Abstract—This paper investigates the stabilization problem for a linear time-invariant (LTI) time-delay system by means of a decentralized finite-dimensional LTI output feedback controller. Both commensurate and incommensurate delays are considered. It is assumed that delay can appear in the state, inputs, and outputs of the system. In this case, using the definition of $\mu$-decentralized fixed modes ($\mu$-DFM) introduced in a recent work necessary and sufficient conditions for the decentralized stabilizability of LTI time-delay systems is obtained. Some algebraic conditions are also provided to determine if a mode of a time-delay system is a $\mu$-DFM. A numerical algorithm is proposed to obtain the set of $\mu$-DFMs of the system, and the notion of $\mu$-approximate decentralized fixed modes ($\mu$-ADFM) is also presented. Finally, three numerical examples are given to illustrate various applications of the results.

I. INTRODUCTION

Interconnected systems have been studied extensively in the past few decades, and several results have been reported on this subject (for example, see [1], [2], [3], [4] and references therein). For these types of systems, since it typically is not feasible to assume that all output measurements can be transmitted to every local controller, there are constraints on information exchange imposed between the different subsystems; i.e., full output access is rarely possible. A special case of constrained control structure is the one with a diagonal (or block-diagonal) information flow, which is often referred to as a decentralized control system where each local control station only has access to measurements of its corresponding subsystems in order to generate the systems local control input [5].

A fundamental question in the analysis and design of decentralized control systems is under what conditions does a set of local feedback control laws exist to achieve stability or arbitrary pole-placement in a given region of the $s$-plane. The notion of a decentralized fixed mode (DFM) was introduced in [6] to address this question for finite-dimensional linear time-invariant (LTI) systems. In this case, it was shown that a DFM remains fixed in the complex plane, using any types of LTI decentralized dynamic controller.

On the other hand, actuators, sensors, and communication networks in feedback control systems often introduce delays in closed-loop dynamics which can deteriorate the stability and performance significantly [7], [8]. The stability analysis for LTI systems subject to commensurate and incommensurate delays has been a topic of longstanding interest (e.g., see [9], [10], [11] and the references therein), and in this case, the analysis can be carried out in either the frequency or time domain. Frequency sweeping and small gain tests have also been proposed for analyzing the stability of LTI time-delay systems in the frequency domain [12], Lyapunov-Krasovskii and Razumikhin theorems, on the other hand, have been popular tools for the stability analysis in the time domain [13]. It is worth mentioning that most of the existing results on this subject have been developed for systems with a centralized control structure. However, there has been a growing interest recently in the problem of decentralized stabilization of large-scale time-delay interconnected systems (see, e.g. [14], [15]).

In a recent work, decentralized stabilizability of LTI time-delay systems with commensurate delays is studied [16]. The notion of $\mu$-DFM is proposed, and it is shown that this notion can be utilized to provide necessary and sufficient conditions for stabilization of the system using decentralized finite-dimensional LTI output feedback controllers.

This paper deals with the stabilization of a LTI time-delay system with commensurate and incommensurate delays in the state variables, inputs and outputs, using decentralized control. It is first shown that the definition of $\mu$-DFM in [16] can be used to present necessary and sufficient conditions for stabilizability of a LTI time-delay system by means of a decentralized finite-dimensional LTI output feedback controller. A computational algorithm is then proposed to obtain the set of $\mu$-DFMs of a LTI time-delay system. Furthermore, some algebraic conditions are provided to determine if a mode of a time-delay system is a $\mu$-DFM. This is followed by presenting the notion of $\mu$-ADFM of the system respect with a decentralized control structure.

The remainder of the paper is organized as follows. In Section II, the notation which is used in the paper is given and the problem statement is introduced. The main result of the paper, i.e. stabilizability conditions for a decentralized LTI time-delay system are then presented in Section III. Three numerical examples are provided in Section IV to illustrate the results. Finally, some concluding remarks are given in Section V.

II. PROBLEM FORMULATION

A. Notation

- The set of real, and complex numbers are denoted by $\mathbb{R}$, and $\mathbb{C}$, respectively.

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Given a matrix $F \in \mathbb{C}^{r \times s}$ with its $i$-th column represented by $f_i$, $i = 1, 2, \ldots, s$, $\text{vec}(F)$ is defined as

$$\text{vec}(F) = \left[ f_1^T \quad f_2^T \quad \ldots \quad f_s^T \right]^T$$

$h$ denotes the delay, and $\lambda$ the delay operator; i.e. $\lambda f(t) = f(t - h)$, where $f$ is a function of time $t$.

$R[\lambda]$ denotes the ring of polynomials in $\lambda_1, \lambda_2, \ldots, \lambda_k$ with real coefficients, where $\lambda_j$ is the delay operator corresponding to the delay $h_j$, $j = 1, 2, \ldots, k$.

$A(\lambda) \in \mathbb{R}^{n \times n}[\lambda]$ denotes the set of $m \times n$ matrices over $R[\lambda]$.

For $A(\lambda) \in \mathbb{R}^{n \times n}[\lambda]$ with degree $\delta_j$ in $\lambda_j$, $j \in \{1, \ldots, k\}$, $A(\lambda)x(t)$ is defined as follows

$$A(\lambda)x(t) = \sum_{i=0}^{k_l} \sum_{j=1}^{\delta_j} A^{(j,i)}x(t - lh_j)$$

where $A^{(0,0)}$ and $A^{(j,i)} \in \mathbb{R}^{n \times n}$ are constant matrices with $l \in \{1, \ldots, \delta_j\}$.

### B. Preliminaries

Consider the following interconnected LTI time-delay system with $\nu$ subsystems subject to commensurate and incommensurate delays [12]

$$\dot{x}(t) = A^{(0,0)}x(t) + \sum_{j=1}^{k_1} \sum_{i=1}^{\delta_i} A^{(j,i)}(x(t - lh_j) + \sum_{l=1}^{\nu} B_l^{(0,0)}u_l(t)$$

$$+ \sum_{i=1}^{\nu} \sum_{j=1}^{k_2} \sum_{k=1}^{\delta_k} B^{(j,k,i)}u_i(t - lh_j)$$

$$y_i(t) = \sum_{j=1}^{k_3} \sum_{k=1}^{\delta_k} C^{j,k,i}x(t - lh_j), \quad i \in \mathcal{V} := \{1, 2, \ldots, \nu\}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u_l(t) \in \mathbb{R}^{n_l}$ and $y_i(t) \in \mathbb{R}^{n_i}$ are the input and output of the $i$-th local subsystem, respectively. The matrices $A^{(j,i)} \in \mathbb{R}^{n \times n}$, $B^{(j,k,i)} \in \mathbb{R}^{n \times n}$ and $C^{j,k,i} \in \mathbb{R}^{n \times n}$ are assumed to be real and constant. It is to be noted that in (1), delays can exist in the input, state and output.

Using the $\lambda$-operator, the system (1) can be rewritten as

$$\dot{x}(t) = A(\lambda)x(t) + \sum_{i=1}^{\nu} B_i(\lambda)u_i(t)$$

$$y_i(t) = C_i(\lambda)x(t), \quad i \in \mathcal{V}$$

where $A(\lambda) \in \mathbb{R}^{n \times n}[\lambda]$, $B_i(\lambda) \in \mathbb{R}^{n \times m}[\lambda]$, and $C_i(\lambda) \in \mathbb{R}^{p_i \times n}[\lambda]$. Let the matrices $B(\lambda), C(\lambda)$ and the vectors $y(t), u(t)$ be constructed as follows

$$B(\lambda) = \left[ \begin{array}{c} B_1(\lambda) \\ B_2(\lambda) \\ \vdots \\ B_{\nu}(\lambda) \end{array} \right]$$

$$C(\lambda) = \left[ \begin{array}{c} C_1(\lambda) \\ C_2(\lambda) \\ \vdots \\ C_{\nu}(\lambda) \end{array} \right]$$

$$y(t) = \left[ \begin{array}{c} y_1(t) \\ y_2(t) \\ \vdots \\ y_{\nu}(t) \end{array} \right]$$

$$u(t) = \left[ \begin{array}{c} u_1(t) \\ u_2(t) \\ \vdots \\ u_{\nu}(t) \end{array} \right]$$

then, the system (2) can be represented as

$$\dot{x}(t) = A(\lambda)x(t) + B(\lambda)u(t)$$

$$y(t) = C(\lambda)x(t)$$

In decentralized control system design, one fundamental problem of interest is to determine whether there exist $\nu$ decentralized controllers with the structure

$$\dot{z}_i(t) = \Gamma_i z_i(t) + R_i \nu(t)$$

$$u_i(t) = Q_i z_i(t) + K_i \nu(t), \quad i \in \mathcal{V}$$

where $z_i(t) \in \mathbb{R}^{n_i}$ is the state of the $i$-th local controller, and $\Gamma_i, R_i, Q_i$ and $K_i$ are the real constant matrices of appropriate size which will stabilize the system.

**Definition 1:** Consider the LTI time-delay system (2), corresponding to $A(\lambda) \in \mathbb{R}^{n \times n}[\lambda]$, the matrix $A(e^{-sh})$ is defined as

$$A(e^{-sh}) := A(\lambda)\big|_{\lambda_1 = e^{-sh_1}, \ldots, \lambda_k = e^{-sh_k}}$$

**Definition 2:** Given the system (2), the matrices $B_i(e^{-sh})$ and $C_i(e^{-sh})$, $i \in \mathcal{V}$, can be defined similarly to $A(e^{-sh})$ as follows

$$B_i(e^{-sh}) := B_i(\lambda)\big|_{\lambda_1 = e^{-sh_1}, \ldots, \lambda_k = e^{-sh_k}},$$

$$C_i(e^{-sh}) := C_i(\lambda)\big|_{\lambda_1 = e^{-sh_1}, \ldots, \lambda_k = e^{-sh_k}}.$$

Define also

$$B(e^{-sh}) = \left[ \begin{array}{c} B_1(e^{-sh}) \\ B_2(e^{-sh}) \\ \vdots \\ B_{\nu}(e^{-sh}) \end{array} \right]$$

$$C(e^{-sh}) = \left[ \begin{array}{c} C_1(\lambda) \\ C_2(\lambda) \\ \vdots \\ C_{\nu}(\lambda) \end{array} \right]$$

**Definition 3:** Consider the $\nu$ local controllers given in (4), and define the following matrices

$$\Gamma := \text{block diagonal}[\Gamma_1, \Gamma_2, \ldots, \Gamma_{\nu}],$$

$$R := \text{block diagonal}[R_1, R_2, \ldots, R_{\nu}],$$

$$Q := \text{block diagonal}[Q_1, Q_2, \ldots, Q_{\nu}],$$

$$K := \text{block diagonal}[K_1, K_2, \ldots, K_{\nu}]$$

Define also

$$K^c = \left[ \begin{array}{c} K \\ Q \end{array} \right]$$

**Definition 4:** Consider the system (2), and assume that there is no delay in the system, i.e. $h = 0$. Then an eigenvalue $\lambda \in \text{sp}(A)$ is called a DFM of the system if it is fixed with respect to any constant decentralized feedback gain matrix $K$ whose $i$-th entry on the main diagonal is an arbitrary $m_i \times p_i$ matrix. In other words, $\lambda$ is a DFM of the system if

$$\lambda \in \bigcap \text{sp}(A + BK), \quad \forall K = \text{block diag}[K_1, K_2, \ldots, K_{\nu}]$$

One can easily determine the DFM of (2) using a random number generator to generate the gain matrices of (7); e.g. see [1]. It is to be noted that a similar approach can be used to characterize the centralized fixed modes (CFM) of a system, which correspond to the unobservable OR/AND uncontrollable modes of a system [17].

**Problem Statement:** The objective is to find necessary and sufficient conditions for the stabilizability of the system (2) under the decentralized output feedback controller (4).
III. MAIN RESULTS

A. Decentralized Stabilizability for Time-Delay Systems

In this subsection, it is shown that the notion of decentralized fixed modes as defined in [16] can be used to obtain decentralized stabilizability conditions for the class of time-delay systems introduced in (1). In the sequel, some basic definitions are given, and Lemmas 1 and 2 are presented which are essential in arriving the main result of this subsection given by Theorem 1.

Definition 5: The system (3) is called spectrally controllable if [18]

\[ \text{rank} \left[ sI - A(e^{-sh}) B(e^{-sh}) \right] = n, \quad \forall s \in \mathbb{C} \]  

(8)

Analogously, the system (3) is called spectrally observable if [18]

\[ \text{rank} \left[ sI - A(e^{-sh}) C(e^{-sh}) \right] = n, \quad \forall s \in \mathbb{C} \]  

(9)

Definition 6: For \( A(\lambda) \in \mathbb{R}^{n \times n}[\lambda] \), let the set \( \Omega_{\mu}(A(\lambda)) \) be defined as

\[ \Omega_{\mu}(A(\lambda)) = \{ s | s \in \mathbb{C}, \Re \{ s \} \geq \mu, \ \ \det (sI - A(e^{-sh})) = 0 \} \]  

(10)

The above set is indeed the set of the modes of \( A(\lambda) \) in the closed right of the line \( \Re \{ s \} = \mu \). It is worth mentioning that \( \Omega_{\mu}(A(\lambda)) \) is a finite set [19].

Definition 7: Consider the system (3), and let \( K_e \) denote the set of all \( m \times p \) matrices with arbitrary real entries. For a constant \( \mu \in \mathbb{R} \), the set of \( \mu \)-centralized fixed modes (\( \mu \)-CFM) of the system (3), denoted by \( \Lambda_{\mu}(C(\lambda), A(\lambda), B(\lambda), K_e) \), is defined as follows [16]

\[ \Lambda_{\mu}(C(\lambda), A(\lambda), B(\lambda), K_e) = \{ s | s \in \mathbb{C}, \Re \{ s \} \geq \mu, \ \ \phi(s) = 0, \ \forall \ K \in \mathbb{K}_e \} \]

where

\[ \phi(s) = \det (sI - A(e^{-sh}) B(e^{-sh}) K C(e^{-sh})) \]

Definition 8: Consider the system (2), and let \( K_d \) denote the set of all block diagonal matrices given below

\[ K_d = \{ K | K = \text{block diagonal } [K_1, K_2, \ldots, K_v], \ K_i \in \mathbb{R}^{m_i \times p_i}, i \in \mathbb{V} \} \]  

(11)

For a constant \( \mu \in \mathbb{R} \), the set of \( \mu \)-decentralized fixed modes (\( \mu \)-DFM) of the system (2), denoted by

\[ \Lambda_{\mu}(C(\lambda), A(\lambda), B(\lambda), K_d) \],

is defined as follows [16]

\[ \Lambda_{\mu}(C(\lambda), A(\lambda), B(\lambda), K_d) = \{ s | s \in \mathbb{C}, \Re \{ s \} \geq \mu, \ \ \phi(s) = 0, \ \forall \ K \in \mathbb{K}_d \} \]

where

\[ \phi(s) = \det (sI - A(e^{-sh}) B(e^{-sh}) K C(e^{-sh})) \]

Lemma 1: Consider the system (3), and choose an arbitrary \( s_0 \in \Omega_{\mu}(A(\lambda)) \) and a finite \( \mu \in \mathbb{R} \). The mode \( s_0 \) is not a \( \mu \)-CFM if and only if it is both spectrally controllable and observable.

Proof: Define

\[ \rho_0(K) = \det \left( s_0 I - A(e^{-s_0 h}) - B(e^{-s_0 h}) K C(e^{-s_0 h}) \right) \]

as a \((m \times p)\)-variable polynomial in entries of \( K \). It is shown in the following that \( \rho_0(K) \) is identically zero if and only if at least one of the statements in this lemma is violated. In the sequel, suppose that for all \( K \in \mathbb{K}_e \),

\[ \rho_0(K) \equiv 0 \]  

(12)

Construct the matrices \( \hat{B}(e^{-s_0 h}) \) and \( \hat{C}(e^{-s_0 h}) \) as follows

\[ \hat{B}(e^{-s_0 h}) = \begin{bmatrix} B(e^{-s_0 h}) & 0_{n \times (p-m)} \end{bmatrix}, \ m \geq p \]

\[ \hat{C}(e^{-s_0 h}) = \begin{bmatrix} C(e^{-s_0 h}) \end{bmatrix}, \ m \geq p \]

where \( 0_{n \times (p-m)} \) and \( 0_{(m-p) \times n} \) are zero matrices of the specified dimensions. Thus, it follows from (12) that for all \( K \in \mathbb{R}^{p \times p} \),

\[ s_0 I - A(e^{-s_0 h}) - \hat{B}(e^{-s_0 h}) \hat{K} \hat{C}(e^{-s_0 h}) \]

is not full-rank, where \( \pi := \max \{ m, p \} \). From [20, Lemma 3], it is concluded that

\[ s_0 I - A(e^{-s_0 h}) - \hat{B}(e^{-s_0 h}) \hat{K} \hat{C}(e^{-s_0 h}) \]

is not full-rank for all \( \pi \times \pi \) constant real matrices \( L \). On the other hand, the above matrix can be written as

\[ \begin{bmatrix} M_1(e^{-s_0 h}) & M_2(e^{-s_0 h}) + N_2 Q_2 \end{bmatrix} \]

where

\[ M_1(e^{-s_0 h}) = \begin{bmatrix} s_0 I - A(e^{-s_0 h}) \end{bmatrix}, \ M_2(e^{-s_0 h}) = \begin{bmatrix} \hat{B}(e^{-s_0 h}) \end{bmatrix} \]

\[ N_2 = \begin{bmatrix} 0 \end{bmatrix}, \ Q_2 = L \]

It is then concluded from [21, Lemma 1] that \( s_0 \) is not both spectrally uncontrollable or unobservable. Since the above argument is reversible, the proof of necessity follows immediately.

In the following, Lemma 5 in [16] is proven for the class of time-delay systems given in (1). The novelty of the proof is due to the fact that Kalman canonical decomposition is not necessarily required to be used. It is to be noted that the Kalman canonical form for time-delay systems with incommensurate delays, to the authors’ knowledge, has not been investigated in the literature.

Lemma 2: Consider the system (2) and define

\[ A^e(\lambda) = \begin{bmatrix} A(\lambda) & 0 \\ 0 & 0 \end{bmatrix}, \ B^e(\lambda) = \begin{bmatrix} B(\lambda) & 0 \\ 0 & I \end{bmatrix}, \]

\[ C^e(\lambda) = \begin{bmatrix} C(\lambda) & 0 \\ 0 & I \end{bmatrix} \]  

(14)
Denote by $K^e_d$ the set of all $(m + p) \times (m + p)$ real constant matrices of the form (6). Then, for any given set of integers $\eta_1 \geq 0, \ldots, \eta_v \geq 0$ and any $\mu \in \mathbb{R}$

$$
\Lambda_\mu (C(\lambda), A(\lambda), B(\lambda), K^e_d) \subseteq \Lambda_\mu (C(\lambda), A^T(\lambda), B^T(\lambda), K^e_d) \quad (16)
$$

**Proof:** The proof is carried out for the special case of $\eta_1 = 1$ and $\eta_i = 0, i = 2, \ldots, v$; the general case easily follows from induction. The matrix $K^e_d$ has the same form as the matrix $K^e$ given in (6), i.e.

$$
K^e_d = \begin{bmatrix}
K_1 & 0 & q_1 \\
K_2 & 0 & \\
\vdots & \vdots & \\
r_1 & 0 & K_v
\end{bmatrix}
$$

In addition, let $K$ be defined as

$$
K = \text{block diagonal}[K_1, K_2, \ldots, K_v] \quad (17)
$$

Similarly to the non-delay case discussed in [6], it can be shown that

$$
\Lambda_\mu (C(\lambda), A(\lambda), B(\lambda), K_d) \subseteq \Lambda_\mu (C(\lambda), A(\lambda) + B(\lambda)KC(\lambda), B(\lambda), K_{c_1}) \quad (18)
$$

where $K_{c_1}$ is the set of all $m_1 \times p_1$ matrices (i.e. $K_{c_1} = \mathbb{R}^{m_1 \times p_1}$). On the other hand,

$$
\Lambda_\mu (C^T(\lambda), A^T(\lambda), B^T(\lambda), K^e_d) = \{ s | s \in \mathbb{C}, \text{ Re}\{s\} \geq \mu, \psi(s) = 0, \forall K \in K_d, q_1 \in \mathbb{R}^{m_1}, r_1 \in \mathbb{R}^{1 \times p_1}, \gamma_1 \in \mathbb{R} \}
$$

with

$$
\psi(s) = \text{det} \begin{bmatrix}
sI - A(e^{-s\theta}) - B(e^{-s\theta})KC(e^{-s\theta}) & -B_1(e^{-s\theta})q_1 \\
-r_1C_1(e^{-s\theta}) & s - \gamma_1
\end{bmatrix}
$$

where $K$ is given in (17). In the sequel, it is shown that any $s_0 \in \Lambda_\mu (C(\lambda), A(\lambda) + B(\lambda)KC(\lambda), B(\lambda), K_{c_1})$ will be a root of $\psi(s)$ for any $K \in K_d$ and for all $q_1 \in \mathbb{R}^{m_1}, r_1 \in \mathbb{R}^{1 \times p_1}, \gamma_1 \in \mathbb{R}$. From Lemma 1, it can be concluded that $s_0$ satisfies at least one of the following conditions

$$
\text{rank} \begin{bmatrix}
s_0I - A(e^{-s_0\theta}) - B(e^{-s_0\theta})KC(e^{-s_0\theta}) & B_1(e^{-s_0\theta}) \\
-r_1C_1(e^{-s_0\theta}) & s_0 - \gamma_1
\end{bmatrix} < n \quad (19a)
$$

$$
\text{rank} \begin{bmatrix}
s_0I - A^T(e^{-s_0\theta}) - B(e^{-s_0\theta})KC(e^{-s_0\theta}) \\
C_1(e^{-s_0\theta})
\end{bmatrix} < n \quad (19b)
$$

Assume that (19a) holds; then there exists a non-zero $\eta \in \mathbb{C}^{1 \times n}$ such that

$$
\eta \begin{bmatrix}
s_0I - A(e^{-s_0\theta}) - B(e^{-s_0\theta})KC(e^{-s_0\theta}) & B_1(e^{-s_0\theta}) \\
-r_1C_1(e^{-s_0\theta}) & s_0 - \gamma_1
\end{bmatrix} = 0_{1 \times n}
$$

It then follows that $[\eta, 0]$ is in the left null space of the following matrix

$$
\begin{bmatrix}
s_0I - A(e^{-s_0\theta}) - B(e^{-s_0\theta})KC(e^{-s_0\theta}) & B_1(e^{-s_0\theta})q_1 \\
-r_1C_1(e^{-s_0\theta}) & s_0 - \gamma_1
\end{bmatrix}
$$

This means that for any $K \in K_d$ and for all $q_1 \in \mathbb{R}^{m_1}, r_1 \in \mathbb{R}^{1 \times p_1}$ and $\gamma_1 \in \mathbb{R}$, $\psi(\lambda_0) = 0$. One can similarly show that this is also true if (19b) holds. Consequently, for any $K \in K_d$

$$
\Lambda_\mu (C(\lambda), A(\lambda) + B(\lambda)KC(\lambda), B(\lambda), K_{c_1}) \subseteq \Lambda_\mu (C^T(\lambda), A^T(\lambda), B^T(\lambda), K^e_d) \quad (20)
$$

Thus, using (18) and (20), one can arrive at (16). Alternatively, one can assume that (19b) holds and similarly shows that (16) is resulted.

Now, using Lemma 2 along with Lemmas 1 and 8 in [16], the following theorem can be concluded.

**Theorem 1:** A necessary and sufficient condition for the existence of an asymptotically stabilizing LTI decentralized controller for the system (2) with the local dynamic control law given by (4) is that

$$
\Lambda_0 (C(\lambda), A(\lambda), B(\lambda), K_d) = \emptyset \quad (21)
$$

where $\Lambda_0$ is given in Definition 8.

**B. Characterization of Decentralized Fixed Modes for Time-Delay Systems**

A numerical algorithm is now presented to find $\Lambda_0 (C(\lambda), A(\lambda), B(\lambda), K_d)$, i.e. the set of unstable decentralized fixed modes of the system (2). Note that this set is required for applying the condition of Theorem 1.

**Algorithm 1:**

1. Compute $\Omega_0 (A(\lambda))$ given by (10) using the MATLAB toolbox DDE-BIFTOO [22], [23] described below in Remark 1.
2. Choose a feedback gain $K_d \in K_d$ by employing a random number generator.
3. Find $\Omega_0 (A(\lambda) + B(\lambda)KC(\lambda))$ using the MATLAB toolbox DDE-BIFTOOL.
4. Obtain

$$
\Omega^e = \Omega_0 (A(\lambda)) \cap \Omega_0 (A(\lambda) + B(\lambda)K_dC(\lambda))
$$

The set $\Omega^e$ resulting from the above algorithm is equal to $\Lambda_0 (C(\lambda), A(\lambda), B(\lambda), K_d)$, for almost all $K_d \in K_d$ (for a detailed description of “almost all” see [24]). It is worth mentioning that a similar algorithm can be employed to obtain $\Lambda_0 (C(\lambda), A(\lambda), B(\lambda), K_c)$ where $\Lambda_0$ is given in Definition 8.

**Remark 1:** Using the toolbox DDE-BIFTOOL, the rightmost roots of a quasi-polynomial characteristic equation can be obtained numerically. In this case, the roots are first approximated using a linear multi-step (LMS) method, and the approximate roots are then adjusted accordingly, using a newton iteration. Convergence is guaranteed under generic conditions.

A set of algebraic conditions will next be provided (analogously to the finite-dimensional case) to characterize the DFMs of the system (2). In a manner similar to the one provided in [21], the following theorem can be obtained using Lemma 3 in [20].

**Theorem 2:** Consider the system (2). The mode $s \in \Omega_d (A(\lambda))$ is a $\mu$-DFM if and only if at least one of the following conditions holds
i.
\[
\text{rank } \begin{bmatrix} sI - A & B_1 & \cdots & B_\nu \\ C_1 & \cdots & \cdots & C_\nu \end{bmatrix} < n
\]

ii.
\[
\begin{bmatrix} sI - A & B_1 \cdots & B_\nu \\ C_1 & \cdots & \cdots \end{bmatrix} < n
\]

iii. There exists at least a partition of the set \( \tilde{\nu} \) into non-empty disjoint subsets \( \{i_1, \ldots, i_k\} \) and \( \{k_{k+1}, \ldots, l_\nu\} \) such that
\[
\begin{bmatrix} sI - A & B_1 \cdots & B_\nu \\ C_{k+1} & \cdots & \cdots \end{bmatrix} < n
\]

C. Characterization of Approximate Decentralized Fixed Modes for Time-Delay Systems

In the sequel, alternative necessary and sufficient conditions are presented for characterizing the \( \mu \)-DFMs of a time-delay system. Using this characterization, one can define a \( \mu \)-ADFM (approximate decentralized fixed mode) for (2). The following lemma is borrowed from [25], and is used in the proof of Theorem 3.

Lemma 3: Consider a singular matrix \( M_0 \in \mathbb{C}^{n \times n} \), and define \( M_i := \theta_i \omega_i^T \), where \( \theta_i \), \( \omega_i \in \mathbb{C}^n \) and \( i \in \bar{\rho} := \{1, 2, \cdots, \rho\} \). Then,
\[
\det \left( M_0 + \sum_{i=1}^\rho \mu_i M_i \right) = 0, \quad \forall \mu_i \in \mathbb{R}
\]

if and only if the following conditions are all satisfied
\[
\det \begin{bmatrix} M_0 & \theta_1 & \cdots & \theta_\rho \\ \omega_1^T & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \omega_\rho^T & 0 & \cdots & 0 \\ \end{bmatrix} = 0
\]

for any non-empty set \( \{i_1, i_2, \cdots, i_\eta\} \) which is a subset of the set \( \bar{\rho} \), where \( \eta = 1, 2, \cdots, \rho \).

Now, let the state-space equations (2) be rewritten as
\[
\begin{align*}
\dot{x}(t) &= A(\lambda)x(t) + \sum_{i=1}^{\nu^*} b_i^*(\lambda)u_i^*(t) \\
y_i^*(t) &= c_i^*(\lambda)x(t), \quad i \in \tilde{\nu}^* := \{1, 2, \cdots, \nu^*\}
\end{align*}
\]

where \( u_i^*, y_i^* \) are scalar inputs and outputs, and \( \nu^* = \sum_{i=1}^{\nu^*} m_i \); this can always be achieved using the Kronecker product procedure as described in [1]. Then, it can be verified that the closed-loop system obtained by applying the controller \( u_i = K_i y_i, \) \( i \in \tilde{\nu} \) to the system (2) is equivalent to the closed-loop system obtained by applying the controller \( u_i^* = k_i y_i^*, \) \( i \in \tilde{\nu}^* \) to (22), where the \( k_i \)'s are defined by
\[
\begin{bmatrix} k_1 & k_2 & \cdots & k_{\nu^*} \\ \text{vec}(K_1)^T & \text{vec}(K_2)^T & \cdots & \text{vec}(K_{\nu^*})^T \\ \end{bmatrix}
\]

Now, using Lemma 3 and the above discussion, the following theorem is obtained.

Theorem 3: Given the system (2), the mode \( s \in \Omega_\mu(A(\lambda)) \) is a \( \mu \)-DFM with respect to \( K_d \) if and only if all of the following conditions hold
\[
\begin{bmatrix} sI - A & b_1^* & \cdots & b_{\nu^*}^* \\ c_1^* & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ c_{\nu^*}^* & 0 & \cdots & 0 \\ \end{bmatrix} < n + \eta
\]

for any non-empty set \( \{i_1, i_2, \cdots, i_\eta\} \) which is a subset of \( \nu^* \), where \( \eta = 1, 2, \cdots, \nu^* \).

This paves the way for defining a \( \mu \)-ADFM of a time-delay system, which provides a measure of how close a mode is to being a \( \mu \)-DFM. Suppose that \( \text{cond}(s) \) denotes the condition measure of the \( l \)-th matrix obtained in the above theorem [26]. Furthermore, let
\[
\kappa = \min\{\text{cond}(s), \ l = 1, 2, \cdots, l^*\}
\]

where \( l^* = 2^{\nu^*} - 1 \). Following an argument similar to the one as presented in [26], the mode \( s \in \Omega_\mu(A(\lambda)) \) is called a \( \mu \)-ADFM of magnitude \( \kappa \) and in the particular case, when \( \kappa = \infty \), \( s \) is a \( \mu \)-DFM.

Remark 2: A mode in \( \Omega_\mu(A(\lambda)) \) which is not a \( \mu \)-DFM, can sometimes be a \( \mu \)-ADFM of “large” magnitude (e.g. \( 10^6 \)) and in such cases, the mode can be regarded as “almost” being a \( \mu \)-DFM. As a result, it may not be possible to design a controller to “shift” such a mode due to numerical problems. Hence, in such a case, the mode should be considered as basically being a \( \mu \)-DFM.

IV. NUMERICAL EXAMPLES

Example 1: Consider the following 2-input 2-output interconnected system
\[
\begin{align*}
\dot{x}(t) &= Ax(t) + A_d x(t-h) + \sum_{i=1}^{2} b_i u(t) + \sum_{i=1}^{2} b_{di} u(t-h) \\
y_1(t) &= c_1 x(t), \quad y_2(t) = c_2 x(t)
\end{align*}
\]

Let the delay \( h \) be equal to 1, and
\[
A = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & 0 \\ 5 & 3 & 1 \end{bmatrix}, \quad A_d = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & -4 & 0 \end{bmatrix}
\]

\[
\begin{align*}
b_1 &= \begin{bmatrix} 1 \\ 0 \\ -5 \end{bmatrix}, \quad b_2 = \begin{bmatrix} -1 \\ 7 \\ 6 \end{bmatrix}, \\
b_{d1} &= \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}, \quad b_{d2} = \begin{bmatrix} -6 \\ 1 \\ -4 \end{bmatrix}, \\
c_1 &= \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \quad c_2 = \begin{bmatrix} 0 \\ 2 & 0 \end{bmatrix}
\end{align*}
\]

Using the MATLAB toolbox DDE-BIFTOOL, one can obtain the unstable open-loop modes of the above system, which are \{1, 1.555\}.
Now, consider a diagonal static output feedback controller $u = Ky$ for the system. The state-space model of the closed-loop system can be written as
\[
\dot{x}(t) = (A + BKC)x(t) + (A_d + B_d KC)x(t - h) \\
y(t) = Cx(t)
\] (25)
and one can apply a diagonal random gain matrix to check the stabilizability of the system with respect to the decentralized LTI finite-dimensional output feedback controllers. For example, using the following gain matrix
\[
K_0 = \begin{bmatrix}
0.769 & 0 \\
0 & 0.232
\end{bmatrix}
\] (26)
the set of unstable modes of the closed-loop system (25) obtained are
\[
\{0.13, 0.329 \pm j0.752, 4.344\}
\]
and so, it is concluded that
\[
\Lambda_0 = \emptyset
\] (27)
This implies that the system does not have any unstable DFM, and hence it can be stabilized using a proper decentralized output feedback controller from Theorem 1. The location of the open-loop and closed-loop modes of the system is sketched in Figure 1.

**Example 2:** Consider the system given in Example 1, with the only difference being that $A_d$ is zero here. In this case, the mode $s = 1$ is an unstable ADFM for this system with magnitude $\kappa(h)$ as defined in (23). The value of $\kappa(h)$ obtained for $h \in (0, 5]$ is plotted in Figure 2(a). For $h = 0$, the system has an unstable DFM at $s = 1$, i.e., $\kappa(0) = \infty$. Figure 2(a) shows that $\kappa(h)$ decreases with $h$, implying that the presence of delay in the dynamics of the system makes the system “easier to stabilize” with respect to decentralized dynamic output feedback controllers (this interesting observation can be regarded as one of the many surprising results reported in the literature for time-delay control systems).

Assume now that $b_i = 0$, $i = 1, 2$, as well. In this case, $\kappa(h)$ increases with $h$ as depicted in Figure 2(b). This implies that the larger the delay in the input is, the more difficult it is to stabilize the system using a decentralized dynamic output feedback controller. It is to be noted that the condition measure used in this example is defined as $\text{cond}(\cdot) = 1/\sigma(\cdot)$, where $\sigma(\cdot)$ is the smallest singular value of a matrix [26]. Likewise, a similar behavior occurs for the other unstable mode, i.e., $s = \sqrt{2}$.

**Example 3:** This example is taken from [27], and presents a trivial case of a decentralized system whose model is expressed as
\[
\dot{x} = ax + u(t - h) \\
y = x
\] (28)
It is shown in [27] that for $h = 1$ and $a = 2$ “the construction of a finite dimensional stabilizing compensator appears to be a nontrivial problem”, using classical design methods. In this case, it is obtained from (23) that the $\mu$-ADF of the mode $a = 2$ is approximately $7 << 10^8$, which implies that the mode $a = 2$ is not a DFM and is not close to being a DFM. Thus, the difficulties of controller design for this system are due mainly to the limitations of the structure of classical controllers.

**V. Conclusions**

This paper deals with the stabilization of LTI time-delay systems using decentralized finite-dimensional output feedback controllers when it is assumed that the system is subject to commensurate or incommensurate delays in its state, input and output. It is first proved that the definition of $\mu$-DFM in [16] can be used to present necessary and sufficient conditions for the existence of stabilizing decentralized controllers for a larger class of time-delay systems than considered in [16]. A numerical algorithm is then proposed to obtain the set of $\mu$-DFMs of the system. This is followed by obtaining some algebraic conditions to determine whether a mode of the system is a DFM or not. The notion of $\mu$-DFM is also defined to provide a measure of how close a mode of a LTI time-delay system is to being a $\mu$-DFM.
is noted that the existing works in the literature on the stabilizability of LTI time-delay systems using decentralized control structure provide sufficient conditions only (e.g., see [28], [29], [30]).

REFERENCES


