Distributed Coordinated Tracking via a Variable Structure Approach - Part II: Swarm Tracking

Yongcan Cao and Wei Ren

Abstract—This is the second part of a two-part paper on distributed coordinated tracking for a group of autonomous vehicles via a variable structure approach. In the first part of this paper [1], we investigated distributed consensus tracking algorithms in the presence of a dynamic virtual leader. In the second part of this paper, we focus on the distributed swarm tracking problem where the followers move cohesively with the virtual leader while avoiding inter-vehicle collision with local interaction. In the case of first-order kinematics, we propose a distributed swarm tracking algorithm without velocity measurements. In the case of second-order dynamics, we first propose a distributed swarm tracking algorithm when the velocity of the virtual leader is constant. We then propose a distributed swarm tracking algorithm based on a distributed estimator when the velocity of the virtual leader is dynamic. For distributed swarm tracking in the case of both first-order kinematics and second-order dynamics, a mild connectivity requirement is proposed by adopting a connectivity maintenance mechanism in which the potential function is defined in a proper way. Several simulation examples are presented as a proof of concept.

I. INTRODUCTION

The second part of this paper is motivated by the study of various flocking and swarm tracking algorithms in the presence/absence of a (virtual) leader. The objective of flocking or swarm tracking with a leader is that a group of followers moves cohesively with the virtual leader while avoiding inter-vehicle collision with local interaction. In [2], the author studied a flocking algorithm under the assumption that the leader’s velocity is constant and is available to all followers. The authors in [3] extended the results in [2] in two aspects. When the leader has a constant velocity, [3] requires accurate position and velocity measurements of the leader. When the leader has a varying velocity, [3] requires that the leader’s position, velocity, and acceleration are available to all followers. In [4], flocking of a group of autonomous vehicles with a dynamic leader was solved by using a set of switching control laws. However, [4] requires the availability of the acceleration of the leader. In [5], the authors studied a swarm tracking algorithm via a variable structure approach using artificial potentials and the sliding mode control technique. However, [5] requires the availability of the leader’s position to all followers and an all-to-all communication pattern among all followers.

Taking into account the limitations in the aforementioned references, we focus on solving a distributed swarm tracking problem via a variable structure approach when there exists a dynamic virtual leader under the following three assumptions: 1) The virtual leader is a neighbor of only a subset of a group of followers; 2) There exists only local interaction among all followers; 3) The velocity measurements of the virtual leader and all followers in the case of first-order kinematics and the acceleration measurements of the virtual leader and all followers in the case of second-order dynamics are not required. In the case of first-order kinematics, we propose and study a distributed swarm tracking algorithm without velocity measurements. In the case of second-order kinematics, we first propose and study a distributed swarm tracking algorithm when the velocity of the virtual leader is constant. We then propose a distributed swarm tracking algorithm based on a distributed estimator when the velocity of the virtual leader is dynamic. For distributed swarm tracking in the case of both first-order kinematics and second-order dynamics, a mild connectivity requirement is proposed by adopting a connectivity maintenance mechanism in which a potential function is defined in a proper way.

The remainder of this paper is organized as follows: In Section II, the graph theory notions used throughout this paper are introduced. Sections III and IV are the main parts of this paper focusing on distributed swarm tracking for, respectively, first-order kinematics and second-order dynamics. Conclusion and future works are given in Section VI.

II. BACKGROUND AND PRELIMINARIES

Suppose that a team consists of n vehicles. We use a weighted undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ to model the interaction among these vehicles, where $\mathcal{V} = \{1, \ldots, n\}$ is the node set, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the edge set, and $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{n \times n}$ is the weighted adjacency matrix. An edge $(i, j)$ in $\mathcal{G}$ denotes that vehicles $i$ and $j$ can obtain information from each other. Vehicle $j$ is a neighbor of vehicle $i$ if $(j, i) \in \mathcal{E}$. The weighted adjacency matrix $\mathcal{A}$ associated with $\mathcal{G}$ is defined such that $a_{ij}$ is a positive weight if $(j, i) \in \mathcal{E}$, and $a_{ij} = 0$ otherwise. Note that here $a_{ij} = a_{ji}$, $\forall i \neq j$, since $(j, i) \in \mathcal{E}$ implies $(i, j) \in \mathcal{E}$.

A path is a sequence of edges in an undirected graph of the form $(i_1, i_2, i_3, \ldots)$, where $i_j \in \mathcal{V}$. An undirected graph is connected if there is an undirected path between every pair of distinct nodes.

Let the Laplacian matrix $\mathcal{L} = [\ell_{ij}] \in \mathbb{R}^{n \times n}$ associated with $\mathcal{A}$ be defined as $\ell_{ii} = \sum_{j=1, j \neq i}^{n} a_{ij}$ and $\ell_{ij} = -a_{ij}$, $i \neq j$. Note that $\mathcal{L}$ is symmetric positive semi-definite. Also note that $\mathcal{L}$ has a simple zero eigenvalue with an associated eigenvector $1$, where $1$ is an all-one column vector with a
compatible size, and all other eigenvalues are positive if and only if $G$ is connected [6].

III. DISTRIBUTED SWARM TRACKING FOR FIRST-ORDER KINEMATICS

In this section, we study distributed swarm tracking for first-order kinematics. Suppose that in addition to the $n$ vehicles, labeled as vehicles 1 to $n$, called followers hereafter, there exists a virtual leader, labeled as vehicle 0, with a (time-varying) position $r_0$ and velocity $\dot{r}_0$. We assume that $|\dot{r}_0| \leq \gamma_t$, where $\gamma_t$ is a positive constant.

Consider followers with first-order kinematics given by

$$\dot{r}_i = u_i, \quad i = 1, \ldots, n,$$

where $r_i \in \mathbb{R}$ is the position and $u_i \in \mathbb{R}$ is the control input associated with the $i$th vehicle. The objective here is to design $u_i$ for (1) such that all followers move cohesively with the virtual leader while avoiding inter-vehicle collision with local interaction in the absence of velocity measurements. Here we have assumed that all vehicles are in a one-dimensional space for the simplicity of presentation. All results hereafter are still valid for the $m$-dimensional ($m > 1$) case by introduction of the Kronecker product.

Before moving on, we need to define potential functions which will be used in the distributed swarm tracking algorithms.

**Definition 3.1:** The potential function $V_{ij}$ is a differentiable, nonnegative function of $||r_i - r_j||^1$ satisfying the following conditions:

1. $V_{ij}$ achieves its unique minimum when $||r_i - r_j||$ is equal to its desired value $d_{ij}$.
2. $V_{ij} \to \infty$ if $||r_i - r_j|| \to 0$.
3. $\alpha ||r_i - r_j|| = 0$ if $||r_i - r_j|| > R$, where $R > \max_{i,j} d_{ij}$ is a positive constant.
4. $V_{ij} = c, i = 1, \ldots, n$, where $c$ is a positive constant.

**Lemma 3.1:** Let $V_{ij}$ be defined in Definition 3.1. The following equality holds

$$\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left( \frac{\partial V_{ij}}{\partial r_i} \dot{r}_i + \frac{\partial V_{ij}}{\partial r_j} \dot{r}_j \right) = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial V_{ij}}{\partial r_i} \dot{r}_i.$$

**Proof:** Note that

$$\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left( \frac{\partial V_{ij}}{\partial r_i} \dot{r}_i + \frac{\partial V_{ij}}{\partial r_j} \dot{r}_j \right) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial V_{ij}}{\partial r_i} \dot{r}_i - \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial V_{ij}}{\partial r_i} \dot{r}_j$$

$$= \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial V_{ij}}{\partial r_i} \dot{r}_i - \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial V_{ij}}{\partial r_j} \dot{r}_j$$

$$= \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial V_{ij}}{\partial r_i} \dot{r}_i + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial V_{ij}}{\partial r_j} \dot{r}_j$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial V_{ij}}{\partial r_i} \dot{r}_i.$$

where we have used the fact that $\frac{\partial V_{ij}}{\partial r_i} = -\frac{\partial V_{ij}}{\partial r_j}$ from Definition 3.1. Therefore, the lemma holds.

Let $\mathcal{N}_i \subseteq \{0, 1, \ldots, n\}$ denote the neighbor set of follower $i$ in the team consisting of the $n$ followers and the virtual leader. We next consider the case of a switching network topology by assuming that $j \in \mathcal{N}_i(t), i = 1, \ldots, n, j = 0, \ldots, n$, if $|r_i - r_j| \leq R$ at time $t$ and $j \notin \mathcal{N}_i(t)$ otherwise, where $R$ denotes the communication/sensing radius of the vehicles. We propose the distributed swarm tracking algorithm for (1) as

$$u_i = -\alpha \sum_{j \in \mathcal{N}_i(t)} \frac{\partial V_{ij}}{\partial r_i} - \beta \text{sgn} \left( \sum_{j \in \mathcal{N}_i(t)} \frac{\partial V_{ij}}{\partial r_i} \right),$$

where $\alpha$ and $\beta$ are positive constants and $V_{ij}$ is defined in Definition 3.1.

**Theorem 3.2:** Suppose that the undirected graph $G(t)$ is connected and the virtual leader is a neighbor of at least one follower (i.e., $0 \in \mathcal{N}_i(t)$ for some $i$) at each time instant. Using (2) for (1), if $\beta > \gamma_t$, the relative distances of all followers and the virtual leader will ultimately converge to local minima and the inter-vehicle collision is avoided.

**Proof:** Consider the Lyapunov function candidate

$$V = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} V_{ij} + \sum_{i=1}^{n} V_{i0}.$$

Taking derivative of $V$ gives that

$$\dot{V} = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left( \frac{\partial V_{ij}}{\partial r_i} \dot{r}_i + \frac{\partial V_{ij}}{\partial r_j} \dot{r}_j \right)$$

$$+ \sum_{i=1}^{n} \left( \frac{\partial V_{i0}}{\partial r_i} \dot{r}_i + \frac{\partial V_{i0}}{\partial r_0} \dot{r}_0 \right)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial V_{ij}}{\partial r_i} \dot{r}_i + \sum_{i=1}^{n} \left( \frac{\partial V_{i0}}{\partial r_i} \dot{r}_i + \frac{\partial V_{i0}}{\partial r_0} \dot{r}_0 \right)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial V_{ij}}{\partial r_i} \dot{r}_i - \alpha \sum_{j=0}^{n} \frac{\partial V_{ij}}{\partial r_i} - \beta \text{sgn} \left( \sum_{j=0}^{n} \frac{\partial V_{ij}}{\partial r_i} \right)$$

$$+ \sum_{i=1}^{n} \frac{\partial V_{i0}}{\partial r_i} \dot{r}_i - \beta \text{sgn} \left( \sum_{j=0}^{n} \frac{\partial V_{i0}}{\partial r_i} \right)$$

$$+ \sum_{i=1}^{n} \frac{\partial V_{i0}}{\partial r_0} \dot{r}_0$$

1 In this definition, $r_i$ can be $m$-dimensional.
= -\alpha \sum_{i=1}^{n} \left( \sum_{j=0}^{n} \frac{\partial V_{ij}}{\partial r_i} \right)^2 - \beta \sum_{i=1}^{n} \left| \sum_{j=0}^{n} \frac{\partial V_{ij}}{\partial r_i} \right| + \sum_{i=1}^{n} \frac{\partial V_{ij}}{\partial r_i} r_i + \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial V_{ij}}{\partial r_i} r_0 \sum_{j=0}^{n} \frac{\partial V_{ij}}{\partial r_i} \right)^2 \leq -\alpha \sum_{i=1}^{n} \left( \sum_{j=0}^{n} \frac{\partial V_{ij}}{\partial r_i} \right)^2 - \beta \sum_{i=1}^{n} \left| \sum_{j=0}^{n} \frac{\partial V_{ij}}{\partial r_i} \right| + \sum_{i=1}^{n} \sum_{j=0}^{n} \frac{\partial V_{ij}}{\partial r_i} r_i \right),

where we have used Lemma 3.1 to derive (3) and the fact that \( \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial V_{ij}}{\partial r_i} = 0 \) to derive (4). Because \( \beta > \gamma \), we get that \( V \leq 0 \), which in turn proves the theorem.

IV. DISTRIBUTED SWARM TRACKING FOR SECOND-ORDER DYNAMICS

In this section, we study distributed swarm tracking for second-order dynamics. Suppose that there exists a virtual leader, labeled as vehicle 0, with a (time-varying) position \( r_0 \) and velocity \( \dot{v}_0 \). We assume that \( |\dot{v}_0| \leq \varphi \), where \( \varphi \) is a positive constant.

Consider followers with second-order dynamics given by

\[
\dot{r}_i = v_i, \quad \dot{v}_i = u_i, \quad i = 1, \ldots, n, \tag{5}
\]

where \( r_i \in \mathbb{R} \) and \( v_i \in \mathbb{R} \) are, respectively, the position and velocity of follower \( i \), and \( u_i \in \mathbb{R} \) is the control input. The objective here is to design \( u_i \) for (1) such that all followers move cohesively with the virtual leader while avoiding inter-vehicle collision with local interaction in the absence of acceleration measurements. We again assume that the network topology switches according to the model described in Section III. Here we only consider the case when all vehicles are in a one-dimensional space. All results hereafter are still valid for the \( m \)-dimensional \( (m > 1) \) case by introduction of the Kronecker product.

A. Constant Virtual Leader’s Velocity

In this subsection, we assume that the virtual leader’s velocity is constant (i.e., the virtual leader’s acceleration is zero.). We propose the distributed swarm tracking algorithm for (5) as

\[
u_i = - \sum_{j \in \mathcal{N}_i(t)} \frac{\partial V_{ij}}{\partial r_i} - \beta \sum_{j \in \mathcal{N}_i(t)} b_{ij} \left\{ \text{sgn} \left[ \sum_{k \in \mathcal{N}_i(t)} b_{ik} (v_i - v_k) \right] - \text{sgn} \left[ \sum_{k \in \mathcal{N}_i(t)} b_{jk} (v_j - v_k) \right] \right\}, \tag{6}
\]

where \( V_{ij} \) is the potential function defined in Definition 3.1, \( \mathcal{N}_i(t) \) is defined as in Section III, \( \beta \) is a positive constant, and \( b_{ij}, b_{ik}, \) and \( b_{jk} \) are positive constants. Note that (6) requires both the one-hop and two-hop neighbors’ information.

Theorem 4.1: Suppose that the undirected graph \( G(t) \) is connected and the virtual leader is a neighbor of at least one follower (i.e., \( 0 \in \mathcal{N}_i(t) \) for some \( i \)) at each time instant. Using (6) for (5), the velocity differences of all followers and the virtual leader will ultimately converge to zero, the relative distances of all followers and the virtual leader will ultimately converge to local minima, and the inter-vehicle collision is avoided.

Proof: Letting \( \ddot{r}_i = r_i - r_0 \) and \( \ddot{v}_i = v_i - v_0 \), it follows that (6) can be written as

\[
u_i = - \sum_{j \in \mathcal{N}_i(t)} \frac{\partial V_{ij}}{\partial r_i} - \beta \sum_{j \in \mathcal{N}_i(t)} b_{ij} \left\{ \text{sgn} \left[ \sum_{k \in \mathcal{N}_i(t)} b_{ik} (v_i - v_k) \right] - \text{sgn} \left[ \sum_{k \in \mathcal{N}_i(t)} b_{jk} (v_j - v_k) \right] \right\}.
\]

Consider the Lyapunov function candidate

\[
V = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} V_{ij} + \frac{1}{2} \dot{v}^T \dot{v},
\]

where \( \dot{v} \) is a column stack vector of \( \dot{v}_i \). Taking derivative of \( V \) gives that

\[
\dot{V} = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left( \frac{\partial^2 V_{ij}}{\partial r_i^2} \ddot{r}_i + \frac{\partial V_{ij}}{\partial r_i} \dot{r}_i + \frac{\partial V_{ij}}{\partial r_i} \dot{r}_0 \right) + \dot{v}^T \dot{v} = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial V_{ij}}{\partial r_i} \ddot{r}_i + \sum_{i=1}^{n} \frac{\partial V_{ij}}{\partial r_i} \dot{r}_i - \sum_{i=1}^{n} \dot{v}_i \sum_{j=1}^{n} \frac{\partial V_{ij}}{\partial r_i} - \beta \dot{v}^T M(t) \text{sgn} \left[ \dot{M}(t) \dot{v} \right] = -\beta \| \dot{M}(t) \|_1 \| \dot{v} \|_1.
\]

(7) is derived by using Lemma 3.1 and the fact that \( \dot{r}_0 = 0 \), and (8) is derived by using the fact that \( \dot{M}(t) \) is symmetric. Note that \( \dot{M}(t) \) is symmetric positive definite at each time instant under the condition of the theorem. Because \( \beta > 0 \), it follows that the derivative of \( V \) is negative semi-definite under the condition of the theorem. When \( V \equiv 0 \), it follows that \( \ddot{v}_i \equiv 0 \). Combining with (6) shows that \( \sum_{j \in \mathcal{N}_i(t)} \frac{\partial V_{ij}}{\partial r_i} \equiv 0 \). It then follows from the LaSalle’s invariance principle for nonsmooth systems [7] that \( v_i(t) \rightarrow v_i \) and \( \sum_{j=0}^{n} \frac{\partial V_{ij}}{\partial r_i} \rightarrow 0 \) as \( t \rightarrow \infty \), which in turn proves the theorem.

B. Dynamic Virtual Leader’s Velocity

In this subsection, we assume that the virtual leader’s velocity is varying (i.e., the virtual leader’s acceleration is, in general, nonzero.). We propose the following distributed
swarm tracking algorithm with a distributed estimator for (5) as
\[ u_i = -\gamma \text{sgn} \left\{ \sum_{j \in N_i(t)} b_{ij} [\hat{v}_{i0} - \hat{v}_j] \right\} - \sum_{j \in N_i(t)} \frac{\partial V_{ij}}{\partial r_i} - \beta \sum_{j \in N_i(t)} b_{ij} \]
\[ \text{sgn} \left\{ \sum_{k \in N_j(t)} b_{ik} (v_i - v_k) \right\} - \text{sgn} \left\{ \sum_{k \in N_j(t)} b_{k} (v_j - v_k) \right\} \]
\[ \text{sgn} \left\{ \sum_{j \in N_i(t)} b_{ij} (\hat{v}_{i0} - \hat{v}_j) \right\}, \quad i = 1, \ldots, n, \]
where
\[ \hat{v}_{i0} = -\gamma \text{sgn} \left\{ \sum_{j \in N_i(t)} b_{ij} [\hat{v}_{i0} - \hat{v}_j] \right\}, \quad i = 1, \ldots, n, \]
with \( \hat{v}_{i0} \) being the i-th vehicle’s estimate of the virtual leader’s velocity and \( \hat{v}_{i0} = v_0 \). Here (11) is a distributed estimator motivated by the results in the first part of this paper [1].

**Theorem 4.2:** Suppose that the undirected graph \( G(t) \) is connected and the virtual leader is a neighbor of at least one follower (i.e., \( 0 \in N_i(t) \) for some \( i \)) at each time instant. Assume that \( \gamma > \varphi_r \). Using (10) for (5), the velocity differences of all followers and the virtual leader will ultimately converge to zero, the relative distances of all followers and the virtual leader will eventually converge to local minima, and the inter-vehicle collision is avoided.

**Proof:** For distributed estimator (11), it follows from Theorem 3.1 in [1] that there exists positive \( 7 \) such that \( \hat{v}_{i0}(t) = v_0(t) \) for any \( t \geq 7 \). Note that \( \hat{v}_{i0} \) in (11) is a switching signal, which is different from \( v_0(t) \) at each time instant. However, for \( t_1, t_2 \geq 7 \), we have that \( \int_{t_1}^{t_2} \hat{v}_{i0}(t)dt = \int_{t_1}^{t_2} v_0(t)dt \) by noting that \( \hat{v}_{i0}(t) = v_0(t) \) for any \( t \geq 7 \). Therefore, \( r_i \) will be unchanged when replacing \( \hat{v}_{i0} \) with \( v_0 \) for \( t \geq 7 \). For \( t \geq 7 \), by replacing \( \hat{v}_{i0} \) with \( v_0 \) and choosing the same Lyapunov function candidate as in the proof of Theorem 4.1, it follows from a similar analysis to that in the proof of Theorem 4.1 and the LaSalle’s invariance principle for nonsmooth systems [7] that \( v_i(t) \rightarrow v_0(t) \) and \( \sum_{j=0}^{n} \frac{\partial V_{ij}}{\partial r_i} \rightarrow 0 \) as \( t \rightarrow \infty \). This completes the proof.

**Remark 4.3:** Note that algorithms (6) and (10) require the availability of both the one-hop and two-hop neighbors’ information. In contrast to the flocking algorithms in [2], [3], [8], (6) and (10) do not require accurate velocity measurements because the velocity measurements are only used to calculate the sign (i.e., ‘+’ or ‘-’) in (6). Therefore, (6) is more robust to internal uncertainties and external disturbances. In contrast to the flocking algorithms in [2], [3], the availability of the virtual leader’s information (i.e., the position, velocity, and acceleration) to all followers is not required in (10) due to the introduction of the distributed estimator.

**V. CONNECTIVITY MAINTENANCE**

In Theorems 3.2, 4.1, and 4.2, it is assumed that the undirected graph \( G(t) \) is connected and the virtual leader is a neighbor of at least one follower at each time instant. However, this poses an obvious constraint in real applications because the connectivity requirement is not necessarily always satisfied. In this section, we propose a mild connectivity requirement for distributed swarm tracking by adopting a connectivity maintenance mechanism. We assume that the communication pattern switches according to the rules described right before (2). To achieve connectivity maintenance, the potential function in Definition 3.1 is redefined as follows:

**Definition 5.1:** 1. When \( ||r_i - r_j|| \geq R \) at the initial time (i.e., \( t = 0 \)), \( V_{ij} \) is defined as in Definition 3.1.
2. When \( ||r_i - r_j|| < R \) at the initial time (i.e., \( t = 0 \)), \( V_{ij} \) is defined satisfying conditions 1), 2), and 4) in Definition 3.1 and condition 3) in Definition 3.1 is replaced with the condition that \( V_{ij} \rightarrow \infty \) as \( ||r_i - r_j|| \rightarrow R \).

The motivation here is to guarantee that the initially existing connectivity patterns always exist for any \( t > 0 \). That is, if two followers are neighbors of each other (correspondingly, if the virtual leader is a neighbor of a follower) at \( t = 0 \), the two followers are guaranteed to be neighbors of each other (correspondingly, the virtual leader is guaranteed to be a neighbor of this follower) at \( t > 0 \). However, if two followers are not neighbors of each other (correspondingly, if the virtual leader is not a neighbor of a follower) at \( t = 0 \), the two followers are not necessarily guaranteed to be neighbors of each other (correspondingly, the virtual leader is not necessarily guaranteed to be a neighbor of this follower) at \( t > 0 \).

Using the potential function defined above, it is easy to show that distributed swarm tracking can be guaranteed for both first-order kinematics (cf. Theorem 3.2) and second-order dynamics (cf. Theorems 4.1 and 4.2) if the undirected graph \( G(t) \) is initially connected (i.e., \( t = 0 \)), the virtual leader is initially a neighbor of at least one follower, and the other conditions for the control gains are satisfied. The proof follows directly from those of Theorems 3.2, 4.1, and 4.2 by choosing the same Lyapunov functions in the proof of Theorems 3.2, 4.1, and 4.2 except that a pair of followers who are neighbors of each other initially will always be the neighbors of each other (correspondingly, if the virtual leader is initially a neighbor of a follower, the virtual leader will always be a neighbor of this follower) because otherwise the potential function will go to infinity based on the Definition 5.1, which contradicts the fact that \( V \leq 0 \). Note that the connectivity maintenance strategy in [9] requires that the number of edges be always nondecreasing. That is, if a pair of followers are neighbors of each other (respectively, the virtual leader is a neighbor of a follower) at some time instant \( T \), then the pair of followers are always neighbors of each

\[ \text{Equivalently, a pair of followers are within the communication range of each other (respectively, the virtual leader is within the communication range of a follower).} \]
other (respectively, the virtual leader is always a neighbor of this follower) at any time \( t > T \). This requirement might not be applicable in reality, especially in large-scale systems where the size of the vehicles cannot be ignored because the group of vehicles will become very compact with the increasing number of edges. Meanwhile, the computation burden will increase significantly as well. In contrast, the connectivity maintenance mechanism with the corresponding potential function proposed in Definition 5.1 takes these practical issues into consideration. In addition, hysteresis is introduced to the connectivity maintenance strategy in [9] to avoid the singularity of the Lyapunov function. However, the hysteresis is not required in the potential function proposed in Definition 5.1.

To illustrate the connectivity maintenance mechanism as proposed in Definition 5.1, we compare two different potential functions \( V_{ij}^1 \) and \( V_{ij}^2 \) whose derivatives satisfy, respectively,

\[
\frac{\partial V_{ij}^1}{\partial r_i} = \begin{cases} 
0, & ||r_i - r_j|| > R, \\
\frac{2\pi(r_i - r_j)\sin[2\pi(||r_i - r_j|| - d_{ij})]}{20 ||r_i - r_j||}, & d_{ij} < ||r_i - r_j|| \leq R, \\
\frac{r_i - r_j ||r_i - r_j|| - d_{ij}}{20 ||r_i - r_j||}, & ||r_i - r_j|| \leq d_{ij},
\end{cases}
\]

and

\[
\frac{\partial V_{ij}^2}{\partial r_i} = \begin{cases} 
\frac{r_i - r_j ||r_i - r_j|| - d_{ij}}{20 ||r_i - r_j||}, & d_{ij} < ||r_i - r_j|| < R, \\
\frac{r_i - r_j ||r_i - r_j|| - d_{ij}}{20 ||r_i - r_j||}, & ||r_i - r_j|| \leq d_{ij},
\end{cases}
\]

where \( R = 2.5 \) and \( d_{ij} = 2 \). Fig. 1 shows the plot of the potential functions \( V_{ij}^1 \) and \( V_{ij}^2 \). It can be seen from Fig. 1(b) that \( V_{ij}^2 \) approaches infinity as the distance \( ||r_i - r_j|| \) approaches \( R \). However, \( V_{ij}^1 \) does not have the property (cf. Fig. 1(a)). In particular, \( V_{ij}^1 \) satisfies condition 3) in Definition 3.1 as shown in Fig. 1(a). In addition, both \( V_{ij}^1 \) and \( V_{ij}^2 \) satisfy conditions 1), 2), and 4) in Definition 3.1. To guarantee that the initially existing communication patterns always exist for any \( t > 0 \), we can choose the potential function as \( V_{ij}^2 \) when \( ||r_i(0) - r_j(0)|| < R \) and \( V_{ij}^1 \) otherwise.

VI. CONCLUSION

In this paper, we studied a distributed swarm tracking problem via a variable structure approach when there exists a dynamic virtual leader who is a neighbor of only a subset of a group of followers, all followers have only local interaction, and only partial measurements of the states of the virtual leader and the followers are available. For first-order kinematics, we proposed and studied a distributed swarm tracking algorithm in the absence of velocity measurements. For second-order dynamics, we first proposed and studied a distributed swarm tracking algorithm when the velocity of the virtual leader is constant. When the virtual leader has a dynamic velocity, we proposed and studied a distributed\(^3\)

\(^3\)Note that neither \( V_{ij}^1 \) nor \( V_{ij}^2 \) is unique because for positive constant \( C \), \( V_{ij}^1 + C \) and \( V_{ij}^2 + C \) are also potential functions satisfying, respectively, (12) and (13). We only plot one possible choice for them.

swarm tracking algorithm by utilizing a distributed estimator. For distributed swarm tracking in the case of both first-order kinematics and second-order dynamics, a mild connectivity requirement was proposed by adopting a connectivity maintenance mechanism in which the potential function was defined in a proper way.

REFERENCES


