On the design of reconfigurable two layer hierarchical control systems with MPC

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Abstract— This paper deals with the design of a hierarchical two-layer controller. At the high level, a robust MPC regulator works at a slow frequency rate and computes the ideal control signals needed to suitably control the plant. At the low level, a number of already controlled actuators drive the plant at a faster time rate and track the desired control signals as accurately as they can, in accordance to their dynamics. The discrepancy between the ideal control actions computed at the high level and those actually provided to the plant gives rise to a robustness issue for the overall control system. To tackle this problem, we propose to model such a discrepancy as a disturbance term to be rejected in the design of the high level controller by means of the small-gain approach. Moreover, setting an optimization procedure at each time step, the proposed algorithm also allows for an easy reconfiguration of the control system in the occurrence of an actuator addition/replacement.

I. INTRODUCTION

This paper deals with the design of a two-layer controller with the Model Predictive Control (MPC) approach. It thus sides with those research lines that attend to hierarchical control, which has been receiving great attention within the technical community in the last decades, see e.g. the books [8], [3], the recent survey papers [19], [14] and the many references therein. Among the numerous contributions to this research theme, we mention the design of controllers for multitime scale systems characterized by clearly separable slow and fast dynamics, see e.g. [3], [6], [2]. Hierarchical control for the coordination of a number of local controllers has been considered in [5], [11], [10]. Finally, cascade control schemes (see e.g. [17]) are the closest hierarchical control applications to the problem studied in this paper.

The approach proposed here stems from the preliminary studies [16], [15], [13], [12], [11], where related MPC results have been stated. In particular, drawing from the rationale developed in [11], where a theoretical background on hierarchical two-level control systems has been developed, in this paper we consider a cascade controller, where the high level provides for a slow dynamics regulator, computing the reference signals the plant would ideally need to be suitably controlled. In turn, at the low layer, a number of faster actuation control loops are in charge of tracking such references as accurately as they can. Because of the dynamic behavior of the actuation equipment, a discrepancy between the ideal control actions determined at the high level and those effectively afforded to the plant arises, so leading to a robustness problem for the overall control system. To tackle this problem, we design the upper level controller resorting to a robust MPC approach (see [7]). The resulting MPC algorithm is then readily extended to cope with the self reconfiguration of the controller, owing to an actuator replacement/addition.

The proposed approach can take a significant role within the “Plug and Play” research community, which studies an emerging control strategy acting as soon as a new device, in general a sensor or actuator, is plugged/substituted into an already functioning control system (see the very recent works [9], [18]). When many actuators are present in the plant, a complete re-design of the controller further to the addition/replacement of only one actuator may often be undesirable for various reasons. Hence, an on-line reconfiguration is advisable, in order to guarantee an incrementally self updatable control apparatus still ensuring desired stability and performance properties.

The paper is organized as follows: in Section II, the statement of the control problem considered is depicted, while in Section III, the two-level hierarchical MPC controller is outlined and the convergence properties of corresponding closed-loop system are discussed. Section IV is devoted to describe the reconfiguration capabilities of the proposed approach. A simulation example is discussed in Section V to highlight the effectiveness of the algorithm proposed. Finally, concluding remarks are reported in Section VI.

Notation. We will consider two time scales: in particular, we will denote the fast discrete-time index by $h$, while we will represent the slow discrete-time index by $k$. By $\| \cdot \|$ we denote the Euclidean vector or induced matrix norm. For $x \in \mathbb{R}^n$ and $\mathbb{R}^{n \times n} \ni P > 0$, we let $\| x \|_P = \sqrt{x^T P x}$.

II. PROBLEM FORMULATION

Let us consider a discrete-time linear system modeling a plant operated by $m$ actuators, whose dynamical behavior in the fast time scale is given by

$$\mathcal{P} : x_f (h + 1) = A_f x_f (h) + \sum_{i=1}^{m} b_i f_i (h)$$

(1)

where $x_f \in \mathbb{R}^{n_f}$ is the measurable state while $u_i \in \mathbb{R}$ stands for the action provided by the $i$-th actuator. Throughout this paper, we suppose that simple (e.g. PI-like) control loops acting on all the systems placed at the low level (i.e. the actuators) are a-priori designed and already working. Moreover, we assume that the resulting closed-loop systems can be described by first order unitary gain SISO models of
the form
\[ S_{act} : \begin{cases} 
\zeta_i(h+1) = f_i\zeta_i(h) + (1 - f_i)u_i(h), \zeta_i(0) = \zeta_{i0} \\
\hat{u}_i(h) = \zeta_i(h) \\
|f_i| < 1 
\end{cases} \tag{2} \]
\[ \forall i = 1, \ldots, m, \] whose output variables \( \hat{u}_i \)'s coincide with the inputs \( u_i \)'s of system (1).

The control objective consists of achieving a stabilizing control law for the cascade interconnection of systems (1) and (2) which is also suitable to handle modifications in the actuation equipment. To this end, we propose a two-level hierarchical regulator. The high level provides for a controller working at a slow time scale and computing the reference signal \( u_i \) to be tracked by each actuator control loop. In turn, at the low level, all the actuators concur to drive the plant, tracking their own reference signal in accordance to their closed loop dynamics (2). For this reason, one in general has \( \hat{u}_i \neq u_i \), at least in transient conditions, so consequently a robustness problem arises. To cope with this problem, we consider the discrepancy between the ideal control actions and those effectively afforded to the process as a disturbance term the high level controller has to be robust to. The design of such a controller is carried out by means of the small-gain paradigm, so allowing to properly manage actuators’ addition/replacement events while still guaranteeing the desired stability properties.

A. Model of the plant in the slow time scale

As for system (1), we consider the control constraints
\[ u_i^f \in U_i = [-\alpha_i, \alpha_i], \alpha_i > 0 \quad \forall i = 1, \ldots, m. \tag{3} \]

In addition, we let \( U = U_1 \times \cdots \times U_m \subset \mathbb{R}^m \) and \( B^f = [b^f_1 \ ... \ b^f_m] \subset \mathbb{R}^{n_x \times m} \).

Assumption 1: The pair \((A^f, B^f)\) is stabilizable.

For some fixed integer \( \tau \geq 1 \), let us decompose the control variables \( u_i^f \)'s of system (1) in the form \( u_i^f(h) = \hat{u}_i(h) + (u_i^f(h) - \hat{u}_i(h)) \), \( \hat{u}_i(h) \in U_i \) being some piecewise constant signals so that, \( \forall k \in \mathbb{N} \) and \( \forall j \in 0, \ldots, \tau - 1 \), it holds that \( \hat{u}_i(\tau k + j) = \hat{u}_i(\tau k) \). Then system (1) can be rewritten as
\[ x^f(h+1) = A^f x^f(h) + \sum_{i=1}^m b^f_i \hat{u}_i(h) + \sum_{i=1}^m b^f_i w^f_i(h) \tag{4} \]
where \( w^f_i(h) = u_i^f(h) - \hat{u}_i(h) \) is considered as a matched disturbance term.

Letting \( x(k) = x^f(\tau k), u_i(k) = \hat{u}_i(\tau k) \) and
\[ A = (A^f)^\tau, \quad b_i = \sum_{j=0}^{\tau-1} (A^f)^{\tau-j} b^f_i, \]
\[ w_i(k) = \sum_{j=0}^{\tau-1} (A^f)^{\tau-j} b^f_i w^f_i(\tau k + j), \tag{5} \]
the sampled version of (4) in the slow sampling rate is
\[ x(k+1) = Ax(k) + \sum_{i=1}^m b_i u_i(k) + \sum_{i=1}^m w_i(k) \tag{6} \]
which, in turn, translates in the vector form
\[ x(k+1) = Ax(k) + B_1 u(k) + B_2 w(k) \tag{7} \]

Once the following definitions have been stated:
\[ u(k)[u_1^f(k) \ ... \ u_m^f(k)]' \in U \subset \mathbb{R}^m \tag{8a} \]
\[ w(k)[w_1^f(k) \ ... \ w_m^f(k)]' \in \mathbb{R}^{m \times n_x} \tag{8b} \]
\[ B_1 = [b_1 \ b_2 \ ... \ b_m] \in \mathbb{R}^{n_x \times m} \tag{8c} \]
\[ B_2 = [I_{n_x} \ I_{n_x} \ ... \ I_{n_x}] \in \mathbb{R}^{n_x \times m} \tag{8d} \]
In conclusion, in view of the linear nature of the problem considered here, the plant to be controlled can be viewed as slow on one hand, by the ideal control commands coming from the high level controller, \( \hat{u}_i(h) \)'s (namely, \( u_i(k) \)'s in the slow sampling rate), and, on the other hand, by the discrepancies between such commands and the effective control signals achieved by the actuators, \( w_i^f(h) \)'s (namely, \( w_i(k) \)'s in the slow sampling rate). Moreover, the latter differ from the usual disturbance term affecting the system dynamics in robust control designs, in that they originate from the high level control action itself, rather than being afforded by the “nature”, as schematically portrayed in Figure 1.

In the light of such a reformulation of the problem, at the high level we address the design of a robustly stabilizing MPC controller for system (7) in the face of the disturbance \( w(k) \), thus achieving the piecewise constant reference signals \( \hat{u}_i(h) \) for the low level actuators’ loops.

B. The hierarchical controller

The hierarchical controller involves a high level robust MPC algorithm, providing the references for the actuators at any slow sampling instant \( k \). Hence, in order to guarantee that the control signals \( \hat{u}_i(h) \)'s, accomplished by the low level systems against such references, satisfy constraints (3), the following hypothesis is stated.

Assumption 2: For all \( i = 1, \ldots, m \)
\[ 1) \ \zeta_i(0) \in U_i; \]
\[ 2) \ \forall k \in \mathbb{N}, \ u_i(k) \in [-\epsilon_i, \epsilon_i], \] where \( \epsilon_i = \frac{\alpha_i(1 - |f_i|)}{2}. \)

We define \( U_i = [-\epsilon_i, \epsilon_i] \) and \( U = U_1 \times \cdots \times U_m \subset \mathbb{R}^m \).

Moreover, the feasibility of the control references computed at the high level needs the following assumption.

Assumption 3: The high level MPC controller incorporates the actuators’ models, i.e. it knows the parameters \( f_i \)'s along with the initial conditions \( \zeta_i(0) \)'s, \( i = 1, \ldots, m \).

Assumption 3 implies that the high level controller can predict the low level systems’ behavior, being thus aware of their current state situation at any time instant. Hence, letting \( \hat{u}_i(k) = \hat{u}_i(\tau k) \) be the slow time rate counterparts of the fast time rate control actions performed by the actuators, from output propagation of systems (2), one has
\[ \hat{u}_i(k+1) = f_i \hat{u}_i(k) + (1 - f_i) u_i(k) \tag{9} \]
\[ u_i(k) \] standing for the \( i \)-th reference entry generated by the upper level controller. Consequently, we can define the following augmented version of system (7):
\[
\chi(k+1) = \bar{A}\chi(k) + \bar{B}_1u(k) + \bar{B}_2w(k),
\]
where
\[
\chi(k) = \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \in \mathcal{X} = \mathbb{R}^{n_x} \times \mathcal{U}, \quad u(k) \in \bar{U},
\]
\[
\bar{A} = \begin{bmatrix} A & 0_{n_x,m} \\ 0_{m,n_x} & F \end{bmatrix}, \quad \bar{B}_1 = \begin{bmatrix} B_1 \\ G \end{bmatrix},
\]
\[
\bar{B}_2 = \begin{bmatrix} B_2 \\ 0_{m,mn} \end{bmatrix},
\]
\[
\bar{u}(k) = \begin{bmatrix} \bar{u}_1(k) & \cdots & \bar{u}_m(k) \end{bmatrix}^T,
\]
\[
F = \text{diag}(f_1, \ldots, f_m),
\]
\[
G = \text{diag}((1-f_1), \ldots, (1-f_m)).
\]

Notice that, in view of Assumption 2.2, if \( \bar{u}(k) \in \bar{U} \) and \( u(k) \in \mathcal{U} \), then \( \chi(k+1) \in \mathcal{X} \). Associated with the dynamic equation (10), we introduce also the output transformation
\[
z(k) = \begin{bmatrix} \chi(k) \\ u(k) \end{bmatrix}.
\]

Finally, to gain versatility in the definition of the cost function the MPC paradigm calls for, we introduce weighted norms in both the state and control spaces. Thus, for fixed symmetric and positive definite matrices \( Q_x \in \mathbb{R}^{n_x \times n_x} \) and \( Q_i \in \mathbb{R}^{i \times i} \), let \( Q_u = \text{diag}(Q_1, \ldots, Q_m) \in \mathbb{R}^{n_u \times n_u} \), \( Q_\chi = \text{diag}(Q_x, Q_i) \in \mathbb{R}^{n_x+m \times n_x+m} \), \( Q_w = \text{diag}(Q_x, \ldots, Q_x) \in \mathbb{R}^{n_x \times n_x} \) and \( Q_\chi = \text{diag}(Q_x, Q_i) \in \mathbb{R}^{n_x+m \times n_x+m} \).

### III. DESIGN AND ANALYSIS OF THE HIGH LEVEL CONTROLLER IN THE BASIC ACTUATION CONFIGURATION

In this Section, we deal with the design of the high level MPC controller for the basic low level configuration, i.e. the one including \( m \) actuators.

According to Assumption 3, iterating (2), the high level controller can easily compute each matched disturbance term \( w_i(k) \) appearing in (6) as follows:
\[
w_i(k) = \sum_{j=0}^{\tau-1} (A^j)^{\tau-j-1}b_i^j f_i \bar{u}_i(\tau) - f_i \bar{u}_i(\tau)\]
\[
= (\bar{u}_i(\tau) - u_i(\tau)) \sum_{j=0}^{\tau-1} (A^j)^{\tau-j-1}b_i^j f_i \\
= (\bar{u}_i(\tau) - u_i(\tau)) \vartheta_i
\]
where \( \vartheta_i = \sum_{j=0}^{\tau-1} (A^j)^{\tau-j-1}b_i^j f_i \) is an \textit{a-priori} known vector for any actuator. Thus, the \( w_i(k) \) terms are linear functions of the discrepancies \( (\bar{u}_i(\tau) - u_i(\tau)) \) and satisfy a gain condition of the form:
\[
\|w_i(k)\|_{Q_x} \leq \sqrt{\lambda_{\text{max}}(Q_x)} \|w_i(k)\| = \sqrt{\lambda_{\text{max}}(Q_x)} \|\vartheta_i\| \cdot \|\bar{u}_i(\tau) - u_i(\tau)\| \\
\leq \sqrt{2\lambda_{\text{max}}(Q_x)} \|\vartheta_i\| \cdot \|\bar{u}_i(\tau) - u_i(\tau)\| \leq \sqrt{2\lambda_{\text{max}}(Q_x)} \|\vartheta_i\| \cdot \|\bar{u}_i(\tau) - u_i(\tau)\|_{Q_i}
\]
where, in view of Assumption 3, the \( \gamma_d(i) \)’s are available to the upper level controller. As a consequence, the high level regulator can be carried out via the small-gain approach by considering \( w \) as a disturbance term satisfying a gain condition of the type \( \|w\|_{Q_x} \leq \gamma \|z\|_{Q_z} \), for a suitable \( \gamma_d \).

In the sequel, we will discuss first a robustly stabilizing auxiliary law and later an MPC controller improving both performance and region of attraction such a law achieves.

**A. The auxiliary law**

Under Assumption 1, we can construct an auxiliary control law for system (10) taking the form
\[
u(k) = K_{\text{aux}} \chi(k), K_{\text{aux}} \in \mathbb{R}^{m \times (n_x+m)}.
\]
Consider \( \gamma > 0 \) such that there exists a symmetric and positive definite matrix \( P \in \mathbb{R}^{(n_x+m) \times (n_x+m)} \) satisfying the Riccati inequality with constraint
\[
-\bar{B}^T P \bar{B} + Q_x - \bar{A}^T P \bar{R} \bar{B}^{-1} \bar{B}^T P \bar{A} < 0
\]
where
\[
\bar{B} = \begin{bmatrix} \bar{B}_1 & \bar{B}_2 \end{bmatrix},
\]
\[
\bar{R} = \begin{bmatrix} \bar{R}_1 & \bar{R}_2 \bar{B}_1 \\ \bar{R}_2 \bar{B}_2 & \bar{R}_1 \end{bmatrix},
\]
and let \( K_{\text{aux}} = -[I_m 0_{m,n_u}] \bar{R}^{-1} \bar{B}^T P \bar{A} \). Consider the function \( V_f(\chi) = \chi^T P \chi \) and for any \( \rho > 0 \), the set \( \Omega_\rho = \{ \chi \in \mathbb{R}_{n_x+m} \mid V_f(\chi) \leq \rho \} \subset \mathbb{R}_{n_x+m} \). The local robust stabilization properties of the auxiliary control law (15) are clarified by the following result.

**Proposition 1:** Define
\[
\bar{\gamma}_d = \max_{i=1,\ldots,m} \gamma_d(i).
\]
Let \( \gamma > 0 \) be such that \( \gamma < \bar{\gamma}_d < 1 \) and assume that a positive definite solution \( P \) for the Riccati inequality (16) exists. Consider system (10) under the corresponding control law (15) and, accordingly, let \( w(k) \) be the vector collecting all the disturbance terms given in (13). Then, for any \( \rho > 0 \) such that the following properties hold \( \forall \chi \in \Omega_{\text{aux}}, \Omega_{\text{aux}} = \Omega_\rho \):

- \( \Omega_{\text{aux}} \subset \mathcal{X} \);
- \( K_{\text{aux}} \chi \in \mathcal{U} \);
- \( \omega(\chi) \equiv \|w(k)\|_{Q_w} \leq \gamma_d \|z(k)\|_{Q_z} \);
- \( \forall \chi(k) \in \Omega_{\text{aux}} \), \( V_f(\chi(k+1)) - V_f(\chi(k)) \leq -\|z(k)\|^2_{Q_z} - \gamma_d \|w(k)\|^2_{Q_w} \);
- \( \Omega_{\text{aux}} \) is positively invariant.

**Proof:** The proof matches the one reported in [11], only minor adjustments being needed.
B. The MPC controller

In this Section, in view of condition (14), we resort to
the small-gain paradigm to derive a robustly stabilizing high
level MPC control law fulfilling the norm bound \( \|w\|_{\infty} \leq \gamma_d\|z\|_{\infty} \) (where \( \gamma_d \) is given in (17)).

In details, we let \( N_p \in \mathbb{N}, N_p \geq 1 \), be the length of the prediction horizon and \( N_c \in \mathbb{N}, N_c \leq N_p \), be the length of the control horizon. Moreover, we define

\[
\mathcal{F}(k, N_c) = \{ u(k) \, u(k+1) \ldots u(k+N_c-1) \}
\]

where \( u(k+j) \in \mathcal{U} \) is the vector of the predicted control signals to be processed by the MPC algorithm at time \( k \). At any time instant \( k \), the control problem consists of solving the following optimization problem:

\[
\min_{\mathcal{F}(k, N_c)} \, J(\chi(k), \mathcal{F}, N_p),
\]

subject to:

(i) system (10), (12), (13) under the control signal (18) and, for \( j = N_c, \ldots, N_p-1 \), \( u(k+j) = K_{aux}\chi(k+j) \);

(ii) the control constraints: \( \forall j = 0, \ldots, N_p-1 \), \( u(k+j) \in \mathcal{U} \);

(iii) the terminal constraint \( \chi(k + N_p) \in \Omega_{aux} \).

If \( \mathcal{F}^{opt}(\chi(k), N_p) = \{ u^o(k) \, u^o(k+1) \ldots u^o(k+N_c-1) \} \) is the optimal solution to (19), according to the Receding Horizon (RH) principle, the MPC law is

\[
u(\chi(k)) = u^o(k).
\]

That is, the optimal solution \( \mathcal{F}^{opt}(\chi(k), N_p) \) is composed of the optimal control at each time instant, which is computed by solving problem (19) for the current state \( \chi(k) \).

C. The overall system: convergence analysis

The following result provides the analysis of the overall control system behavior.

**Theorem 2:** Under the assumptions of Theorem 1, consider the closed-loop system (1), (2), with \( u^i(h) = \hat{u}_i(h) \), and \( u_i(h) = \hat{u}_i(h) \), where \( \hat{u}_i(h) = u_i(\lfloor \frac{h}{1} \rfloor) \) is provided by the upper level controller defined in (20). Assume that, at time \( h = 0 \), the initial states of the actuators (2) fulfill Assumption 2.1, \( i = 1, \ldots, m \) and let \( \mu_0 = [\, \hat{u}_1(0) \, \hat{u}_2(0) \ldots \, \hat{u}_m(0) \,]' \). Assume also that the MPC controller at the upper level is initialized with \( \chi(0) = [x(0) \, \mu_0]' \in X_{MPC}(N_c, N_p) \). Then it holds that

\[
\lim_{h \to +\infty} x^f(h) = 0, \quad \lim_{h \to +\infty} \zeta_i(h) = 0, \quad \forall i = 1, \ldots, m.
\]

Proof: See [12], [11].

IV. CONTROL SYSTEM RECONFIGURATION

According to the RH paradigm, the high level MPC controller stated in Theorem 1 calls for solving problem (19) at each time step. It can thus easily allow for a "plug and play" flexibility in the face of actuators’ addition/replacement. To this end, assuming that only one actuator per (slow) time step can be plugged/replaced into the overall system, plug and play skills are guaranteed as follows.

A. Actuator addition. As an actuator addition happens, variables \( \chi, u, \hat{u}, w \) and \( z \) contain additional components accounting for such a new actuator. Hence, we introduce the notation \( u^{(m+1)} \) to denote the variable \( u \) related to the new low level configuration, i.e. the one composed of \( m+1 \) actuators. Accordingly, we let \( U^{(m+1)} = \mathcal{U} \times [-\epsilon_{m+1}, +\epsilon_{m+1}] \) be the corresponding set of admissible control commands. Let us discuss how the high level controller, hence problem (19), can reconfigure owing to the actuator addition.

The auxiliary control law (15) is largely independent of the low level dynamics. As a consequence, even if it has to feed also the plugged actuator, it can send it a null reference, so keeping unchanged. In details, letting

\[
u^{(m+1)} = K_{aux}^{(m+1)} \chi^{(m+1)} = \begin{bmatrix} K_{aux}^{(m+1)} & 0_m \times 1 \end{bmatrix} \chi^{(m+1)}
\]

and \( \Omega_{aux}^{(m+1)} = \Omega_{aux} \times \{0\} \), it is straightforward to show that the feasibility of the auxiliary law (22) holds in \( \Omega_{aux}^{(m+1)} \). Moreover, problem (19) has to be rephrased in terms of the enlarged variables and the auxiliary law (22). Care has only to be payed in the following issues:

1) \( V_f \) should be replaced by \( V_f^{(m+1)} \), where

\[
V_f^{(m+1)}(\chi^{(m+1)}) = \chi^{(m+1)'P^{(m+1)}\chi^{(m+1)}},
\]

\( P^{(m+1)} \) being \( P \) bordered with a null row and column;

2) constraint (ii) turns into \( u_i^{(m+1)}(k+j) \in \mathcal{U}^{(m+1)}, \forall j = 0, \ldots, N_p-1 \), and (iii) becomes \( \chi^{(m+1)}(k+N_p) \in \Omega_{aux}^{(m+1)} \);

3) if \( \gamma_d(m+1) > \gamma_d \), in order to still guarantee the
exponential stability of the origin, the optimization problem has to be solved under the further constraint
$$\|w(m+1)(k + j)\|^2_{Q^{(m+1)}} \leq \gamma^2 d_j \|z^{(m+1)}(k + j)\|^2_{Q^{(m+1)}}$$
$$\forall j = 0, \ldots, N_c - 1.$$  
(23)

With this position, Theorem 1 holds true and the overall control system self reconfigures, only requiring the minor formal adjustments mentioned above.

As for the feasibility of problem (19), if at time $k$ a new actuator is plugged into the system, the enlarged state is $\chi^{(m+1)}(k) = [\chi(k)', \tilde{u}_{m+1}(k)']'$, where $\tilde{u}_{m+1}(k) = \zeta_{m+1}(k)$ is the internal state of such an actuator. If $\tilde{u}_{m+1}(k) = 0$, then $\chi^{(m+1)}(k)$ belongs to the feasibility region of problem (19) for the new actuation configuration.

As a matter of fact, problem (19) for $k < \bar{k}$ can be seen as a particular instance of the same problem for the enlarged system, under the additional constraint
$$\tilde{u}_{m+1}(k + j) = 0, \forall j = 0, \ldots, N_c - 1.$$  
(24)

We hence introduce the following assumption.

Assumption 4: The value of the control action afforded by the new actuator in correspondence to the addition time instant is null, namely $\tilde{u}_{m+1}(\bar{k}) = 0$.

Finally, considering the stability issue, we let $V^{(m+1)}_{\text{constr}}(\chi^{(m+1)}(k), N_c, N_p)$ be the optimal value of the optimization problem for the enlarged system with constraint (24) and $V^{(m+1)}(\chi^{(m+1)}(k), N_c, N_p)$ be the same quantity for the corresponding unconstrained problem. It thus turns out that $V(\chi(k), N_c, N_p) = V^{(m+1)}_{\text{constr}}(\chi^{(m+1)}(k), N_c, N_p)$. Also, since at time $\bar{k}$ one has
$$V^{(m+1)}(\chi^{(m+1)}(k), N_c, N_p) \leq V^{(m+1)}_{\text{constr}}(\chi^{(m+1)}(k), N_c, N_p),$$
then, by inequality (21c) applied at time $\bar{k} - 1$, one obtains
$$V^{(m+1)}(\chi^{(m+1)}(\bar{k}), N_c, N_p) - V(\chi(\bar{k} - 1), N_c, N_p) \leq - (1 - \gamma^2 d_j) \|z(\bar{k} - 1)\|^2_{Q_j}.$$  
(25)

This inequality stands for the counterpart of (21c) in the instant of the actuator addition. Consequently, since all the above arguments iteratively apply to any of such occurrences, one can conclude that stability is preserved irrespective of any actuator plugging event.

B. Actuator replacement. Differently from the previous case, when the $i$-th actuator is replaced with a new one, dimensionality of system (10) is not an issue, whilst matrices $F$ in (11d) and $G$ in (11e), and hence $\bar{A}$ and $\bar{B}_1$ in (11), change in view of the value $f^\text{new}$ characterizing the new device (matrices $A$ in (5) and $\bar{B}_1$ in (8c) remain instead unaltered).

Let us hence discuss the reconfigurability problem of the high level controller.

The auxiliary control law can be maintained. However, its robust stabilization properties given in Proposition 1 still hold, provided that the following conditions are satisfied: 1) letting $\gamma^\text{new}_d(i)$ be the gain defined in (14) for the new actuator, inequality $\gamma \cdot \gamma^\text{new}_d(i) < 1$ is valid; 2) the left-hand-side of the Riccati inequality (16), evaluated with the same $P$ but with the matrices $\bar{A}$ and $\bar{B}_1$ replaced by those associated with the new low level configuration, is negative definite; 3) property b) of Proposition 1 still holds, even if the set $\mathcal{U}$ becomes $\mathcal{U}^\text{new}$, which differs from the former in the $i$-th component $\mathcal{U}^\text{new}_i = [-\varepsilon^\text{new}_i, \varepsilon^\text{new}_i]$, $\varepsilon^\text{new}_i = \alpha(1 - f^\text{new}(|x|))/\bar{f}$. By continuity arguments, properties 2) and 3) are ensured by the choice of a new actuator such that $|f^\text{new}_i - f_i| < \epsilon$, for a sufficiently small $\epsilon > 0$.

Problem (19) keeps unchanged except for the control constraint (ii) which translates into $u(k + j) \in \mathcal{U}^\text{new}$, $\forall j = 0, \ldots, N_p - 1$.

In the replacement time instant $\bar{k}$, the internal state of the new actuator may differ from the one of the replaced device, we hence denote by $\chi^\text{new}(\bar{k})$ the corresponding state in the new configuration. We hence consider the following assumption.

Assumption 5: The state $\chi^\text{new}(\bar{k})$ belongs to the feasibility region of the modified version of problem (19).

This assumption is guaranteed if the value of the control action afforded by the new actuator at time $\bar{k}$ equals the one the old actuator was giving at $\bar{k}$, i.e. $\tilde{u}^\text{new}_i(\bar{k}) = \tilde{u}_i(\bar{k})$, and if $|f^\text{new}_i - f_i| < \epsilon$, with $\epsilon > 0$ yielding a sufficiently small variation in the feasibility region of problem (19). If all the above conditions hold, the result of Theorem 1 is guaranteed without any further manipulations. However, it is worth noting that the optimal value $V^\text{new}(\chi^\text{new}(\bar{k}))$ of problem (19) in the new configuration may be larger than the optimal value $V(\chi(\bar{k}))$ in the old configuration. Therefore, the counterpart of relation (25) for the replacement case is not ensured. Nevertheless, stability of the overall process can be preserved as long as a sufficiently large time interval between two consecutive replacement events is held on. This corresponds to the well-known concept of dwell-time in switching control [4], thereby a sufficient decrease of the optimal value function is ensured and, as a consequence, possible increases in the replacement instants are counteracted.

V. SIMULATION EXAMPLE

Let us consider the unstable linear discrete-time system described by the state equation
$$x^f(h + 1) = A^f x^f(h) + B^f u^f(h)$$  
(26)
where $A^f = \text{diag}(0.4, -0.8, 1.1)$ and $B^f = [b^f_1 \ b^f_2]$, $b^f_1 = [1 \ 0 \ 1]^t$ and $b^f_2 = [3 \ 1 \ 0]^t$. Also, let us require that the control variables satisfy the constraints $|u^f_1| \leq 0.1$ and $|u^f_2| \leq 0.001$. Moreover, in view of the notation used in (2), let us account for the actuators described by the quantities $f_1 = 0.7$ and $f_2 = 0.5$.

In the implementation of the hierarchical controller discussed in Section III, we have defined the oversampling constant $\tau = 4$, and the weighting matrices $Q_x = 10^{-4}I_3$ and $Q_u = \text{diag} \{10^{-1}, 10^{-2}\}$. In such a formulation, the constant $\gamma_d$ defined in (17) is 0.1610. Hence, we have solved the Riccati inequality with constraint (16) by fixing the attenuation level $\gamma = 0.2033$.

Accordingly, we have defined the final set $\Omega_{aux}$ by picking $\rho = 10$. Finally, we have implemented the MPC algorithm with prediction and control horizons $N_p = 10$ and $N_c = 3$, respectively, and we have taken into account the initial conditions $x^{f}(0) = [1 \ -2 \ 0.8]^t$ and $z_i(0) = \zeta_i(0) = 0$.

A. Basic actuation configuration. The simulation of the
A hierarchical controller applied to the basic actuation configuration has led to the results shown in Figures 2 and 3 (on the left). In particular, Figure 2 witnesses that the closed-loop system is properly stabilized, while Figure 3 confirms the control constraints fulfillment.

**B. Actuator addition.** In the actuator addition case, we have supposed that, at time instant $k = 1$, the actuator computed by the parameter $f_3 = 0.4$ and the high level quantities $b_1 = \begin{bmatrix} 0 & 2 \end{bmatrix}^T$, $|u_1^d| < 0.02$ and $Q_3 = 0.01$ has been plugged into the system. Note that, since $\gamma_d(3) = 0.9626 > \gamma_d$, constraint (23) is needed. As can be seen in Figures 2 and 3 (on the right), the hierarchical controller accounting for the little modifications reported in Section IV-A correctly stabilizes the plant, as expected, in a tighter time interval and, indeed, satisfies all the control constraints on the process.

**C. Actuator replacement.** As for the actuator replacement issue (see Section IV-B), we have assumed that, at $k = 3$, the second actuator has been replaced with the one modeled by $f_2^\text{new} = 0.4$, yielding the gain $\gamma_2^\text{new} = 0.1177$. Hence, $\gamma_2 \cdot \gamma_2^\text{new} < 1$. Figure 4 shows that the closed-loop system exhibits similar performance to the one achieved by the basic actuation configuration.

**VI. CONCLUSION**

A two-level hierarchical control problem has been addressed. At the high level a robust MPC regulator computes the ideal control actions needed to govern the plant. A number of already controlled actuators placed at the low level drive the plant by tracking such desired actions. A convergence result for the overall control system has been derived by resorting to the small-gain approach. The algorithm proposed has also been used in a Plug and Play framework to face actuators’ addition/replacement events.

**REFERENCES**


