is referred as a non-rational controller, as pointed in [19]. Of course, rational controllers can also be considered. In this case it suffices to set \( A_{c,d} = 0 \) and \( C_{c,d} = 0 \).

The nominal interconnection of the controller (4) with the plant (1) is \( u_c(t) = y(t) \) and \( u(t) = y_c(t) \). In consequence of the control bounds, the \textit{de facto} control signal to be injected in the system considering the controller output \( y_c(t) \) is:
\begin{equation}
u(t) = \text{sat}(y_c(t))
\end{equation}

Considering that the controller was designed disregarding the control bounds, the following anti-windup compensator is proposed to mitigate the performance degradation induced by the control saturation and to assure the system closed-loop stability:
\begin{align*}
\dot{x}_a(t) &= A_a x_a(t) + A_{a,d} x_a(t - \tau) + B_a \psi(y_c(t)) \\
y_a(t) &= C_y x_a(t) + C_{a,d} x_a(t - \tau) + D_a \psi(y_c(t)) \\
z_a(t) &= E_a x_a(t) + E_{a,d} x_a(t - \tau) + F_a \psi(y_c(t))
\end{align*}
(5)
with vectors \( \psi(y_c(t)) = \text{sat}(y_c(t)) - y_c(t), x_a(t) \in \mathbb{R}^{n + m_c}, y_a(t) \in \mathbb{R}^{m_c}, z_a(t) \in \mathbb{R}^m \) being, respectively, the input, the state and the output of the compensator. In particular, \( y_a(t) \) compensates the controller dynamics, and \( z_a(t) \) the controller output, [7], [13]. As the controller (4), the anti-windup (5) presents a non-rational structure. The presence of delayed terms in (5) offers additional degrees of freedom in the synthesis procedure through matrices \( A_{a,d}, C_{a,d}, E_{a,d} \), as it will be seen in the sequel. Hence, the controller (4) is modified as:
\begin{align*}
\dot{x}_c(t) &= A_c x_c(t) + A_{c,d} x_c(t - \tau) + B_c u_c(t) + y_a(t) \\
y_c(t) &= C_y x_c(t) + C_{c,d} x_c(t - \tau) + D_c u_c(t) + z_a(t)
\end{align*}
(6)

In this work, we are interested in the synthesis of the dynamic anti-windup compensator (5) (i.e.: in computing matrices \( A_a, A_{a,d}, B_a, C_y, C_{a,d}, D_a, E_a, E_{a,d}, F_a \) to address the following problems:

- **External stabilization**: the compensator should ensure that the closed-loop trajectories of the system remain bounded for any disturbance satisfying (3) for a certain \( \delta \), i.e. the input-to-state stability should be ensured. Moreover, the controller should ensure an upper bound for the \( \mathcal{L}_2 \)-gain between the disturbance \( w(t) \) and the regulated output \( z(t) \), which corresponds to a disturbance rejection problem.

- **Internal Stabilization**: in the absence of disturbances, the internal asymptotic stability of the closed-loop system (1)-(5)-(6) should be guaranteed for all initial conditions belonging to a certain set \( \mathcal{D}_0 \) of functions defined on interval \([-\tau, 0]\), i.e. the set \( \mathcal{D}_0 \) is included in the basin of attraction of the origin of the closed-loop system (1)-(5)-(6).

III. MAIN RESULTS

A. Preliminaries

Let us define the following matrices:
\begin{align*}
A &= \begin{pmatrix} A + BD_c C_y & BC_c \\ B_c C_y & A_c \end{pmatrix}, & A_d &= \begin{pmatrix} A_d & BC_{c,d} \\ 0 & A_{c,d} \end{pmatrix} \\
B &= \begin{pmatrix} B \\ 0 \end{pmatrix}, & \bar{I} &= \begin{pmatrix} 0 \\ I \end{pmatrix}, & B_w &= \begin{pmatrix} B_w \\ 0 \end{pmatrix} \\
C_y &= \begin{pmatrix} C_y \\ 0 \end{pmatrix}, & C_z &= \begin{pmatrix} C_z + D_z D_c C_y \\ D_z C_c \end{pmatrix} \\
C_{c,d} &= \begin{pmatrix} 0 \\ D_z C_{c,d} \end{pmatrix}, & D_z &= \begin{pmatrix} D_z \\ 0 \end{pmatrix} \\
K &= \begin{pmatrix} D_c C_y & C_c \end{pmatrix}, & K_a &= \begin{pmatrix} 0 \\ C_c \end{pmatrix}
\end{align*}
(7)
\[
A = \begin{pmatrix}
A & BE_a + \tilde{I}C_a \\
0 & A_d
\end{pmatrix},
B = \begin{pmatrix}
B + BFa + \tilde{I}D_a \\
Ba
\end{pmatrix},
A_d = \begin{pmatrix}
A_d & BE_{a,d} + \tilde{I}C_{a,d} \\
0 & A_{d,d}
\end{pmatrix},
B_w,
\]
\[
\mathbb{K} = \begin{pmatrix}
K & E_a \\
0 & K_d
\end{pmatrix},
\mathbb{K}_w = B_a
\]

In the sequel, we show that (12), (13) and (14) imply that \( J(t) < 0 \), provided \( \phi_\xi \in B(\beta) \) and \( w(t) \) satisfies (3).

The presence of the saturation nonlinearity in the control loop.

Remark: It is noted that the set \( \mathcal{U}(\gamma t, \varphi(t)) \) is defined as the set of all possible values of the control input \( \varphi(t) \) at time \( t \) that are consistent with the constraint \( |\varphi(t)| \leq \gamma \) for all \( t \in [0, T] \). This set is used to determine the feasible region for the control input, ensuring that the control action is bounded and remains within the specified limits.

The following lemma, concerning the deadzone nonlinearity \( \psi_\xi(t) \), can be stated:

**Lemma 3.1:** [14], [20] If \( \xi(t) \in \mathcal{S}(u_\alpha) \) then the relation:

\[
\psi_\xi(t) = \psi_\xi(t) - \mathbf{G} \xi(t)
\]

is verified for any diagonal positive definite matrix \( T \in \mathbb{R}^{m \times m} \).

In the sequel, Lemma 3.1 will be used to take into account the presence of the saturation nonlinearity in the control loop.

**B. Local Stabilization Results**

The following theorem regards the synthesis problem in a regional (local) context. In this case, the stability is ensured provided that the initial conditions of the system and the disturbances belong to certain admissible sets.

**Theorem 3.1:** If there exist symmetric positive definite matrices \( Y, X, H_1, H_3 \in \mathbb{R}^{(n + n_z) \times (n + n_z)} \), a diagonal positive definite matrix \( S \in \mathbb{R}^{m \times m} \), matrices \( E, D_t, U, V \in \mathbb{R}^{n \times (n + n_z)} \), \( H_2, A_d \in \mathbb{R}^{(n + n_z) \times (n + n_z)} \), \( \tilde{B} \in \mathbb{R}^{n \times (n + n_z)} \), \( \tilde{C}_d \in \mathbb{R}^{m \times (n + n_z)} \), \( \tilde{D} \in \mathbb{R}^{m \times m} \), \( \tilde{F} \in \mathbb{R}^{m \times m} \) and positive scalars \( \mu \) and \( \gamma \) such that the LMIs (12) (in the top of the next page), (13) and (14) are verified:

\[
\begin{pmatrix}
X & * & * \\
KX + U & KY + \tilde{E} + V & \mu u^2_{\alpha(i)}
\end{pmatrix}
\]

is satisfied. Then, there exists a dynamic anti-windup compensator as defined in (5), which ensures that:

1. the trajectories of the system (9) are bounded for every initial condition in the ball \( B(\beta) \):

\[
B(\beta) = \{ \phi_\xi \in C^\ast_\nu : \| \phi_\xi \|_2^2 \leq \beta \langle \lambda(P) + \tau \lambda(R) \rangle \},
\]

with \( 0 \leq \beta \leq \mu^{-1} - \delta^{-1} \).

2. when \( w(t) = 0 \), the closed-loop system is locally asymptotically stable, and for all initial conditions belonging to

\[
B(\mu) = \{ \phi_\xi \in C^\ast_\nu : \| \phi_\xi \|_2^2 \leq \mu^{-1} - \delta^{-1} \}
\]

the corresponding trajectories converge asymptotically to the origin, where matrices \( P = P' > 0 \) and \( R = R' > 0 \) are given by

\[
P = \begin{pmatrix}
X^{-1} & * \\
M & E
\end{pmatrix},
Q = P^{-1} = \begin{pmatrix}
Y & * \\
N & F
\end{pmatrix},
\]

with \( M, N \) such that \( N'M = I - YX^{-1} \).

**Proof:** Consider the Lyapunov-Krasovskii candidate functional [10], [11]:

\[
V(t) = \xi(t)' \mathbf{P} \xi(t) + \int_{t-\tau}^{t} \xi(t)'R \xi(t) d\theta
\]

In particular, it satisfies:

\[
\nu \langle \lambda(P) \| \xi(t) \|_2^2 \leq V(t) \leq \langle \lambda(P) + \tau \lambda(R) \| \xi(t) \|_2^2, \]

where \( \xi(t) \) denotes the restriction of \( \xi(t) \) to the interval \([t-\tau,t] \) [10]. Hence, if the initial condition \( \phi_\xi \in B(\beta) \), it follows that \( V(0) \leq \beta \).

Define now \( \mathcal{J}(t) = \dot{V}(t) - w(t)'w(t) + \frac{1}{2} z(t)'z(t) \). If \( \mathcal{J}(t) < 0 \), it follows that

\[
\frac{1}{T} \int_{0}^{T} \mathcal{J}(t) dt = \dot{V}(0) - \int_{0}^{T} \dot{w}(t)'w(t) dt
\]

which, \( \forall \phi_\xi \in B(\beta) \), with \( \beta + \frac{1}{2} \leq \mu^{-1} \), implies that

\[
V(T) \leq V(0) + \| w(t) \|_2^2 \leq \beta + \delta^{-1} \leq \mu^{-1}
\]

Hence, one gets \( \xi(T)' \mathbf{P} \xi(T) \leq V(T) \leq \mu^{-1} \), i.e. for all \( T > 0 \) the trajectories of the system do not leave the set \( \mathcal{E}(u, \mu) = \{ \xi(t) \in \mathbb{R}^{2(n + n_z)} : \xi(t)' \mathbf{P} \xi(t) \leq \mu^{-1} \} \) for all \( w(t) \) satisfying (3). Moreover, for \( T \to +\infty \), (16) yields:

\[
\| z(t) \|_2^2 \leq \gamma \| w(t) \|_2^2 + \gamma V(0)
\]

In the sequel, we show that (12), (13) and (14) imply that \( \mathcal{J}(t) < 0 \), provided \( \phi_\xi \in B(\beta) \) and \( w(t) \) satisfies (3).
Considering (9), (15) and from Lemma 3.1, provided that \( \xi(t) \in S(u_0) \), it follows that:

\[
\mathcal{J}(t) < 0
\]

\( \forall t > 0 \), and by pre and post multiplying (18) by blockdiag\((Q, P, S, I, I)\), respectively, we get:

\[
\begin{bmatrix}
He(PA) + R
B'P + TG
\begin{pmatrix}
\mathbb{B}
\mathbb{C}_z
\mathbb{C}_{z,d}
\mathbb{D}_z
\end{pmatrix}
\end{bmatrix}
< 0
\]

From (17) and the Schur's complement, once the following condition is verified,

\[
\begin{bmatrix}
He(PA) + R
B'P + TG
\begin{pmatrix}
\mathbb{B}
\mathbb{C}_z
\mathbb{C}_{z,d}
\mathbb{D}_z
\end{pmatrix}
\end{bmatrix}
< 0
\]

it follows that (19) is equivalent to LMI (12).

From (19) we can conclude that the trajectories of \( \xi(t) \) never leave the ellipsoid \( \varepsilon(P, \mu^{-1}) \), for all \( t > 0 \), provided that \( \xi(t) \in S(u_0) \). In this sense, the inclusion of \( \varepsilon(P, \mu^{-1}) \) in \( S(u_0) \) assures that \( \xi(t) \in S(u_0) \), i.e. it enforces the validity of (11). This is assured through the verification of the following condition:

\[
\begin{bmatrix}
P
Q
\end{bmatrix}
< 0
\]

which is equivalent to:

\[
\begin{bmatrix}
P
Q
\end{bmatrix}
< 0
\]

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\begin{pmatrix}
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\mathbb{D}_z
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\[
\begin{bmatrix}
P
Q
\end{bmatrix}
< 0
\]

where \( Z = \mathbb{GQ} = \begin{pmatrix} V & \bar{U} \end{pmatrix} \), and \( \Gamma = QRQ \), with \( \Gamma = \begin{pmatrix} \Gamma_1 & * \\ \Gamma_2 & \Gamma_3 \end{pmatrix} \).

Similarly as [18], we define a matrix \( \Upsilon = \begin{pmatrix} I & MX \\ 0 & I \end{pmatrix} \).

Multiplying now (19) by blockdiag\((\Upsilon', \Upsilon', I, I, I)\) and its transpose by the left and the right side respectively and considering the following variable change:

\[
\begin{pmatrix}
\hat{A}
\hat{A}_d
\hat{B}
\hat{C}
\hat{C}_d
\hat{D}
\hat{E}
\hat{E}_d
\hat{F}
\end{pmatrix}
= \begin{pmatrix}
\begin{pmatrix} X&M' \\ 0 & 0 \end{pmatrix}
0 & I
\begin{pmatrix} N & 0 \\ 0 & N \end{pmatrix}
\begin{pmatrix} E_a & E_{a,d} \\ F_a & F_{a,d} \end{pmatrix}
\begin{pmatrix} I & XM' \\ 0 & I \end{pmatrix}
\end{pmatrix}
\]

it follows that (19) is equivalent to LMI (12).

From (19) we can conclude that the trajectories of \( \xi(t) \) never leave the ellipsoid \( \varepsilon(P, \mu^{-1}) \), for all \( t > 0 \), provided that \( \xi(t) \in S(u_0) \). In this sense, the inclusion of \( \varepsilon(P, \mu^{-1}) \) in \( S(u_0) \) assures that \( \xi(t) \in S(u_0) \), i.e. it enforces the validity of (11). This is assured through the verification of the following condition:

\[
\begin{bmatrix}
P
Q
\end{bmatrix}
< 0
\]

where \( Z = \mathbb{GQ} = \begin{pmatrix} V & \bar{U} \end{pmatrix} \), and \( \Gamma = QRQ \), with \( \Gamma = \begin{pmatrix} \Gamma_1 & * \\ \Gamma_2 & \Gamma_3 \end{pmatrix} \).

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\[
\begin{pmatrix}
\hat{A}
\hat{A}_d
\hat{B}
\hat{C}
\hat{C}_d
\hat{D}
\hat{E}
\hat{E}_d
\hat{F}
\end{pmatrix}
= \begin{pmatrix}
\begin{pmatrix} X&M' \\ 0 & 0 \end{pmatrix}
0 & I
\begin{pmatrix} N & 0 \\ 0 & N \end{pmatrix}
\begin{pmatrix} E_a & E_{a,d} \\ F_a & F_{a,d} \end{pmatrix}
\begin{pmatrix} I & XM' \\ 0 & I \end{pmatrix}
\end{pmatrix}
\]

it follows that (19) is equivalent to LMI (12).
Corollary 3.1: If there exist symmetric positive definite matrices \( Y, X, H_1, H_2 \in \mathbb{R}^{(n+n_s) \times (n+n_s)} \), a diagonal positive definite matrix \( S \in \mathbb{R}^{m \times m} \), matrices \( H_2, A, \dot{A} \in \mathbb{R}^{(n+n_u) \times (n+n_s)} \), \( \dot{B} \in \mathbb{R}^{(n+n_u) \times m} \), \( \dot{C}, \dot{C}_d \in \mathbb{R}^{n_c \times (n+n_u)} \), \( D \in \mathbb{R}^{n_c \times m} \), \( E, E_d \in \mathbb{R}^{m \times (n+n_u)} \), \( F \in \mathbb{R}^{m \times m} \) and a positive scalar \( \gamma \) satisfying LMI (12) with \( V = -KY - E \) and \( U = -KX \) and
\[
Y - X > 0 \tag{22}
\]
then there exists a dynamic anti-windup compensator as defined in (5), which ensures that:

1) the trajectories of the system (9) are bounded for every initial condition \( \phi_0 \in \mathcal{C}_2' \) and any \( \omega(t) \in \mathcal{L}_2 \)
2) \( \|z(t)\|_2^2 < \gamma \|w(t)\|_2^2 + \gamma V(0) \)
3) when \( w(t) = 0 \), the closed-loop system origin is globally asymptotically stable.

Proof: In (10) consider \( \mathcal{G} = -\mathcal{K} \). It follows that the sector condition (10) of Lemma 3.1 is verified for all \( \xi(t) \in \mathbb{R}^{1 \times (n+n_u)} \). In this case variables \( V \) and \( U \) of LMI conditions (12) and (13), become \( V = -KY - E \) and \( U = -KX \).

D. Compensator Determination

Once the LMIs of Theorem 3.1 (or Corollary 3.1) are verified, we can determine the corresponding anti-windup compensator (5) from (20), through the following relation:

\[
\begin{pmatrix}
A_a & A_{a,d} & B_a \\
C_a & C_{a,d} & D_a \\
E_a & E_{a,d} & F_a
\end{pmatrix}
= \begin{pmatrix} X & M' & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}^{-1}
\begin{pmatrix} \dot{A} & \dot{A}_d & \dot{B} \\ \dot{C} & \dot{C}_d & \dot{D} \\ \dot{E} & \dot{E}_d & \dot{F} \end{pmatrix}
\begin{pmatrix} N & 0 & 0 \\ 0 & N & 0 \\ 0 & 0 & S \end{pmatrix}^{-1}
= \begin{pmatrix} A & \dot{A}_d & \dot{B} \\ \dot{C} & \dot{C}_d & \dot{D} \\ \dot{E} & \dot{E}_d & \dot{F} \end{pmatrix}
\begin{pmatrix} N & 0 & 0 \\ 0 & N & 0 \\ 0 & 0 & S \end{pmatrix}^{-1}
\]
with \( N'M = I - YX^{-1} \).

Comment 1: Note that from (23) and (20), the results of Theorem 3.1 and Corollary 3.1 allow the synthesis of a full non-rational compensator as well as a rational one, within the same framework. This last case can be addressed by setting \( \dot{A}_d = 0, \dot{C}_d = 0, \dot{E}_d = 0 \) in the LMI (12). In particular, as pointed in [17] the use of rational anti-windup compensators is interesting when the delay is not precisely known or time varying.

Comment 2: LMI (12) is equivalent to (19). In this latter inequality, the first matrix block is \( A \cdot F + FA' \). This fact ensures the resulting synthesized rational anti-windup compensator is stable. Exploiting this fact, it is possible to constrain the dynamic of the anti-windup compensator as performed in [9].

IV. OPTIMIZATION PROBLEMS

Verifying LMIs of Theorem 3.1 ensures that the closed-loop system (9) presents bounded trajectories for any admissible disturbance, provided that the initial conditions belong to the set \( \mathcal{B}(\beta) \). Since the proposed conditions are in LMI form, they can be considered in convex optimization problems. We present in the sequel two problems of interest.

A. Maximization of the disturbance tolerance

The idea is to maximize the \( \mathcal{L}_2 \) norm bound on the disturbance for which it can be ensured that the system trajectories remain bounded. Considering that the initial condition is null (i.e., \( \phi_2(\theta) = 0, \forall \theta \in [-\tau, 0] \)), this can be accomplished by the following convex optimization problem:

\[
\min \mu \\
\text{subject to} \ (12) - (13) \tag{24}
\]

Note that since the initial condition is assumed to be null, we have \( \beta = 0 \), then \( \mu = \delta \).

B. Maximization of the disturbance attenuation

For a non-null bound on the \( \mathcal{L}_2 \) norm of the admissible disturbances (given by \( \mu^{-1} = \delta^{-1} \)), the idea is to minimize the upper bound for the \( \mathcal{L}_2 \) gain of \( w(t) \) on \( z(t) \). Considering that the initial condition is null (i.e., \( \phi_2(\theta) = 0, \forall \theta \in [-\tau, 0] \)), this can be obtained from the solution of the following convex optimization problem:

\[
\min \gamma \\
\text{subject to} \ (12) - (13) \tag{25}
\]

V. NUMERICAL EXAMPLES

Example 5.1: Consider system (1) given by the following matrices:

\[
A = \begin{pmatrix} -1.5 & 1 \\ 1 & 0 \end{pmatrix}, \ A_d = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\]

and a PI stabilizing controller described by the matrices:

\[
A_c = 0, \ B_c = -1, \ C_c = 0.15, \ D_c = -3,
\]

The control bound is given by \( u_o = 1 \). Since the open-loop system is unstable we apply the local stability results proposed in Theorem 3.1. We consider both the non-rational and rational cases (i.e. with matrices \( A_{a,d}, C_{a,d}, E_a, E_{a,d}, F_a \) forced to be null). These results are compared with the ones obtained with a static anti-windup compensator, computed from an adaptation of the technique proposed in [21].

Considering the optimization problem (24) in Table I it is shown the obtained optimal value for the 3 cases above. Note that the value obtained with the non-rational compensator is smaller, which means that the bound on the tolerable \( \mathcal{L}_2 \) disturbance is greater with this compensator.

\[
\begin{array}{|c|c|}
\hline
\text{Method} & \text{optimal value} \\
\hline
\text{Nonrational AW} & 8.6 \\
\text{Rational AW} & 13.8 \\
\text{Static AW [21]} & 17.1 \\
\hline
\end{array}
\]

TABLE I

RESULTS OF PROBLEM (24)

In Table II, it is shown the value of \( \gamma \) obtained from the solution of problem (25) for the same 3 cases, considering different values of \( \mu \). Note that the smaller \( \mu \) greater is the bound on the \( \mathcal{L}_2 \) disturbance and, consequently, greater is
the $L_2$-gain bound. N/A means that the optimization problem was unfeasible.

Example 5.2: Consider system (1) given by the following matrices:

\[
A = -0.1, \ A_d = 0.08, \ B = 1, \ B_w = 0.1
\]
\[
C_y = C_z = 1, \ D_z = 0
\]

Consider a PI stabilizing controller described by the following matrices:

\[
A_c = 0, \ B_c = -0.4, \ C_c = 1, \ D_c = -2,
A_{c,d} = 0, \ C_{c,d} = 0.
\]

The control bound is given by $u_n = 1$. Now, the open-loop system is stable, therefore we can apply the global stability results presented in Subsection III-C. We then solve the optimization problem (25).

For simulation purposes consider the following $L_2$ disturbance:

\[
w(t) = \begin{cases} \bar{w} & 0 \leq t < 1 \\ 0 & t \geq 1 \end{cases}
\]

with $\bar{w} = 100$ and $\tau = 5$.

In Figure 1, it is depicted the response of the closed-loop system considering: a rational anti-windup compensator (obtained from Corollary 3.1), a static anti-windup (obtained from [21]), and no anti-windup compensation. As expected, the regulated output response with the dynamic rational anti-windup is more damped and presents a smaller settling time.

![Comparison between different methodologies for global stability scope](image)

VI. CONCLUSION

In this work we have presented a methodology for synthesizing dynamic anti-windup compensators for systems subject to state delays and input saturation. The conditions that ensure the existence of a solution are obtained in LMI form. The proposed framework encompasses previous results in the sense that rational as well as non-rational anti-windup compensators can be synthesized within the same convex framework. Additionally, restrictions on the anti-windup dynamics can be considered through slight changes in the LMIs.

REFERENCES