On Consensus among Identical Linear Systems using Input-Decoupled Functional Observers

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Abstract—The consensus problem among identical linear systems under relative sensing is considered. We propose a method to design dynamic feedback laws depending on relative output measures between the individual systems, that ensure temporal coincidence of the output trajectory of all members of the group. Our method is based on functional input-decoupled observers and a static feedback. The two design steps can be performed independently with independent robustness features.

I. INTRODUCTION

The consensus problem in groups of linear systems attracts considerable attention in the literature since quite some years. The main reasons are probably the advances in networking techniques as key enabling technology for interacting groups and the huge variety of possible applications ranging from cooperative control of unmanned vehicles and formation control to the understanding of swarms and flocks of different animals. In addition, the challenges related to the consensus problem are definitely interesting from a purely theoretic point of view. See e.g. [1], [2], [3], [4], [5], [6], [7] for overviews over different aspects of the consensus problem.

Initially, mainly individual systems modeled as integrators have been considered. More recently, focus is primarily on individual systems model as integrator chains (cf. [8]) or general high order linear systems (cf. [9], [10], [11], [7]). The case of static feedback of relative states is essentially solved (cf. [12], [7] and Section III).

Therefore, current research addresses the solution of consensus problems among identical linear systems by static (e.g. [9]) or dynamic (e.g. [11], [10]) output feedback. It is in this area that our paper contributes some ideas: We consider a network of \( N \) identical linear systems in a quite general form. The interconnections between the individual systems are modeled by some, possibly directed and weighted, graph. The only information available to the individual systems are relative outputs, i.e. differences between their own and their neighbor’s outputs. In particular, individual systems do not share information about their inputs. Such information is e.g. obtained using relative sensing. Our approach is to design exact asymptotic observers for the biggest possible part of the relative states without knowledge of the input of the neighboring systems in conjunction with static feedbacks designed with methods known from the consensus by relative state feedback.

The main advantages of our approach is simplicity stemming from the fact that the observer and the static feedback can be designed independently and the robustness features of the two are, unlike in the classical observer-based designs for stabilization, independent of each other owing to the input-decoupled nature of the observer. Admittedly, our approach contains some conservatism, which is the price to pay for the aforementioned advantages.

The remainder of this paper is organized as follows: We thoroughly introduce the problem under consideration in Section II. In Section III, we review some important facts about static feedback laws ensuring consensus before we state our main results in Section IV. The results are illustrated on an example in Section V and the paper is summarized and concluded in section VI.

II. PROBLEM SETUP

A. Notation

Throughout the paper, we use the following notation: We write \( I_n \) for an \( n \times n \) identity matrix and \( 1_n \) for the all ones column vector of size \( n \). The complex conjugate of some complex number \( \lambda \) is written as \( \bar{\lambda} \), the closed right-half complex plane is abbreviated as \( \mathbb{C}_R^+ \). Given some complex numbers \( \nu_1, \ldots, \nu_r \) we denote the convex hull of these numbers in \( \mathbb{C} \) as conv(\( \nu_1, \ldots, \nu_r \)). Finally, given some matrix \( M \), we write \( M^+ \) for the Moore-Penrose generalized inverse (cf. [13]), \( M^T \) for the transpose of \( M \) and \( M^H \) for the complex conjugate transpose of \( M \).

B. Individual Systems

We consider a group of \( N \) identical linear systems modeled as

\[
\Sigma_i: \begin{cases}
\dot{x}_i = A x_i + B u_i \\
y_i = C x_i
\end{cases}
\]

with state \( x_i \in \mathbb{R}^n \), input \( u_i \in \mathbb{R}^p \), and output \( y_i \in \mathbb{R}^q \) for \( i = 1, \ldots, N \). Without loss of generality, we assume that \( \text{rank}(B) = p \) and \( \text{rank}(C) = q \). The information available to each system is the relative output defined as

\[
\delta_i = \sum_{j=1}^{N} a_{ij} (y_i - y_j),
\]

with scalar coefficients \( a_{ij} \geq 0 \).

C. Interconnections

Usually, the coefficients \( a_{ij} \) are given a graph theoretic interpretation: Consider a graph \( G \), where the \( i \)th vertex represents system \( \Sigma_i \) and a directed arc connecting the \( j \)th to the \( i \)th vertex represents an information flow weighted by \( a_{ij} \) with \( a_{ij} > 0 \) if and only if there exists an arc connecting the \( j \)th vertex to the \( i \)th vertex. In the sequel, we call \( G \) the communication graph. The adjacency matrix of
the communication graph $G$ is given as $A = [a_{ij}] \in \mathbb{R}^{N \times N}$. The Laplacian matrix is defined as $L = [l_{ij}] \in \mathbb{R}^{N \times N}$ with

$$l_{ij} = \begin{cases} -a_{ij} & \text{if } i \neq j, \\ \sum_{k=1}^{N} a_{ik} & \text{if } i = j. \end{cases}$$

Define the stacked vectors $\delta = (\delta_{1}^T, \ldots, \delta_{N}^T)^T \in \mathbb{R}^{Nq}$, $y = (y_{1}^T, \ldots, y_{N}^T)^T \in \mathbb{R}^{Nq}$, and $x = (x_{1}^T, \ldots, x_{N}^T)^T \in \mathbb{R}^{Nn}$, to obtain the simple notation $\delta = (\delta \otimes I_{0}) y = (\delta \otimes C) x$, where “$\otimes$” denotes the Kronecker product. Note that all relevant information on the communication graph $G$ is encoded in $L$. Later on, we will make use of well-known algebraic properties of $L$ (see e.g. [14], [15], [7]). In particular, 0 is an eigenvalue of $L$ which is simple if and only if $G$ is quasi strongly connected (cf. [16]). The corresponding left-eigenvector is $1_N$; the corresponding right-eigenvector $p^T$ can be chosen to have non-negative entries that sum up to 1. We denote the eigenvalues of $L$ as $\lambda_i(L)$, $i = 1, \ldots, N$ with $0 = \text{real}(\lambda_1(L)) \leq \text{real}(\lambda_2(L)) \leq \cdots \leq \text{real}(\lambda_N(L))$.

D. Control Objective

The objective addressed in this paper is to find local controllers

$$\begin{align*}
\Omega_i : \quad \dot{z}_i &= E z_i + F \delta_i \\
\delta_i &= G z_i + H \delta_i
\end{align*}$$

with controller states $z_i \in \mathbb{R}^m$, such that the group of systems $\Sigma_i$, $i = 1, \ldots, N$ reaches output consensus, i.e.

$$\lim_{t \to \infty} \|y_i(t) - y_j(t)\| = 0, \quad i, j = 1, \ldots, N,$$

independently of initial conditions $x_i(0)$ and $z_i(0)$. The trivial consensus shall be excluded, i.e. $(x_i, z_i) = (0, 0)$, $i = 1, \ldots, N$, must not be an asymptotically stable equilibrium of the closed loop system.

E. Remarks on the Problem Setup

We conclude this section by some remarks on the problem defined in sections II-B to II-D:

1) The information used by the local controllers $\Omega_i$, $i = 1, \ldots, N$ is relative in the sense that it only depends on pairwise differences between the systems $\Sigma_i$, $i = 1, \ldots, N$. As a consequence, $\delta = 0$ if $y_i = y_j$ for all $i, j = 1, \ldots, N$, i.e. if consensus is reached.

2) The information used by the local controllers $\Omega_i$, $i = 1, \ldots, N$, is local with respect to the topology defined by the communication graph $G$ because $\delta_i$ depends only on the outputs of the direct predecessors of $\Sigma_i$ in $G$ for $i = 1, \ldots, N$.

3) The above definition of consensus does not imply that all systems asymptotically reach a constant value. It does not even imply boundedness of solutions. The type of consensus reached will actually depend on the individual system dynamics as will be seen later.

III. CONSENSUS BASED ON RELATIVE STATE FEEDBACK

To make this paper self-contained, we first summarize some important results for consensus by static feedback of relative states, i.e. the case when $q = n$ and $C = I_n$. The results in this section are taken from [2], [7], [12]. In this case, the controllers $\Omega_i$ defined in (2) take the form

$$u_i = K \delta_i, \quad i = 1, \ldots, N.$$  

The objective is to find $K$ such that state consensus is achieved. Such a $K$ exists if the pair $(A, B)$ is stabilizable and $G$ is quasi strongly connected, i.e. it contains at least one spanning tree, as suggested by the following proposition:

**Proposition 1 (cf. [12]):** Assume $(A, B)$ is detectable and $G$ is quasi strongly connected. Choose a scalar $\mu$ such that $0 < \mu < 2 \text{real}(\lambda_2(L))$, a matrix $Q = M^T M \in \mathbb{R}^{n \times n}$ such that the pair $(A, M)$ is detectable and some matrix $R = R^T \in \mathbb{R}^{p \times p}$, $R > 0$. Then $A^T P + PA + Q - \mu^2 P B T R^{-1} B P = 0$ has a unique solution satisfying $P = P^T > 0$ and $K = -R^{-1} B P$ solves the consensus problem.

**Proposition 1** gives, in a constructive way, sufficient conditions for existence of some $K$ solving the consensus problem based on the LQR design. Excluding asymptotically stable systems (1), these conditions are in fact also necessary. If the pair $(A, B)$ is not stabilizable or $G$ does not contain a spanning tree, it is a simple consequence of the results presented in [2], [7], that there exists no $K$ that solves the consensus problem.

If some feedback gain $K$ solving the consensus problem has been found, the following result states how the individual systems behave at consensus:

**Proposition 2 (cf. [7]):** Assume $K$ is chosen such that consensus is reached, then

$$\lim_{t \to \infty} x_i(t) = e^{At} (I_N \otimes p^T) x,$$

where $p^T$ is uniquely defined by $p^T L = 0$ and $p^T 1_N = 1$.

All individual systems continue evolving according to the individual system dynamics. They converge to a particular solution corresponding to an initial condition given as a weighted average of initial conditions of individual systems. Interestingly, this solution is independent of $K$. As can be seen from Proposition 2, excluding trivial consensus is equivalent to the requirement that $A$ is not Hurwitz.

Next, we review an LMI based design method to find the feedback gain $K$ ensuring consensus:

**Proposition 3 (cf. [7]):** Define the matrix functions $C_0 : \mathbb{R}^{n \times n} \times \mathbb{R} \to \mathbb{R}^{2n \times 2n}$, $C_R : \mathbb{R}^{p \times n} \to \mathbb{R}^{2n \times 2n}$, and $C_I : \mathbb{R}^{p \times n} \otimes \mathbb{R}^{2n \times 2n}$ as

$$C_0(Q, \chi) := I_2 \otimes (QA^T + AQ + 2\chi Q),$$

$$C_R(\kappa) := I_2 \otimes (B\kappa + \kappa^T B^T),$$

$$C_I(\kappa) := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes (B\kappa - \kappa^T B^T),$$

and let

$$C(\lambda, Q, \kappa, \chi) := C_0(Q, \chi) + \text{real}(\lambda) C_R(\kappa) + \text{imag}(\lambda) C_I(\kappa).$$

There exists $K$ solving the consensus problem with convergence rate $\chi$ if and only if there exist matrices $Q = Q^T \in \mathbb{R}^{n \times n}$, $Q > 0$ and $\kappa \in \mathbb{R}^{p \times n}$ such that

$$C(\lambda_i(L), Q, \kappa, \chi) \prec 0, \quad i = 2, \ldots, N.$$  

The feedback gain sought-after is then given as $K = \kappa Q^{-1}$.

The idea of Proposition 3 is to construct a Lyapunov function for the whole group in some special coordinates. Despite the special structure of the Lyapunov function, feasibility is necessary for existence of $K$ to reach consensus. This can be seen using Proposition 1: in fact, solving the LQR problem of Proposition 1, $Q = P^{-1}$ and $K = KP^{-1}$ is feasible for the LMIs (4).
Note that the eigenvalues of $L$ enter the LMIs (4) affinely and $C(\lambda, Q, \kappa, \chi) \prec 0 \iff C(\nu_i, Q, \kappa, \chi) \prec 0$. Therefore, it is sufficient to check $C(\nu_i, Q, \kappa, \chi) \prec 0$, $i = 1, \ldots, r$ for some values $\nu_i \in \mathbb{C}$, $i = 1, \ldots, r$ such that $\lambda_i(L) \in \text{conv}(\nu_{i_1}, \ldots, \nu_{i_r}, \overline{\nu_{i_r}})$, $i = 2, \ldots, N$. As shown in [7], it is possible to choose $\nu_i$, $i = 1, \ldots, r$ with very limited knowledge on the actual graph $\mathcal{G}$.

With these preliminary results, we are ready to proceed with our original problem, namely consensus based on relative output feedback.

IV. CONSENSUS BASED ON RELATIVE OUTPUT FEEDBACK

As described in the previous section, the consensus problem using relative state feedback is quite well understood. There exist easy to check solvability conditions and computationally very tractable design methods. Extending the problem to consensus by relative output feedback, a straightforward idea would be to design observers to estimate the relative states based on the relative outputs and rely on the separation principle to obtain a dynamic controller which ensures consensus.

In order to apply the methods in Section III the quantity to estimate at the $i$th system is $\rho_i := \sum_{j=1}^N a_{ij}(x_i - x_j)$. Defining $\gamma_i := \sum_{j=1}^N a_{ij}(u_i - u_j)$, we have

$$\begin{align*}
\dot{\rho}_i &= A\rho_i + B\gamma_i \\
\dot{\delta}_i &= C\rho_i
\end{align*}$$

(5)

where $\delta_i$ is known and $\gamma_i$ is unknown as it involves the inputs to neighboring systems. Therefore, the standard Luenberger observer design cannot be applied directly.

One possibility to overcome this problem is to consider the unknown input as disturbance and design some observer which is robust to that disturbance (cf. e.g. [10], [11]). A major drawback of this approach is that the separation principle does not hold anymore. Therefore, the relative state feedback cannot be designed independently of the observer. To overcome this disadvantage, we follow a different direction. Namely, we aim at estimating the relative states $\rho_i$ or parts of it exactly using input decoupled observers.

A. Full order input decoupled observers

We first consider the case of full order observers for unknown inputs as described e.g. in [17]. The existence conditions for such an observer are stated in the following proposition:

**Proposition 4 (cf. [17]):** Given system (5), an observer

$$\begin{align*}
\dot{z}_i &= E z_i + F \delta_i \\
\dot{\rho}_i &= R z_i + S \delta_i
\end{align*}$$

asymptotically estimating $\rho_i$ exists if and only if

$$\text{rank}(CB) = \text{rank}(B) = p$$

(6)

and

$$\text{rank} \left( \begin{array}{cc} A - sI & B \\ -C - sI & 0 \end{array} \right) = n + p, \quad \forall s \in \mathbb{C}_0^+.$$ 

(7)

The conditions given in the above proposition have direct system theoretic interpretations. Namely, condition (6) corresponds to a relative degree one requirement while condition (7) is equivalent to asymptotically stable zero dynamics. If $p = q$, i.e. system (5) has the same number of inputs and outputs, $(A, B)$ is controllable, and $(A, C)$ is observable, the two conditions are satisfied if and only if system (5) is feedback equivalent to a strictly passive system [18], [19].

If conditions (6), (7) happen to be satisfied for the individual systems, the design method described in [17] can be applied to obtain the observer. We will not go into details on the observer design here, as the full observer cited here will turn out to be a special case of the design proposed in the sequel. The relative state feedback design described in Section III can be applied to obtain the controller (2) setting $G = KR$ and $H = KS$.

In many cases, conditions (6), (7) are too restrictive. Though, this is not the end of what can be achieved yet. Since we actually do not need all the information contained in $\rho_i$, but only the feedback law $u_i = K \rho_i$, there is no need to estimate the whole state of (5). Therefore, in the sequel, we investigate the possibility to asymptotically estimate the biggest possible part of $\rho_i$ without knowledge of $\gamma_i$ by means of functional observers. Once the best – in an appropriate sense to be defined – possible observer determined, we seek for feedback gains $K$ reaching consensus. The next section is devoted to the observer design. The next but one section deals with the design of the feedback gain $K$.

B. Maximal functional input decoupled observers

The question addressed in this section is the following: which parts of the state $\rho_i$ of system (5) can be asymptotically estimated without knowledge of the input $\gamma_i$. More formally, we are interested in the question for which $\eta_i = D \rho_i$, there exists an observer

$$\begin{align*}
\dot{z}_i &= E z_i + F \delta_i \\
\dot{\eta}_i &= R z_i + S \delta_i
\end{align*}$$

(8)

such that $\|\eta(t) - \tilde{\eta}(t)\| \leq a_0\|\eta(0) - \tilde{\eta}(0)\|e^{-\chi t}$ for some $a_0 > 0$ and $\chi > 0$, independently of initial conditions $\rho_i(0)$ and $z_i(0)$. If an observer exists for some $\eta_i = D \rho_i$, we say that $\eta_i$ is input-decoupled observable.

For our purposes, we are interested in the best such observer in the following sense: Assume $\eta_i = D \rho_i$ is input-decoupled observable. Then $\eta_i$ is maximal if any other input-decoupled observable $\tilde{\eta}_i = D \tilde{\rho}_i$ is such that $\tilde{\eta}_i = \Xi \eta_i$ for some matrix $\Xi$ or equivalently ker $D \subset \text{ker} \tilde{D}$. Figuratively speaking, this means that any information contained in $\tilde{\eta}_i$ is also contained in $\eta_i$. The maximal input-decoupled observable is uniquely defined (modulo transformations) as stated in the following lemma.

**Lemma 1:** For a system (5), there exists some input-decoupled observable $\eta_i = D \rho_i$, $D$ having full row rank, such that any input-decoupled observable $\tilde{\eta}_i = D \tilde{\rho}_i$ satisfies $\tilde{\eta}_i = \Xi \eta_i$ for some matrix $\Xi$.

**Proof:** For any input-decoupled observable $\eta_i = D \rho_i$, the corresponding complementary input-decoupled observable subspace is $S = \text{ker}(D) \subset \mathbb{R}^n$. Denote $\mathbb{S}$ the set of all complementary input-decoupled observable subspaces. Let $S_1, S_2 \in \mathbb{S}$, then $S_1 \cap S_2 \in \mathbb{S}$, i.e. $\mathbb{S}$ is closed under subspace intersection. To see this, assume $S_1 = \text{ker}(D_{i_1})$, $i = 1, 2$. Then $\eta_i = (D_{i_1}^T, D_{i_2}^T, \ldots)^T \rho_i$ is clearly input-decoupled observable and ker$(D_{i_1}^T, D_{i_2}^T, \ldots)$ is $S_1 \cap S_2$. Since $\mathbb{R}^n$ is finite-dimensional and ker $C \in \mathbb{S}$, there exists an infimal member $S_{\text{min}} \in \mathbb{S}$ and any matrix $D$ with rank$(D) =$
\( n - \dim(S_{\text{min}}) > 0 \) such that \( \ker(D) = S_{\text{min}} \) yields the maximal input-decoupled observable \( \eta_i \).

The next paragraphs explain how to determine the maximal \( \eta_i \) and construct the observer (8) asymptotically estimating this quantity. In order to do so, we first need to shortly introduce factor spaces and some related concepts and notation.

Given a map \( A : \mathbb{R}^n \rightarrow \mathbb{R}^n \) and a subspace \( \mathcal{X} \subset \mathbb{R}^n \) invariant under \( A \), i.e. \( A\mathcal{X} \subset \mathcal{X} \). We denote the map induced by \( A \) on the factor space \( \mathbb{R}^n/\mathcal{X} \), defined as \( x + \mathcal{X} \mapsto A(x) + \mathcal{X} \), as \( A/\mathcal{X} : \mathbb{R}^n/\mathcal{X} \rightarrow \mathbb{R}^n/\mathcal{X} \). Let \( P : \mathbb{R}^n \rightarrow \mathbb{R}^n/\mathcal{X} \) be the canonical projection defined as \( x \mapsto x + \mathcal{X} \), then the map \( A/\mathcal{X} \) is uniquely defined by \( (A/\mathcal{X})P = PA \). For details, the reader is referred to [20].

Furthermore, we need some geometric concepts from linear control (cf. [21], [20]). All definitions refer to the next section.

Consider the sequence \( \{S_{\text{im}}^{\mu}\} \) defined as \( S_{\text{im}}^{0} = \{0\} \), \( S_{\text{im}}^{\mu} = \text{im } B + A(S_{\text{im}}^{\mu-1} \cap \ker C), \mu \geq 0 \). (9)

This sequence converges in a finite number of steps and \( S_{\text{im}}^{\mu} = S_{\text{im}}^{\mu-1} \cap \ker C, \mu \geq 0 \). (10)

This sequence converges in a finite number of steps and \( S_{\text{im}}^{\mu} = S_{\text{im}}^{\mu-1} \cap \ker C, \mu \geq 0 \). (11)

With these results, we are ready to determine the maximal input-decoupled observable.

Lemma 2: An input-decoupled observable \( \eta_i = D\rho_i \) is maximal if and only if \( \ker D = S_{\text{im}}^{\mu} \cap \ker C \).

Proof: The proof is an immediate consequence of [21, Proposition 4].

To conclude this section, we give a design method for the observer (8).

Proposition 5: Determine the set \( S_{\text{im}}^{\mu} \) according to (9), (10), (11). Let \( P : \mathbb{R}^n \rightarrow \mathbb{R}^n/S_{\text{im}}^{\mu} \) be the canonical projection and choose \( D \) such that \( \ker D = S_{\text{im}}^{\mu} \cap \ker C \) and \( D \) has full rank. The observer (8) is given as

\[
E = P(A + LC)P^+, \quad F = -PL, \quad R = DP^+, \quad S = (I - P^+P)C^+.
\]

The observer error is \( \varepsilon_i := \hat{\eta}_i - \eta_i = R(z_i - P\rho_i) \). Defining \( \varepsilon_i := z_i - P\rho_i \), the error evolves according to

\[
\dot{\varepsilon}_i = R\dot{\varepsilon}_i = RE\varepsilon_i
\]

where \( E \) can be chosen to have eigenvalues with real part less than \( \chi \).

Proof: Observe that \( \eta_i = D\rho_i = RP\rho_i + S\delta_i \). With \( P(A + LC) = P(A + LC)P^+P \), the expression for the error dynamics follows. Assignability of the eigenvalues of \( E \) is a direct consequence of the definition of complementary detectability subspaces.

Note that restricting \( L \) to be contained in \( L(S_{\text{im}}^{\mu}) \) is a matter of solving a set of linear equations while choosing \( L \in L(S_{\text{im}}^{\mu}) \) such that the eigenvalues of \( E \) have real part less than \( \chi \) is solved by standard pole placement.

Having solved the problem of estimating the biggest possible part of \( \rho_i \), the next step is to use this estimate in order to achieve consensus. This problem is addressed in the next section.

C. Design of the consensus feedback

In the previous section, we derived methods to determine the maximal input-decoupled observable \( \eta_i = D\rho_i \) and showed how to construct an observer that asymptotically estimates \( \eta_i \). Since the observer is independent of the system input, the separation principle holds despite unknown parts of the input. In fact system (1) together with the observer (8) and \( u_i = K\hat{\eta}_i = K\eta_i + K\rho_i \) can be written as

\[
\begin{pmatrix}
\dot{x}_i \\
\dot{\varepsilon}_i
\end{pmatrix} = \begin{pmatrix} A & BKR \\ 0 & E \end{pmatrix} \begin{pmatrix} x_i \\
\varepsilon_i \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} KD\rho_i
\]

which is equivalent to system (1) with state feedback \( u_i = K\rho_i \), i.e. \( \dot{x}_i = Ax_i + BKD\rho_i \), modulo uncontrollable stable parts, namely the error dynamics \( \dot{\varepsilon}_i = E\varepsilon_i \). Therefore, the problem to solve is the static output feedback problem \( u_i = K\eta_i \) for the virtual output \( \eta_i \).

Similar problems are dealt with in [9], where the focus is on neutrally stable systems. We rely on results from [7], [22] adapting the LMIs presented in Proposition 3 to obtain an iterated LMI condition for the feedback gain \( K \).

To simplify notation, we adopt the convention that a matrix \( A \in \mathbb{C}^{n \times n} \) is Hurwitz if and only if all eigenvalues of \( A \) have negative real part. A necessary and sufficient condition for this to be the case is existence of some matrix \( P \in \mathbb{C}^{n \times n} \) such that \( PH = P, P > 0, \) and \( A^HT + PA < 0 \). Equivalently, \( A \) has eigenvalues with real parts less than \( \chi \) if and only if there exists \( P \in \mathbb{C}^{n \times n} \) such that \( PH = P, P > 0, \) and \( A^HT + PA + 2\chi P < 0 \) (cf. [13], [23]).

With these preliminaries at hand, we are ready to state the main results of this section, following the ideas presented in [22]. Define the matrices \( \Gamma_1(P, X, \chi), \Gamma_2(P, K, \lambda) \in \mathbb{C}^{n \times n} \) as

\[
\Gamma_1(P, X, \chi) := A^TP + PA + 2\chi P + XBB^TP - PBB^TX,
\]

\[
\Gamma_2(P, K, \lambda) := PB + (\lambda KD)^H.
\]

and let

\[
\Gamma(P, K, \lambda, \chi) := \begin{pmatrix}
\Gamma_1(P, X, \chi) & \Gamma_2(P, K, \lambda) \\
\Gamma_2^H(P, K, \lambda) & -I
\end{pmatrix}.
\]
Proposition 6: There exists a feedback gain $K$ such that the matrix $A + \lambda BKH$ has eigenvalues with real parts less than $\chi$ if and only if there exist matrices $P = P^H > 0$ and $X = X^H > 0$ such that $\Gamma(P, K, \lambda, \chi) < 0$.

Proof: $\Gamma(P, K, \lambda, \chi) < 0$ can be equivalently written as $\Gamma_1(P, X, \chi) + \Gamma_2(P, K, \lambda) \prec 0$ using the Schur complement. Define
\[
\Gamma(P, K, \lambda, \chi) := (A + \lambda BKH)^H P + P(A + \lambda BKH) + 2\chi P,
\]
\[
\Gamma_1(P, X, \chi) := |\lambda|^2 (K^T K) + \chi^2 P,
\]
\[
\Gamma_3(K, \lambda) := |\lambda|^2 (K^T K) + \chi^2 P,
\]
\[
\Gamma_4(P, X) := (X - P)BB^T (X - P).
\]

Note that $\Gamma_1(P, X, \chi) + \Gamma^H_2(P, K, \lambda) \Gamma_2(P, K, \lambda) < 0$ for any $X$. This proves sufficiency. To show necessity, assume there exists a solution $P = P^H > 0$ to $\Gamma(P, K, \lambda, \chi) < 0$.

Then it is possible to choose $P$ such that $\Gamma_i(P, X, \chi) < 0$ for some scalar $X > |\lambda|$. Choosing $X = P$ (i.e. $\Gamma_4(P, X) = 0$), this implies $\Gamma_1(P, X, \chi) < 0$.

The matrix $\Gamma(P, X, K, \lambda, \chi)$ possesses some convenient properties: While it is quadratic in $X$ and bilinear in $X$ and $P$, fixed $X$ is linear in the matrix variables $P$ and $K$. Furthermore, the complex number $\lambda$ enters $\Gamma(P, X, K, \lambda, \chi)$ affinely. This makes $\Gamma(P, X, K, \lambda, \chi)$ amenable to the iterative solution approach presented in the sequel. First we need to relate the matrix inequalities derived above to the consensus problem. To that end, we define the index set
\[
\mathcal{I}(L) := \{i \in \mathbb{N} : i > 2 \wedge \text{imag}(\lambda_i(L)) \geq 0\},
\]
then the controllers (2) with $G = KR$ and $H = KS$ and $E, F, R, S$ as given in Proposition 5 ensure that the group reaches consensus with
\[
\|y_i(t) - y_j(t)\| \leq a_0 \|y_i(0) - y_j(0)\| e^{\lambda t}, \quad i, j = 1, \ldots, N
\]
for some $a_0 > 0$.

Proof: We already showed that the observer error $E$ has eigenvalues with real part less than $\chi$ in Proposition 5 and explained in the beginning of this section that the separation principle holds. Therefore, it only remains to show that the static feedback $\nu_i = KD\rho_i$ has the desired properties. This has been shown to be the case if and only if the matrices $(A + \lambda_i(L)BKH)$, $i \in \mathcal{I}(L)$ have eigenvalues with real part less than $\chi$ (cf. [2], [7]).

The matrix inequalities given in the theorem represent a sufficient condition for this to be the case.

The result stated in Theorem 3 contains two sources of conservatism. Firstly, the same $P$ is assumed for all $\lambda_i(L), i \in \mathcal{I}(L)$. This is non-restrictive only for $D = I$, as shown in Proposition 1. Secondly – and more importantly – the use of input-decoupled observers is of course restrictive. This is the price to pay for the convenience to design the observer and a static feedback gain independently with non-interacting robustness features.

The reason why we use the same $P$ for all $\lambda_i(L), i \in \mathcal{I}(L)$ is of course simplicity but most notably the possibility to obtain a condition which is mostly independent of the actual communication graph.

Corollary 4: Given some complex numbers $\nu_i, i = 1, \ldots, r$ such that $\lambda_i(L) \in \text{conv}(\nu_1, \ldots, \nu_r), i \in \mathcal{I}(L)$, it is sufficient to replace (12) in Theorem 3 by $\Gamma(P, X, K, \mu_i, \chi) < 0$ for $i = 1, \ldots, r$.

Proof: The proof is a direct consequence of $\Gamma(P, X, K, \lambda, \chi) < 0$ being convex in $\lambda$.

A method to choose $\nu_i, i = 1, \ldots, r$ based on a lower bound for $\text{real}(\lambda_2(L))$ and an upper bound on $a_{ij}$ is described in [7].

In order to solve the matrix inequalities given in Theorem 3 and Corollary 4, the iterative procedure described in [22] can be applied:

1. Set $\mu = 1$ and $X_1$ the positive definite solution to $A^T X_1 + X_1 A - X_1 BB^T X_1 + P_0 = 0$.

2. Solve the optimization problem
\[
\min_{P_\mu, K_\mu, \alpha_\mu} \alpha_\mu
\]
\[
s.t. \quad P_\mu = P_\mu^H > 0
\]
\[
\Gamma(P_\mu, X_\mu, K_\mu, \nu_i, \alpha_\mu) < 0, \quad i = 1, \ldots, r
\]
for $P_\mu$, $X_\mu$, and $\alpha_\mu$. Perform a bisection on $\alpha_\mu$. Denote $\alpha_\mu^*$ the optimal value.

3. If $\alpha_\mu^* \leq \chi$, $K_\mu$ is the feedback gain we are looking for. The iteration stops.

4. Denote $P_\mu^*$ the solution to the optimization problem
\[
\min_{P_\mu, K_\mu} \text{trace} P_\mu
\]
\[
s.t. \quad P_\mu = P_\mu^H > 0
\]
\[
\Gamma(P_\mu, X_\mu, K_\mu, \nu_i, \alpha_\mu^*) < 0, \quad i = 1, \ldots, r
\]
in $P_\mu$ and $X_\mu$.

5. If $\|X_\mu - P_\mu^*\| < \delta$ – a prescribed tolerance – the iteration stops without solution, else set $X_{\mu + 1} = P_\mu^*$ and $\mu = \mu + 1$ and go to step 2.

This iterative algorithm completes the design method of the consensus feedback. Together, in sections IV-B and IV-C, we showed how to estimate the maximal possible part of the relative states $\rho_i$ and how to design a feedback based on this estimate in order to reach consensus.

Before illustrating the results on an example in the next section, we conclude this section by an implementation remark: The matrix inequalities involved in the methods described in this section are expressed in terms of complex matrices while most numerical solvers can only deal with real matrices. A real formulation of a complex matrix inequality can be easily obtained making use of the fact that given $P = P^H \in \mathbb{C}^{n \times n}$,
\[
P > 0 \iff \begin{pmatrix}
\text{real}(P) & -\text{imag}(P) \\
\text{imag}(P) & \text{real}(P)
\end{pmatrix} > 0.
\]

The latter matrix is real symmetric of dimension $2n$ since $P = P^H$ implies $\text{imag}(P)^T = -\text{imag}(P)$. 

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V. EXAMPLE
As an example we consider a group of six individual systems modeled as
\[
\dot{x}_i = \begin{pmatrix}
1 & 1 & -1 & 0 \\
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 \\
1 & 1 & 0 & 1
\end{pmatrix} x_i + \begin{pmatrix}
0 \\
1 \\
0 \\
1
\end{pmatrix} u_i,
\]
for \(i = 1, \ldots, 6\). The group is assumed to be connected in an undirected ring topology. The objective is to reach consensus with a convergence rate given by \(\chi = -2\). As a first step, we compute \(S^0_{\text{im}} B, N^0_{\text{im}} B, \) and \(S^*_{\chi, \text{im}} B\) using the algorithms described in Section IV-B. The subspace \(S^*_{\text{im}} B\) is computed as
\[
S^0_{\text{im}} = \{0\}, \quad S^1_{\text{im}} = \text{im} B, \quad S^2_{\text{im}} = \text{im} B,
\]
The set \(N^0_{\text{im}} B\) is computed as
\[
N^0_{\text{im}} = \mathbb{R}^4, \quad N^1_{\text{im}} = \text{im} B + \ker C, \quad N^2_{\text{im}} = \text{im} B.
\]
Since \(S^*_{\chi, \text{im}} B \subset S^0_{\chi, \text{im}} B \subset N^0_{\text{im}} B\), it follows that \(S^*_{\chi, \text{im}} B = \text{im} B\). A matrix \(D\) such that \(\ker D = S_{\chi, \text{im}} B \cap \ker C\) is given as
\[
D = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}.
\]
First, we determine the set of friends of \(S^*_{\chi, \text{im}} B\) as the matrices \(L\) of the form
\[
L = \begin{pmatrix}
l_{11} & -1 \\
l_{21} & l_{22} \\
l_{31} & 0 \\
l_{41} & l_{42}
\end{pmatrix}.
\]
The eigenvalues of the observer error dynamics can be freely assigned by performing a pole placement for the system \(PA P^T + (P L)(C P^T)\). Requiring a double eigenvalue at \(-4\) yields \(l_{11} = -10\) and \(l_{31} = 24\). The resulting observer is
\[
E = \begin{pmatrix}
-9 & -1 \\
25 & 1
\end{pmatrix}, \quad F = \begin{pmatrix}
10 & 1 \\
-24 & 0
\end{pmatrix},
\]
\[
R = \begin{pmatrix}
1 & 0 \\
0 & 0 \\
0 & 1
\end{pmatrix}, \quad S = \begin{pmatrix}
0 & 0 \\
0 & 1
\end{pmatrix}.
\]
To design the feedback, it is possible to choose \(r = 2, \nu_1 = 1\), and \(\nu_2 = 4\). Solving the iterated LMIs of Theorem 3 yields
\[
K = -\begin{pmatrix}
504 & 82 & 862 \\
2620 & 414 & 4410
\end{pmatrix}.
\]
Consensus is reached with a convergence rate of \(\chi \approx -2.18\).

VI. CONCLUSION
In this paper, we showed an approach to design dynamic relative-output feedback for consensus of identical linear systems. The approach is based on maximal functional input-decoupled observers and static relative-output feedback or static relative-state feedback in the case when a full order input-decoupled observer exists. The result represents a necessary condition for consentability by relative-output feedback for arbitrary interconnection topologies. It can be applied without precise knowledge of the interconnection topology and may be easily extended to changing interconnection topologies. Its main advantages are a fairly simple design and very good robustness properties due to the input-decoupled structure of the observer. An example is provided to illustrate the techniques proposed in this paper.

REFERENCES