Distributed Coordinated Tracking via a Variable Structure Approach - Part I: Consensus Tracking

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Abstract—This is the first part of a two-part paper on distributed coordinated tracking for a group of autonomous vehicles via a variable structure approach. Here, the term coordinated tracking is used to refer to both consensus tracking and swarm tracking. In the first part of this paper, we focus on the consensus tracking problem where a group of autonomous vehicles can track a (time-varying) virtual leader when the state of the virtual leader is available to only a subset of the group of vehicles. In the case of first-order kinematics, we propose a distributed consensus tracking algorithm without velocity measurements under both fixed and switching network topologies. In particular, we show that distributed consensus tracking can be achieved in finite time. In the case of second-order dynamics, we propose two distributed consensus tracking algorithms without acceleration measurements under, respectively, a fixed and switching network topology. In particular, we show that the proposed algorithms guarantee at least global exponential tracking. For distributed consensus tracking in the case of both first-order kinematics and second-order dynamics, a mild connectivity requirement is proposed by adopting a connectivity maintenance mechanism in which the adjacency matrix is defined in a proper way.

I. INTRODUCTION

In the past decade, multi-vehicle cooperative control has received significant attention in the systems and controls society. The motivation behind multi-vehicle cooperative control is that a group of vehicles working cooperatively can achieve great benefits including low cost, high adaptivity, and easy maintenance [1].

One distributed approach used in multi-vehicle cooperative control is consensus, which means that a group of vehicles reaches an agreement on a common value by interacting with their local (time-varying) neighbors. Consensus has been studied for systems with both first-order kinematics and second-order dynamics. Recent study of consensus and its applications in distributed multi-vehicle cooperative control can be found in [2], [3]. Existing consensus algorithms were often studied either when there does not exist a leader or when the leader is static. Although consensus without a leader is useful in applications such as cooperative rendezvous of a group of vehicles, there are many applications that require a dynamic leader. Examples include formation flying, body guard, and coordinated tracking applications.

Consensus with a dynamic leader, called consensus tracking hereafter, has been studied from different perspectives. The objective of consensus tracking is that a group of followers tracks a dynamic leader with local interaction. The authors in [4], [5] proposed and analyzed a consensus tracking algorithm under a variable undirected network topology. However, [4], [5] require the availability of the leader’s acceleration input to all followers and/or the design of distributed observers. In [6]–[8], the authors proposed a proportional-and-derivative-like consensus tracking algorithm under a directed network topology in both continuous-time and discrete-time settings. However, [6]–[8] require either the availability of the leader’s velocity and the followers’ velocities or their estimates, or a small sampling period. In [9], the authors studied a leader-follower consensus tracking problem with time-varying delays. However, [9] requires the velocity measurements of the followers and an estimator to estimate the leader’s velocity.

In the first part of this paper, we focus on solving a distributed consensus tracking problem via a variable structure approach when there exists a dynamic virtual leader under the following three assumptions: 1) The virtual leader is a neighbor of only a subset of a group of followers; 2) There exists only local interaction among all followers; 3) The velocity measurements of the virtual leader and all followers in the case of first-order kinematics or the acceleration measurements of the virtual leader and all followers in the case of second-order dynamics are not required. In contrast to the assumptions that appeared in the aforementioned references, the three assumptions are more general and practical. In the case of first-order kinematics, we propose a distributed consensus tracking algorithm without velocity measurements under both fixed and switching network topologies. In particular, we show that distributed consensus tracking can be achieved in finite time. In the case of second-order dynamics, we propose two distributed consensus tracking algorithms without acceleration measurements under, respectively, a fixed and switching network topology. In particular, we show that the proposed algorithms guarantee at least global exponential tracking. For the proposed distributed consensus tracking algorithms, a mild connectivity requirement is proposed by adopting a connectivity maintenance mechanism in which the adjacency matrix is defined in a proper way.

II. BACKGROUND AND PRELIMINARIES

Suppose that a team consists of $n$ vehicles. We use a weighted undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ to model the interaction among these vehicles, where $\mathcal{V} = \{1, \ldots, n\}$ is the node set, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the edge set, and $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{n \times n}$ is the weighted adjacency matrix. An edge $(i, j)$ in $\mathcal{G}$ denotes that vehicles $i$ and $j$ can obtain information...
from each other. Vehicle \( j \) is a neighbor of vehicle \( i \) if \((j, i) \in \mathcal{E}\). The weighted adjacency matrix \( \mathcal{A} \) associated with \( \mathcal{G} \) is defined such that \( a_{ij} \) is a positive weight if \((j, i) \in \mathcal{E}\), and \( a_{ij} = 0 \) otherwise. Note that here \( a_{ij} = a_{ji}, \forall i \neq j \), since \((j, i) \in \mathcal{E}\) implies \((i, j) \in \mathcal{E}\).

A path is a sequence of edges in an undirected graph of the form \((i_1, i_2), (i_2, i_3), \ldots \), where \(i_j \in \mathcal{V}\). An undirected graph is connected if there is an undirected path between every pair of distinct nodes.

Let the Laplacian matrix \( \mathcal{L} = [\ell_{ij}] \in \mathbb{R}^{n \times n} \) associated with \( \mathcal{A} \) be defined as \( \ell_{ii} = \sum_{j=1, j \neq i}^{n} a_{ij} + \ell_{ij} = -a_{ij}, i \neq j \). Note that \( \mathcal{L} \) is symmetric positive semi-definite. Also note that \( \mathcal{L} \) has a simple zero eigenvalue with an associated eigenvector \( \mathbf{1} \), where \( \mathbf{1} \) is an all-one column vector with a compatible size, and all other eigenvalues are positive if and only if \( \mathcal{G} \) is connected [10].

### III. Distributed Consensus Tracking for First-order Kinematics

In this section, we study distributed consensus tracking for first-order kinematics. Suppose that in addition to the \( n \) vehicles, labeled as vehicles \( 1 \) to \( n \), called followers hereafter, there exists a virtual leader, labeled as vehicle \( 0 \), with a (time-varying) position \( r_0 \) and velocity \( \dot{r}_0 \). We assume that \(|\dot{r}_0| \leq \gamma \), where \( \gamma \) is a positive constant.

Consider followers with first-order kinematics given by

\[
\dot{r}_i = u_i, \quad i = 1, \ldots, n, \tag{1}
\]

where \( r_i \in \mathbb{R} \) is the position and \( u_i \in \mathbb{R} \) is the control input associated with the \( i \)th vehicle. The objective of this section is to design \( u_i \) for (1) such that all followers track the virtual leader with local interaction in the absence of velocity measurements. Here we have assumed that all vehicles are in a one-dimensional space for the simplicity of presentation. However, all results hereafter are still valid for the \( m \)-dimensional (\( m > 1 \)) case by introduction of the Kronecker product.

**A. Fixed Network Topology**

In this subsection, we assume that the network topology is fixed. We propose the distributed consensus tracking algorithm for (1) as

\[
\dot{u}_i = -\alpha \sum_{j=0}^{n} a_{ij} (r_i - r_j) - \beta \text{sgn} \left[ \sum_{j=0}^{n} a_{ij} (r_i - r_j) \right], \tag{2}
\]

where \( a_{ij}, i, j = 1, \ldots, n \), is the \((i,j)\)th entry of the adjacency matrix \( \mathcal{A} \), \( a_{ij}, i = 1, \ldots, n, j \) is a positive constant if the virtual leader’s position is available to follower \( i \) and \( a_{ij} = 0 \) otherwise, \( \alpha \) is a nonnegative constant, \( \beta \) is a positive constant, and \( \text{sgn}(\cdot) \) is the signum function.

**Theorem 3.1:** Suppose that the fixed undirected graph \( \mathcal{G} \) is connected and at least one \( a_{0i} \) is nonzero (and hence positive). Using (2) for (1), if \( \beta > \gamma \), then \( r_i(t) \to r_0(t) \) in finite time. In particular, \( r_i(t) = r_0(t) \) for any \( t \geq \tau \), where

\[
\tau = \frac{\sqrt{r^T(0)M\dot{r}(0)} \sqrt{\lambda_{\max}(M)}}{(\beta - \gamma)\lambda_{\min}(M)}, \tag{3}
\]

where \( \dot{r} \) is the column stack vector of \( \dot{r}_i, i = 1, \ldots, n, \) with \( \dot{r}_i = r_i - r_0, M = \mathcal{L} + \text{diag}(a_{01}, \ldots, a_{0n}) \) with \( \mathcal{L} \) being the Laplacian matrix, and \( \lambda_{\min}(\cdot) \) and \( \lambda_{\max}(\cdot) \) denote, respectively, the smallest and the largest eigenvalue of a symmetric matrix.

**Proof:** Noting that \( \dot{r}_i = r_i - r_0 \), we can rewrite the closed-loop system of (1) using (2) as

\[
\dot{\tilde{r}}_i = -\alpha \sum_{j=0}^{n} a_{ij} (\tilde{r}_i - \tilde{r}_j) - \beta \text{sgn} \left[ \sum_{j=0}^{n} a_{ij} (\tilde{r}_i - \tilde{r}_j) \right] - \dot{r}_0. \tag{4}
\]

Equation (4) can be written in matrix form as

\[
\dot{\tilde{r}} = -\alpha \mathcal{M}\tilde{r} - \beta \text{sgn}(\mathcal{M}\tilde{r}) - \mathbf{r}_0, \tag{5}
\]

where \( \dot{r} \) and \( \mathcal{M} \) are defined in (3), and \( \text{sgn}(\cdot) \) is defined componentwise. Because the fixed undirected graph \( \mathcal{G} \) is connected and at least one \( a_{0i} \) is nonzero (and hence positive), \( \mathcal{M} \) is symmetric positive definite.

Consider the Lyapunov function candidate \( V = \frac{1}{2} \tilde{r}^T \mathcal{M}\tilde{r} \).

The derivative of \( V \) is

\[
\dot{V} = \tilde{r}^T \mathcal{M} ( -\alpha \mathcal{M}\tilde{r} - \beta \text{sgn}(\mathcal{M}\tilde{r}) - \mathbf{r}_0) \leq -\alpha \tilde{r}^T \mathcal{M}^2 \tilde{r} - (\beta - \gamma) \| \mathcal{M}\tilde{r} \|_1, \tag{6}
\]

where we have used the Holder’s inequality and \( |\dot{r}_0| \leq \gamma \) to obtain (5). Note that \( \mathcal{M}^2 \) is symmetric positive definite, \( \alpha \) is nonnegative, and \( \beta > \gamma \). Therefore, it follows that \( \dot{V} \) is negative definite.

We next show that \( V \) will decrease to zero in finite time (i.e., \( \tilde{r}_i(t) \to 0 \) in finite time). Note that \( V \leq \frac{1}{2} \lambda_{\max}(\mathcal{M}) \| \tilde{r} \|^2_2 \). It then follows from (5) that the derivative of \( V \) satisfies

\[
\dot{V} \leq - (\beta - \gamma) \| \mathcal{M}\tilde{r} \|_2 \\
= - (\beta - \gamma) \sqrt{\tilde{r}^T \mathcal{M}^2 \tilde{r}} \\
\leq - (\beta - \gamma) \sqrt{\lambda_{\max}(\mathcal{M})} \| \tilde{r} \|_2^2 \\
\leq - (\beta - \gamma) \sqrt{\lambda_{\max}(\mathcal{M})} \sqrt{V}.
\]

After some manipulation, we can get that

\[
2 \sqrt{V(t)} \leq 2 \sqrt{V(0)} - (\beta - \gamma) \sqrt{\frac{2}{\lambda_{\min}(\mathcal{M})}} t.
\]

Therefore, we have \( V(t) = 0 \) when \( t \geq \tau \), where \( \tau \) is given by (3). This completes the proof.

**B. Switching Network Topology**

In this subsection, we assume that the network topology is switching (i.e., \( a_{ij} \) is not necessarily constant.). Let \( \mathcal{N}_i \subseteq \{0, 1, \ldots, n\} \) denote the neighbor set of follower \( i \) in the team consisting of the \( n \) followers and the virtual leader. We next consider the case of a switching network topology by assuming that \( j \in \mathcal{N}_i(t), i = 1, \ldots, n, j = 0, \cdots, n, \) if \( |r_i - r_j| \leq R \) at time \( t \) and \( j \notin \mathcal{N}_i(t) \) otherwise, where \( R \) denotes the communication/sensing radius of the vehicles.
In this case, we consider the distributed consensus tracking algorithm for (1) as
\[ u_i = -\alpha \sum_{j \in \mathcal{N}_i(t)} b_{ij}(r_i - r_j) - \beta \text{sgn}\left[ \sum_{j \in \mathcal{N}_i(t)} b_{ij}(r_i - r_j) \right], \tag{6} \]
where \( b_{ij}, i = 1, \ldots, n, j = 0, \ldots, n, \) are positive constants, and \( \alpha, \beta, \) and \( \text{sgn}(\cdot) \) are defined as in (2).

**Theorem 3.2:** Suppose that the undirected graph \( \mathcal{G}(t) \) is connected and the virtual leader is a neighbor of at least one follower (i.e., \( 0 \in \mathcal{N}_i(t) \) for some \( i \)) at each time instant. Using (2) for (1), if \( \beta > \gamma_c \), then \( r_i(t) \to r_0(t) \) as \( t \to \infty \).

**Proof:** Let \( V_{ij} = \frac{1}{2} b_{ij}(r_i - r_j)^2, i, j = 1, \ldots, n, \) when \( |r_i - r_j| \leq R \) and \( V_{ij} = \frac{1}{2} b_{ij} R^2 \) when \( |r_i - r_j| > R \). Also let \( V_0 = \frac{1}{2} b_{00}(r_i - r_0)^2, i = 1, \ldots, n, \) when \( |r_i - r_0| \leq R \) and \( V_0 = \frac{1}{2} b_{00} R^2 \) when \( |r_i - r_0| > R \). Consider the right-hand vector candidate \( V = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} V_{ij} + \sum_{i=1}^{n} V_0. \) Note that \( V \) is not smooth but is regular. We use differential inclusions [11], [12] and nonsmooth analysis [13], [14] to analyze the stability of (1) using (6). Therefore, the closed-loop system of (1) using (6) can be written as
\[ \dot{r}_i \in \mathbb{R}^n. \]
\[ -K \left[ \sum_{j \in \mathcal{N}_i(t)} b_{ij}(r_i - r_j) + \beta \text{sgn}\left[ \sum_{j \in \mathcal{N}_i(t)} b_{ij}(r_i - r_j) \right] \right], \tag{7} \]
where \( K[\cdot] \) is the differential inclusion [12] and a.e. stands for “almost everywhere”.

The generalized derivative of \( V \) is given by
\[ \dot{V} = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij} \left[ \frac{\partial V_{ij}}{\partial r_i} \dot{r}_i + \frac{\partial V_{ij}}{\partial r_j} \dot{r}_j \right] + \sum_{i=1}^{n} b_{00} \left[ \frac{\partial V_{00}}{\partial r_i} \dot{r}_i + \frac{\partial V_{00}}{\partial r_0} \dot{r}_0 \right] = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij} [(r_i - r_j) \dot{r}_i + (r_j - r_i) \dot{r}_j] + \dot{r}_0 \sum_{i=0}^{n} b_{00}(r_i - r_0) = -\alpha \dot{\bar{M}}(t) \dot{\bar{r}} - \beta \| \dot{M}(t) \bar{r} \|_1 + \dot{r}_0 \sum_{i=0}^{n} b_{00}(r_i - r_0) \tag{8} \]
and \( \dot{M}(t) = [\dot{m}_{ij}(t)] \in \mathbb{R}^{n \times n} \) is defined as
\[ \dot{m}_{ij}(t) = \begin{cases} -b_{ij}, & j \in \mathcal{N}_i(t), j \neq i, \\ 0, & j \notin \mathcal{N}_i(t), j \neq i, \end{cases} \tag{10} \]
Note that \( \dot{M}(t) \) is symmetric positive definite at each time instant under the condition of the theorem. Because \( \beta > \gamma_c \), it then follows that the generalized derivative of \( V \) is negative definite under the condition of the theorem, which implies that \( V(t) \to 0 \) as \( t \to \infty \). Therefore, we can get that \( r_i(t) \to r_0(t) \) as \( t \to \infty \).

**Remark 3.3:** Under the condition of Theorem 3.2, distributed consensus tracking can be achieved in finite time under a switching network topology. However, in contrast to the result in Theorem 3.1, it is not easy to explicitly compute the bound of the time (i.e., \( \bar{t} \) in Theorem 3.1) because the switching pattern of the network topology also plays an important role in determining the bound of the time.

**IV. DISTRIBUTED CONSENSUS TRACKING FOR SECOND-ORDER DYNAMICS**

In this section, we study distributed consensus tracking for second-order dynamics. Suppose that there exists a virtual leader, labeled as vehicle 0, with a (time-varying) position \( r_0 \) and velocity \( v_0 \). We assume that \( |v_0| \leq \varphi_\ell \), where \( \varphi_\ell \) is a positive constant.

Consider followers with second-order dynamics given by
\[ \dot{r}_i = v_i, \quad \ddot{r}_i = u_i, \quad i = 1, \ldots, n, \tag{11} \]
where \( r_i \in \mathbb{R} \) and \( v_i \in \mathbb{R} \) are, respectively, the position and velocity of follower \( i \), and \( u_i \in \mathbb{R} \) is the control input. Again we only consider the case when all vehicles are in a one-dimensional space. All results hereafter are still valid for the \( m \)-dimensional (\( m > 1 \)) case by introduction of the Kronecker product.

**A. Fixed Network Topology**

In this subsection, we assume that the network topology is fixed. The objective here is to design \( u_i \) for (11) such that all followers track the virtual leader with local interaction in the absence of acceleration measurements. We propose the distributed consensus tracking algorithm for (11) as
\[ u_i = -\sum_{j=0}^{n} a_{ij} [(r_i - r_j) + \alpha (v_i - v_j)] - \beta \text{sgn}\left( \sum_{j=0}^{n} a_{ij} [(r_i - r_j) + (v_i - v_j)] \right), \tag{12} \]

where \( a_{ij}, i, j = 1, \ldots, n, \) is the \((i, j)\)th entry of the adjacency matrix \( \mathcal{A} \), \( a_{0i}, i = 1, \ldots, n, \) is a positive constant if the virtual leader’s position and velocity are available to follower \( i \) and \( a_{00} = 0 \) otherwise, and \( \alpha, \beta, \) and \( \gamma \) are positive constants. Before moving on, we need the following lemma.

**Lemma 4.1:** Suppose that the fixed undirected graph \( \mathcal{G} \) is connected and at least one \( a_{0j} \) is nonzero (and hence positive). Let \( P = \begin{bmatrix} \frac{1}{2} M^T & \frac{1}{2} M \\
\frac{1}{2} M^T & \frac{1}{2} M \end{bmatrix} \) and \( Q = \)
where $\alpha$ and $\gamma$ are positive constants and $M = L + \text{diag}(a_{10}, \ldots, a_{n0})$. If $\gamma$ satisfies
\begin{equation}
0 < \gamma < \min\left\{ \sqrt{\lambda_{\min}(M)}, \frac{4\alpha \lambda_{\min}(M)}{4 + \alpha^2 \lambda_{\min}(M)} \right\},
\end{equation}
then both $P$ and $Q$ are symmetric positive definite.

**Proof:** When the fixed undirected graph $G$ is connected and at least one $a_{ij}$ is nonzero (and hence positive), $M$ is symmetric positive definite. It follows that $M$ can be diagonalized as $M = \Gamma^{-1} \Lambda \Gamma$, where $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$ with $\lambda_i$ being the $i$th eigenvalue of $M$. It then follows that $P$ can be written as
\begin{equation}
P = \begin{bmatrix} \Gamma^{-1} & 0_{n \times n} \\ 0_{n \times n} & \Gamma^{-1} \end{bmatrix} \begin{bmatrix} \frac{1}{2} \Lambda^2 & \frac{1}{2} \Lambda \\ \frac{1}{2} \Lambda & \frac{1}{2} \Lambda \end{bmatrix} \begin{bmatrix} \Gamma & 0_{n \times n} \\ 0_{n \times n} & \Gamma \end{bmatrix},
\end{equation}
where $0_{n \times n}$ is the $n \times n$ zero matrix. Let $\mu$ be an eigenvalue of $F$. Because $\Lambda$ is a diagonal matrix, it follows from (14) that $\mu$ satisfies
\begin{equation}
\mu^2 - \frac{1}{2}(\lambda_i + \lambda_i)\mu + \frac{1}{4}(\lambda_i^3 - \gamma^2 \lambda_i^2) = 0.
\end{equation}
Because $F$ is symmetric, the roots of (15) are real. Therefore, all roots of (15) are positive if and only if \( \frac{1}{2}(\lambda_i + \lambda_i) > 0 \) and \( \frac{1}{4}(\lambda_i^3 - \gamma^2 \lambda_i^2) > 0 \). Because $\lambda_i > 0$, it follows that \( \frac{1}{4}(\lambda_i^3 - \gamma^2 \lambda_i^2) > 0 \). It thus follows that when $\gamma^2 < \lambda_i$, the roots of (15) are positive. Noting that $P$ has the same eigenvalues as $F$, we can get that $P$ is positive definite if $0 < \gamma < \sqrt{\lambda_{\min}(M)}$.

By following a similar analysis, we can get that $Q$ is positive definite if $0 < \gamma < \frac{4\alpha \lambda_{\min}(M)}{4 + \alpha^2 \lambda_{\min}(M)}$. Combining the above arguments proves the lemma.

**Theorem 4.1:** Suppose that the fixed undirected graph $G$ is connected and at least one $a_{ij}$ is nonzero (and hence positive). Using (12) for (11), if $\beta > \varphi_\ell$ and $\gamma$ satisfies (13), then $r_l(t) \to 0_n$ and $v_l(t) \to 0_n$ globally exponentially as $t \to \infty$. In particular, it follows that
\begin{equation}
\| [\hat{v}_l(t) \quad \hat{v}_l(t)] \|^2 \leq \kappa_1 e^{-\kappa_2 t},
\end{equation}
where $\hat{r}_l$ and $\hat{v}_l$ are, respectively, the column stack vectors of $\hat{r}_i$ and $\hat{v}_i$, $i = 1, \ldots, n$, with $\hat{r}_i = r_i - r_0$ and $\hat{v}_i = v_i - v_0$. $P$ and $Q$ are defined in Lemma 4.1, $\kappa_1 = \frac{\alpha \gamma}{\lambda_{\min}(P)}$, and $\kappa_2 = \frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)}$.

**Proof:** Noting that $\hat{r}_i = r_i - r_0$ and $\hat{v}_i = v_i - v_0$, we rewrite the closed-loop system of (11) using (12) as
\begin{equation}
\begin{aligned}
\dot{\hat{r}}_i &= \hat{v}_i \\
\dot{\hat{v}}_i &= -\sum_{j=0}^{n} a_{ij} [(\hat{r}_i - \hat{r}_j) + \alpha(\hat{v}_i - \hat{v}_j)] \\
&\quad - \beta \text{sgn} \left\{ \sum_{j=0}^{n} a_{ij} [\gamma(\hat{r}_i - \hat{r}_j) + (\hat{v}_i - \hat{v}_j)] \right\} - \hat{v}_0.
\end{aligned}
\end{equation}
Equation (17) can be written in matrix form as
\begin{equation}
\begin{aligned}
\dot{\hat{r}} &= \hat{v} \\
\dot{\hat{v}} &= -M \hat{r} - \alpha M \hat{v} - \beta \text{sgn}[M(\gamma \hat{r} + \hat{v})] - \hat{v}_0,
\end{aligned}
\end{equation}
where $\hat{r}$ and $\hat{v}$ are defined in (16) and $M = L + \text{diag}(a_{10}, \ldots, a_{n0})$.

Consider the Lyapunov function candidate
\begin{equation}
V = [\hat{r}^T \quad \hat{v}^T] P [\hat{r} \quad \hat{v}] = \frac{1}{2} \hat{r}^T M^2 \hat{r} + \frac{1}{2} \hat{v}^T M \hat{v} + \gamma \hat{r}^T M \hat{v}.
\end{equation}
Note that according to Lemma 4.1, $P$ is symmetric positive definite when $\gamma$ satisfies (13). The derivative of $V$ is
\begin{equation}
\dot{V} = \hat{r}^T M^2 \hat{v} + \hat{v}^T M \hat{v} + \gamma \hat{r}^T M \hat{v} \\
&\leq - \hat{r}^T \hat{v} Q [\hat{r} \quad \hat{v}] - (\beta - \varphi_\ell) \| M(\gamma \hat{r} + \hat{v}) \|^2_1,
\end{equation}
where the last inequality follows from the fact that $|v_0| \leq \varphi_\ell$. Note that according to Lemma 4.1, $Q$ is symmetric positive definite when $\gamma$ satisfies (13). Also note that $\beta > \varphi_\ell$. It follows that $\dot{V}$ is negative definite. Therefore, it follows that $\hat{r}(t) \to 0_n$ and $\hat{v}(t) \to 0_n$ as $t \to \infty$, where $0_n$ is the $n \times 1$ zero vector. Equivalently, it follows that $r_l(t) \to 0_n$ and $v_l(t) \to 0_n$ as $t \to \infty$.

We next show that distributed consensus tracking is at least achieved globally exponentially. Note that $\dot{V} \leq \lambda_{\max}(P) \| [\hat{r}^T \quad \hat{v}^T] \|^2_2$. It then follows from (19) that
\begin{equation}
\dot{V} \leq - \lambda_{\min}(Q) \| [\hat{r}^T \quad \hat{v}^T] \|^2_2 \\
\leq - \lambda_{\min}(Q) \lambda_{\max}(P).
\end{equation}
Therefore, we can get that $V(t) \leq V(0)e^{-\frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)} \lambda_{\max}(P)}$. Note also that $V \geq \lambda_{\min}(P) \| [\hat{r}^T \quad \hat{v}^T] \|^2_2$. After some manipulation, we can get (16).

**Remark 4.2:** In the proof of Theorem 4.1, the Lyapunov function is chosen as (18). Here $P$ can also be chosen as $P = \begin{bmatrix} \frac{1}{2}M & \frac{1}{2}M \\ \frac{1}{2}M & \frac{1}{2}M \end{bmatrix}$ and the derivative of $V$ also satisfies (19) with $Q = \begin{bmatrix} \alpha^2 & \gamma^2 \gamma M \gamma \gamma M \\ \gamma^2 & \gamma M \gamma \gamma M \end{bmatrix}$.

By following a similar analysis to that of Lemma 4.1, both $P$ and $Q$ are symmetric positive definite when $\alpha$ and $\gamma$ are chosen properly. In particular, sufficient conditions for $\alpha$ and $\gamma$ are $\alpha \gamma = 1$ and $\gamma < \frac{\lambda_{\min}(M)}{4\lambda_{\max}(M)^2}$.

### B. Switching Network Topology

In this subsection, we assume that the network topology is switching. We further assume that the network topology switches according to the same model as described right
before (6). In this case, we propose the distributed consensus tracking algorithm for (11) as
\[ u_i = -\sum_{j \in \mathcal{N}_i(t)} b_{ij} [(r_i - r_j) + \alpha (v_i - v_j)] \]
\[ -\beta \sum_{j \in \mathcal{N}_i(t)} b_{ij} \left( \text{sgn} \left( \sum_{k \in \mathcal{N}_i(t)} b_{ik} [\gamma (r_i - r_k) + (v_i - v_k)] \right) \right) - \text{sgn} \left( \sum_{k \in \mathcal{N}_i(t)} b_{ik} [\gamma (r_j - r_k) + (v_j - v_k)] \right), \] \tag{20}
where \( \mathcal{N}_i(t) \) is defined as in Section III-A, \( b_{ij}, i = 1, \ldots, n \), are positive constants, and \( \alpha, \beta \), and \( \gamma \) are positive constants.\(^1\) Before moving on, we need the following lemma.

**Lemma 4.2:** Suppose that the undirected graph \( G(t) \) is connected and the virtual leader is a neighbor of at least one follower (i.e., \( 0 \in \mathcal{N}_i(t) \) for some \( i \)) at each time instant. Let \( \bar{M}(t) \) be defined as in (10). Let \( \bar{P}(t) = \left[ \frac{1}{2} \bar{M}(t) - \frac{1}{2} I_n \right] \) and \( \bar{Q}(t) = \left[ \frac{\gamma \bar{M}(t)}{2} - \frac{\alpha \bar{M}(t) - \gamma I_n}{2} \right] \), where \( \gamma \) and \( \alpha \) are positive constants. If \( \gamma \) satisfies
\[ 0 < \gamma < \min_{t} \left\{ \sqrt{\frac{\lambda_{\min}(\bar{M}(t))}{4}} \frac{4\alpha \lambda_{\min}(\bar{M}(t))}{4 + 4\alpha^2 \lambda_{\min}(\bar{M}(t))} \right\}, \] \tag{21}
then both \( \bar{P}(t) \) and \( \bar{Q}(t) \) are symmetric positive definite at each time instant.

**Proof:** The proof is similar to that of Lemma 4.1 and is therefore omitted here. \( \blacksquare \)

**Theorem 4.3:** Suppose that the undirected graph \( G(t) \) is connected and the virtual leader is a neighbor of at least one follower (i.e., \( 0 \in \mathcal{N}_i(t) \) for some \( i \)) at each time instant. Using (20) for (11), if \( \beta > \varphi_{\epsilon} \) and (21) is satisfied, then \( r_i(t) \to r_0(t) \) and \( v_i(t) \to v_0(t) \) as \( t \to \infty \).

**Proof:** Let \( V_i = \frac{1}{2} b_{ij} (r_i - r_j)^2 \), \( i, j = 1, \ldots, n \), when \( |r_i - r_j| \leq R \) and \( V_i = \frac{1}{2} b_{ij} R^2 \) when \( |r_i - r_j| > R \). Also let \( V_0 = \frac{1}{2} b_{00} (r_0 - r_j)^2 \), \( i = 1, \ldots, n \), when \( |r_i - r_0| \leq R \) and \( V_0 = \frac{1}{2} b_{00} R^2 \) when \( |r_i - r_0| > R \). Consider the Lyapunov function candidate \( V = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} V_{ij} + \sum_{i=1}^{n} V_{ii} + \frac{1}{2} \bar{v}^T \bar{v} + \frac{1}{2} \bar{v}_{T} \bar{v} \), where \( \bar{v} = [\bar{r}_1, \ldots, \bar{r}_n]^T \) with \( \bar{r}_i = r_i - r_0 \) and \( \bar{v} = [\bar{v}_1, \ldots, \bar{v}_n]^T \) with \( \bar{v}_i = v_i - v_0 \). Note that \( V \) can be written as
\[ V = \left[ \bar{r}^T \right] \bar{P}(t) \left[ \bar{r} \right] + \frac{1}{2} \sum_{i=1}^{n} \sum_{j \in \mathcal{N}_i(t), j \neq 0} b_{ij} R^2 \]
\[ + \frac{1}{2} \sum_{i \in \mathcal{N}_i(t)} b_{0i} R^2. \] \tag{22}
Note also that according to Lemma 4.2, \( \bar{P}(t) \) is symmetric positive definite when (21) is satisfied. By following a similar line to the proof of Theorem 4.1 and using nonsmooth analysis, we can obtain that the generalized derivative of \( V \) is negative definite under the condition of the theorem. Therefore, we have that \( r_i(t) \to r_0(t) \) and \( v_i(t) \to v_0(t) \) as \( t \to \infty \).

**Remark 4.4:** It can be noted that (20) requires the availability of the information from both the neighbors (i.e., one-hop neighbors) and the neighbors’ neighbors (i.e. two-hop neighbors). However, accurate measurements of the two-hop neighbors’ information are not necessary because, by only the signs (i.e. ‘+’ or ‘−’), we require them in (20). In fact, (20) can be easily implemented in real systems in the sense that follower \( i, i = 1, \ldots, n \), shares both its own state (i.e., position and velocity) and the sign of \( \sum_{j \in \mathcal{N}_i(t)} b_{ij} [\gamma (r_i - r_j) + (v_i - v_j)] \) with its neighbors. Note that follower \( i \) has to compute \( \sum_{j \in \mathcal{N}_i(t)} b_{ij} (r_i - r_j) \) and \( \sum_{j \in \mathcal{N}_i(t)} b_{ij} (v_i - v_j) \) (20) (correspondingly, \( \sum_{j=0}^{n} a_{ij} (r_i - r_j) \) and \( \sum_{j=0}^{n} a_{ij} (v_i - v_j) \) in (12)) in order to derive the corresponding control input for itself. Therefore, compared with (12), (20) does not significantly increase the computation complexity.

**Remark 4.5:** Under the condition of Theorem 4.3, (20) guarantees at least global exponential tracking under a switching network topology. However, in contrast to the result in Theorem 4.1, it might not be easy to explicitly compute the decay rate (i.e., \( \kappa \) in Theorem 4.3) because the switching pattern of the network topology will play an important role in determining the decay rate.

**Remark 4.6:** Similar to the analysis in Remark 4.2, in Lyapunov function (22), we can choose \( \bar{P}(t) = \left[ \frac{1}{2} \bar{M}(t) - \frac{1}{2} I_n \right] \). Then it follows that
\[ \dot{Q}(t) = \left[ \frac{\alpha \gamma}{2} \bar{M}(t) + \frac{\bar{M}(t) - \gamma I_n}{2} \right] \frac{\alpha \bar{M}(t) - \gamma I_n}{2} \]
Accordingly, both \( \bar{P}(t) \) and \( \dot{Q}(t) \) are positive definite if \( \alpha \) and \( \gamma \) are chosen properly. In particular, sufficient conditions for \( \alpha \) and \( \gamma \) are \( \alpha \gamma = 1 \) and \( \gamma < \min_{t} \frac{\lambda_{\min}(\bar{M}(t))}{4 \lambda_{\min}(\bar{M}(t)) + 1} \).

V. CONNECTIVITY MAINTENANCE

In Theorems 3.2 and 4.3, it is assumed that the undirected graph \( G(t) \) is connected and the virtual leader is a neighbor of at least one follower at each time instant. However, this poses an obvious constraint in real applications because the connectivity requirement is not necessarily always satisfied. In this section, we propose an adaptive connectivity maintenance mechanism in which the adjacency matrix (i.e., \( b_{ij} \) in (6) and (20)) is redefined as follows:

1. When \( |r_i(0) - r_j(0)| \geq R, b_{ij}(t) = 1 \) if \( |r_i(t) - r_j(t)| < R \) and \( b_{ij}(t) = 0 \) otherwise.
2. When \( |r_i(0) - r_j(0)| < R, b_{ij}(t) \) is defined satisfying: 1) \( b_{ij}(0) > 0 \); 2) \( b_{ij}(t) \) is nondecreasing; 3) \( b_{ij}(t) \) is differentiable (or differentiable almost everywhere); 4) \( b_{ij}(t) \) goes to infinity if \( |r_i(t) - r_j(t)| \to R \).

The motivation here is to maintain the initially existing connectivity patterns. That is, if two followers are neighbors of each other (correspondingly, the virtual leader is a neighbor of a follower) at \( t = 0 \), the two followers are guaranteed to be neighbors of each other (correspondingly, the virtual
leader is guaranteed to be a neighbor of this follower) at $t > 0$. However, if two followers are not neighbors of each other (correspondingly, the virtual leader is not a neighbor of a follower) at $t = 0$, the two followers are not necessarily guaranteed to be neighbors of each other (correspondingly, the virtual leader is not guaranteed to be a neighbor of this follower) at $t > 0$.

Using the proposed adaptive adjacency matrix, the consensus tracking algorithm for (1) can be chosen as

$$ u_i = -\alpha \sum_{j \in \mathcal{N}_i(t)} b_{ij}(t)(r_i - r_j) - \beta \sum_{j \in \mathcal{N}_i(t)} b_{ij}(t) \left\{ \text{sgn} \left[ \sum_{k \in \mathcal{N}_j(t)} b_{jk}(t)(r_i - r_k) \right] - \text{sgn} \left[ \sum_{k \in \mathcal{N}_j(t)} b_{jk}(t)(r_j - r_k) \right] \right\} $$

with the Lyapunov function chosen as $V = \frac{1}{2} \tilde{r}^T \tilde{r}$ while the consensus tracking algorithm for (11) can be chosen as (20) with the Lyapunov function chosen as $V = \left[ \tilde{r}^T \ \tilde{v}^T \right] \hat{P}(t) \left[ \begin{array}{c} \tilde{r} \\ \tilde{v} \end{array} \right]$ with $\hat{P}(t)$ chosen as in Remark 4.6. Note that there always exist $\alpha$ and $\gamma$ satisfying the conditions in Remark 4.6 because $\lambda_{\min}(\hat{M}(t))$ is nondecreasing under the connectivity maintenance mechanism. When the control gains are chosen properly (i.e., $\alpha > 0$ and $\beta > \gamma$ for single-integrator kinematics and $\alpha$ and $\gamma$ satisfies Remark 4.6 and $\beta > \varphi_\ell$ for double-integrator dynamics), it can be shown that distributed consensus tracking can be guaranteed for both first-order kinematics and second-order dynamics if the undirected graph $\hat{G}(t)$ is initially connected and the virtual leader is initially a neighbor of at least one follower (i.e., at $t = 0$). The proof follows a similar analysis to that of the corresponding algorithm in the absence of connectivity maintenance mechanism except that the initially existing connectivity patterns can be maintained because otherwise $\dot{V} \to -\infty$ as $\|r_i(t) - r_j(t)\| \to R$ by noting that $\dot{V} = -\alpha \tilde{r} \hat{M}(t) \tilde{r} - (\beta - \gamma) \|\hat{M}(t)\tilde{r}\|$ for single-integrator kinematics and $\dot{V} = -\left[ \tilde{r}^T \ \tilde{v}^T \right] \hat{Q}(t) \left[ \begin{array}{c} \tilde{r} \\ \tilde{v} \end{array} \right]$ for double-integrator dynamics, where $\hat{Q}(t)$ is defined in Remark 4.6.

VI. Conclusion

In this paper, we studied a distributed consensus tracking problem via a variable structure approach when there exists a dynamic virtual leader who is a neighbor of only a subset of a group of followers, all followers have only local interaction, and only partial measurements of the states of the virtual leader and the followers are available. For first-order kinematics, we proposed a distributed consensus tracking algorithm without velocity measurements and showed that distributed consensus tracking can be achieved in finite time. For second-order dynamics, we proposed two distributed consensus tracking algorithms without acceleration measurements and showed that the proposed algorithms guaranteed at least global exponential tracking. A mild connectivity requirement was proposed by adopting an adaptive connectivity maintenance mechanism.

References