Towards optimal actuator placement for dissipative PDE systems in the presence of uncertainty

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Abstract—We consider the issue of actuator placement for transport-reaction processes when there is significant time-varying disturbance present. Such processes are commonly mathematically modeled by perturbed linear dissipative partial differential equations (PDEs). The proposed method is based on previous work by the authors on actuator placement for PDEs, however the presence of noise and/or model uncertainty precludes their direct application. By Using modal decomposition for space discretization and employing the concept of spatial and modal norms, an optimization problem is formulated that considers the controllability of specific modes, minimizes the spillover effects to the fast modes and takes explicitly into consideration the spatial distribution of noise or model uncertainty. The proposed method is successfully applied to a representative one-dimensional parabolic PDE, where the optimal location of multiple actuators is computed.

I. INTRODUCTION

The issue of actuator choice and placement has long been recognized as an important controller design aspect. The location of the actuator is especially critical for a large number of industrially important chemical processes which exhibit both variation in space and excitation due to disturbances of the process variables. Such processes can be categorized as transport-reaction processes and examples include chemical vapor deposition reactors and plasma etching processes [1] as well as the more traditional thermal processes. Mathematical models can be derived from dynamic conservation equations and usually involve parabolic partial differential equation (PDE) systems.

The traditional approach to actuator placement is to select the locations based on open-loop considerations to ensure that the necessary controllability, reachability or power factor requirements are satisfied [2], [3], [4], [5], [6]. For further work on different aspects of actuator and sensor placement based on controllability/observability, the reader is directed to the survey papers of van de Wal and de Jager in [7] and of Kebrusly and Malebranque in [8].

The issue of integrating feedback control and optimal actuator placement has been examined for linear [9] and quasi-linear [10], [11] parabolic PDEs, and in actively controlled structures [12], [13], [14], [15], [16]. Using quadratic performance measures, parameterized by the actuator and sensor locations, the optimal position was found as the one yielding the minimum of the optimal control measures. For the linear parabolic PDEs this amounts to parameterizing the solution to the associated Algebraic Riccati Equation by the candidate actuator locations and minimizing the optimal value of the trace of the location-parameterized Riccati solution. For an in-depth exposure please see [17]. Associated with the above performance measures for actuator placement, is the issue of spatial robustness, in which one considers closed-loop H² costs, as opposed to LQR ones. In such cases, the spatial distribution of disturbances is directly embedded in the actuator optimization problem, and thus one seeks locations that yield certain levels of controllability, provide a form of performance enhancement and also render the locations spatially robust, [18].

Results have also been reported on the identification of optimal strategies for the activation of actuators (from a pool of available ones) according to predetermined criteria for distributed processes [19], [20], and the integration of these strategies with controller design [21], [22].

The present work considers the optimal placement of actuators for transport-reaction processes, mathematically modeled by linear parabolic partial differential equations in the presence of disturbance. Using model decomposition to discretize the spatial coordinate, and based on the definitions of spatial and modal controllability [23], [24], the infinite optimization problem is formulated as a nonlinear optimization problem in appropriate L² spaces. Standard optimal search algorithms are subsequently used to obtain the optimal locations. The method presented here is successfully applied to a representative diffusion process, modeled by a one-dimensional parabolic PDE, where one actuator is placed at a position to preferentially counter the effects of disturbances for the processes. Concurrently, minimum requirements on spatial controllability, modal controllability and spillover suppression are explicitly satisfied.

II. MATHEMATICAL FORMULATION

We consider the problem of computationally identifying the optimal locations of actuators for processes that can be mathematically described by parabolic partial differential equation (PDE) systems of the form:

\[ \frac{\partial}{\partial t} x(t, \xi) = \mathcal{A}(\xi) x(t, \xi) + b(\xi; \xi_0) u(t) + \mathcal{W}(\xi) \theta(t), \]

where \( x(t, \xi) \in \mathbb{R} \) is the state, \( t \in \mathbb{R}^+ \) is the time, \( \xi \in \Omega \) is the spatial coordinate and \( \Omega \) is a bounded smooth domain in \( \mathbb{R}^n \) (\( n = 1, 2 \) or 3), \( u(t) \in \mathbb{R}^K \) is the manipulated variable vector, and \( \theta(t) \in \mathbb{R}^{K_d} \) represents the disturbance vector. \( \mathcal{A} \) is
a second order (strongly) elliptic operator [25] of the form:
\[ \mathcal{A} \phi = \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial}{\partial \xi_j} (\alpha_{jk}(\xi) \frac{\partial \phi}{\partial \xi_k}) + \sum_{j=1}^{n} \alpha_{j}(\xi) \frac{\partial \phi}{\partial \xi_j} + \alpha_0(\xi) \phi, \]
for \( \xi \in \Omega \), \( \mathcal{W}(\xi) = [w_1(\xi) \ldots w_K(\xi)] \), where \( w_j(\xi) \) denotes the spatial distribution of the \( j^{th} \) disturbance in the spatial domain and \( b_j(\xi, \xi_a) = [b_{1j}(\xi, \xi_a, 1) \ldots b_{Kj}(\xi, \xi_a, K)] \), where \( b_j(\xi, \xi_a, j) \) denotes the spatial distribution of the \( j^{th} \) actuating device (e.g., boundary, distributed and/or pointwise) placed at location \( \xi_a, j \), the \( j^{th} \) component of the vector \( \xi_a = [\xi_{a,1}, \xi_{a,2}, \ldots, \xi_{a,K}] \in \Omega_a \). \( \Omega_a \subseteq \prod K \Omega \) denotes the domain of permissible actuator locations. All three cases of boundary conditions for the above PDE system may be considered: mixed (Robin), Neumann or Dirichlet [26], where the boundary, denoted by \( \partial \Omega \), can be decomposed to \( \partial \Omega = \Gamma_a \cup \Gamma_b \), with \( \Gamma_a \) denoting the part of the boundary where the actuator(s) may be placed and \( \Gamma_b (= \partial \Omega \setminus \Gamma_a) \) the remainder of the boundary where actuators are not desired or allowed to be placed. Defining an appropriate Hilbert space \( \mathcal{H} = L_2(\Omega) \) with inner product
\[ \langle \psi_1, \psi_2 \rangle_{L_2} = \int_{\Omega} \psi_1(\xi) \psi_2(\xi) \, d\xi, \quad (2) \]
and norm \( \| \psi \|_{L_2} = \sqrt{\langle \psi_1, \psi_2 \rangle_{L_2}} \), the PDE system can be equivalently written in the following abstract form [25]:
\[ x(t) = \mathcal{A} x(t) + B(\xi_a) u(t) + W \theta(t), \quad (3) \]
where \( x(t) = x(\xi, t) \in \mathcal{H} \) is the state and \( B(\xi_a) \) denotes the input operator when the actuators are in positions \( \xi_a \). Similarly, \( W \) denotes the input operator of the disturbance.

Using the eigenfunctions of the operator \( \mathcal{A} \) as a basis function set for \( \mathcal{H} \), \( x(t, \xi) \) can be equivalently expressed via the expansion \( x(t, \xi) = \sum_{i=1}^{\infty} \phi_i(\xi) x_i(t) \) where \( x_i \) denotes the \( i^{th} \) eigenmode. The PDE of (1) can be solved independently for each eigenmode by using the orthogonality properties of the eigenfunctions of the spatial operator \( \mathcal{A} \) [27]. In this case
\[ \int_{\Omega} \phi_j^*(\xi) \mathcal{A} \phi_i(\xi) \, d\xi = \lambda_i \delta_{ji}, \quad \int_{\Omega} \phi_j^*(\xi) \phi_i(\xi) \, d\xi = \delta_{ji}, \quad (4) \]
where \( \lambda_i \) denotes the \( i^{th} \) eigenvalue, and \( \delta_{ji} \) denotes the Kronecker delta. We assume that the eigenvalues are ordered such that \( \lambda_{i+1} \leq \lambda_i, \forall i = 1, \ldots, \infty \). Employing Laplace transforms [28] with \( L[x_i(t)] = X_i(s) \), the system is represented in the \( s \) domain
\[ X_i(s; \xi_a) = \frac{1}{s-\lambda_i} [B_i(\xi_a) U(s) + W_i \Theta(s)] \quad (5) \]
with \( X(s, \xi, \xi_a) = \sum_{i=1}^{\infty} \phi_i(\xi) X_i(s; \xi_a) \), and \( B_i(\xi_a) \) is defined as the \( K \)-dimensional row vector for the manipulated variables
\[ B_i(\xi_a) \triangleq \int_{\Omega} \phi_i^*(\xi) b(\xi, \xi_a) \, d\xi, \quad (6) \]
and \( W_i \) is defined as the \( K_i \)-dimensional row vector for the disturbances
\[ W_i \triangleq \int_{\Omega} \phi_i^*(\xi) W(\xi) \, d\xi. \]
We may also define the transfer function matrix of the \( i^{th} \) eigenmode as
\[ G_{ui}(s; \xi_a) \triangleq \frac{1}{s-\lambda_i} B_i(\xi_a), \quad G_{di}(s) \triangleq \frac{1}{s-\lambda_i} W_i. \quad (6) \]

The Laplace transform of the spatial distributed state can then be represented as the infinite sum
\[ X(s, \xi_a) = \sum_{i=1}^{\infty} \phi_i(\xi) X_i(s; \xi_a) = \sum_{i=1}^{\infty} \phi_i(\xi) \left[ \frac{1}{s-\lambda_i} B_i(\xi_a) U(s) + W_i \Theta(s) \right] \quad (7) \]
The resulting \( K \)-input/distributed(infinite)-output transfer functions are then given by
\[ G_{ui}(s, \xi_a) = \sum_{i=1}^{\infty} \phi_i(\xi) G_{ui}(s; \xi_a) = \sum_{i=1}^{\infty} \phi_i(\xi) G_{ui}(s; \xi_a), \quad G_{di}(s) = \sum_{i=1}^{\infty} \phi_i(\xi) \frac{1}{s-\lambda_i} W_i = \sum_{i=1}^{\infty} \phi_i(\xi) G_{di}(s). \quad (8) \]

A property of strongly elliptic operators is that their eigen-spectrum can be decomposed into a finite number of eigenvalues that are close to the imaginary axis and an infinite one that lies far in the left half complex plane, which implies that the long term dynamics of the process can be accurately captured by only a finite number of eigenmodes, while the infinite complement eigenmodes relax to their steady-state values fast. This property will be employed in the next subsection to formulate a computationally tractable optimization problem.

### III. ACTUATOR PLACEMENT

In this section, the question of optimal actuator placement for processes with significant disturbances is now posed as a constrained nonlinear optimization problem. We initially introduce the mathematical background leading to the definition of the objective function and constraint functions. We then present the optimization problem.

#### A. Spatial norms and controllability measures

In the proposed approach we seek to place the actuators at locations of high control authority with respect to the distributed process state, the individual eigenmodes of the mathematical description and the disturbance vector. To facilitate this objective we first define metrics of the actuators’ authority. The spatial \( H_2 \) norm [24] of the transfer function \( G_{ui}(s, \xi_a) \) in (8) is defined as
\[ \| G_{ui} \|_{H_2} \triangleq \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{\xi_a} \{ G_{ui}(j\omega, \xi_a) G_{ui}(j\omega, \xi_a) \} \, d\xi \, d\omega. \]
The spatial \( H_2 \) norm at location \( \xi_a \) is a measure of controller authority placed at location \( \xi_a \) over the entire spatial domain in an average sense. Using the orthogonality property of the eigenfunctions, the above norm simplifies to
\[ \| G_{ui} \|_{H_2} = \sum_{i=1}^{\infty} \left\| G_{ui}(s; \xi_a) \right\|_{L_2}^2 = \sum_{i=1}^{\infty} \int_{\Omega} \| G_{ui}(\xi_a; \xi) G_{ui}(\xi_a; \xi) \| \, d\xi. \quad (9) \]
With the aid of this spatial \( H_2 \) norm, a number of measures of actuator placement effectiveness can be proposed. Note that the spatial \( H_2 \) norm has now become the sum of the modal norms. We define the \( i^{th} \) modal norm
\[ f_i(\xi_a) \triangleq \| G_{ui}(s; \xi_a) \|_2 = \left\| \frac{1}{s-\lambda_i} B_i(\xi_a) \right\|_2. \quad (10) \]
}\]
\[ f_{i,j}(\xi_a) \triangleq \left\| \frac{1}{(s - \lambda_i)} \int_{\Omega} \phi_i(\xi) b_j(\xi;\xi_a) d\xi \right\|_2, \quad (11) \]

which describes the controller authority of the \( j \)th actuator over the \( i \)th mode. If the modal norm at a specific actuator location of a given mode is zero, it means that the specific controller has no authority over that mode. Note that from the definition of \( f_{i,j}(\xi_a) \) and \( f_i(\xi_a) \) we have

\[ f_i^T(\xi_a) = \sum_{j=1}^{K} f_{i,j}(\xi_a). \]

The component spatial controllability may be defined similarly as the \( j \)th actuator authority over the entire spatial domain in an average sense.

Capitalizing on the previously mentioned property of strong elliptic operators that the higher modes become progressively more stable \((\lambda_{i+1} \leq \lambda_i \text{ as } i \to \infty)\), we only need consider a finite number of modes to compute an accurate approximation of the spatial \( \mathcal{H}_2 \) norm:

\[ \|G_u(s;\xi_a)\|_{\mathcal{H}_2}^2 \equiv \sum_{i=1}^{N} \|G_u(s;\xi_a)\|_{\mathcal{H}_2}^2 = \sum_{i=1}^{N} \left\| \frac{1}{(s - \lambda_i)} \int_{\Omega} \phi_i(\xi) b_j(\xi;\xi_a) d\xi \right\|_2, \quad (12) \]

where \( \mathbf{H}^2(\xi_a) \equiv \sum_{i=1}^{N} \|G_u(s;\xi_a)\|_{\mathcal{H}_2}^2 = \sum_{i=1}^{N} f_i^T(\xi_a) \) denotes the truncation of the \( \mathcal{H}_2 \) spatial norm to the first \( N \) modes and \( \mathbf{S}^2(\xi_a) \equiv \sum_{i=N+1}^{\infty} \|G_u(s;\xi_a)\|_{\mathcal{H}_2}^2 = \sum_{i=N+1}^{\infty} f_i^T(\xi_a) \) denotes the “spillover” of the actuator authority to the higher modes.

Note that \( \mathbf{H}^2(\xi_a) \) at location \( \xi_a \) is a measure of controller authority placed at location \( \xi_a \) over the entire spatial domain in an average sense (averaged over the first \( N \) modes).

Following [23], we may now define the \( i \)th modal controllability at actuator locations \( \xi_a \) as follows.

**Definition 1:** [\( i \)th modal controllability] The \( i \)th modal controllability at actuator locations \( \xi_a = \{\xi_a,1,\xi_a,2,\ldots,\xi_a,K\} \) is defined as

\[ \mathcal{M}_i(\xi_a) = \frac{f_i(\xi_a)}{\max_{\xi_a \in \Omega_a} f_i(\xi)} \times 100\%, \quad i = 1, 2, \ldots, N, \quad (13) \]

and describes the total controller authority of all actuators at locations \( \xi_a \) over the \( i \)th mode.

If the modal controllability at some locations \( \xi_a \) of a given mode is zero, it means that none of the controllers has any authority over that mode. This also coincides with the notion of approximate controllability for the class of PDEs with Riesz-spectral operators [25]. The requirement for that in this case is \( \langle b, \phi \rangle_{L^2} \neq 0, \forall i \). When the \( i \)th mode has a zero modal controllability at location \( \xi_a \), it means that \( f_i(\xi_a) \equiv 0 \) and hence \( B_i(\xi_a) = 0 \), or that

\[ \int_{\Omega} \phi_i(\xi) b_i(\xi;\xi_a) d\xi = 0. \]

Even though the above definitions allow us to formulate the optimal actuator placement problem, in order to gain better control over the placement of each individual actuator, we also define the \( i \)th modal controllability of the \( j \)th actuator at location \( \xi_a,j \). The \( i \)th modal controllability of the \( j \)th actuator at location \( \xi_a,j \) is defined as

\[ \mathcal{M}_{i,j}(\xi_a) = \frac{f_{i,j}(\xi_a)}{\max_{\xi_a \in \Omega_a} f_{i,j}(\xi)} \times 100\%, \quad i = 1, \ldots, N. \quad (14) \]

The \( i \)th modal controllability of the \( j \)th actuator at location \( \xi_a,j \) describes the controller authority of the \( j \)th actuator over the \( i \)th mode. If the modal controllability at a specific actuator location of a given mode is zero, it means that the specific controller has no authority over that mode.

**B. Spillover effects**

While the notions of spatial \( \mathcal{H}_2 \) norm and modal controllability allow one to choose actuator locations that cater to specific (primarily dominant low) modes, care must be exercised in order to avoid choosing locations that might excite the higher modes of this system representation. This issue is more pronounced when considering medium range modes, which are the prominent modes that lead to performance deterioration; medium range modes are modes that are at immediate proximity of the first \( N \) modes. The effect of the actuators on the higher modes is mathematically described by the term \( \mathbf{S}(\xi_a) \) in (12), which, in general, cannot be computed since it is an infinite sum of modal controllabilities. Capitalizing again on the property of strongly elliptic operators, it can be similarly assumed that the effect on these medium range modes \( i = N + 1, \ldots, M \) need only be considered when computing \( \mathbf{S}(\xi_a) \), i.e. \( \mathbf{S}^2(\xi_a) \cong \sum_{i=N+1}^{\infty} f_i^T(\xi_a) \). Thus, \( \mathbf{S}^2(\xi_a) \) at location \( \xi_a \) is a measure of the destabilizing effect of the controller placed at location \( \xi_a \) over the entire spatial domain in an average sense.

**C. Multiple location issues**

A problem commonly encountered during the placement of multiple actuators for a distributed process is the issue of clusterization, i.e. placing actuators at or near the same location. An excellent exposure of the optimization and computational issues associated with multiple actuators, including that of actuator clusterization, can be found in [29].

We maintain, one must also avoid placing actuators at locations that provide certain spatial symmetry (i.e., they affect the system modes in a similar fashion) while placed at different locations. This issue was exposed and addressed in detail in [30]. For completeness we present the mathematical foundation; the interested reader may refer to [30] for further details. We define the spatial controllability matrix \( \mathcal{M} \in \mathbb{R}^{N \times K} \) as:

\[ \mathcal{M}(\xi_a) = \begin{bmatrix} \mathcal{M}_{1,1}(\xi_a) & \ldots & \mathcal{M}_{1,K}(\xi_a) \\ \vdots & \ddots & \vdots \\ \mathcal{M}_{N,1}(\xi_a) & \ldots & \mathcal{M}_{N,K}(\xi_a) \end{bmatrix}. \quad (15) \]

We assume that the number of modes that are of interest are higher than the number of available actuators \( N \geq K \). The proximity, in an \( L_2 \) sense, of the actuator locations can then be identified through the singular values of \( \mathcal{M} \). For example, if two actuators are placed at the same position, two columns of \( \mathcal{M} \) are linearly dependent, which in turn implies that at least one singular value becomes zero. The proximity of the columns of \( \mathcal{M} \) to linear dependence can be
identified through the ratio of the largest singular value to the smallest one, which represents the condition number of $\mathcal{M}$, [31]. This implies that there is a direct relationship between the value of the condition number and the redundancy in the actuator network. An in depth presentation of the physical interpretation of the specific measure can be found in [30].

D. Effect of disturbance

In the proposed approach we assume that the spatial distribution of the disturbance is known. Based on this we can similarly define metrics of the disturbance effects. Specifically, we define the $i$th disturbance modal norm, a measure of the effect of all the disturbances over the specific $i$th eigenmode of the system,

$$d_i \triangleq \|G_{d,i}(s)\|_2 = \left\| \frac{1}{(s-\lambda_i)} W_i \right\|_2.$$

(16)

Capitalizing again on the property of strong elliptic operators that the higher modes become progressively more stable ($\lambda_{i+1} \leq \lambda_i$ and $\lambda_i \to \infty$ as $i \to \infty$), and assuming that the frequency of the disturbance term $\theta(t)$ is bounded, we only need consider a finite number of modes to compute an accurate approximation of the effect of disturbances.

Remark 1: Cases may arise that the disturbance distribution function $W$ and the frequency bound on $\theta(t)$ is such that it excites specific higher modes to the system. In such a case, the dimensionality of the finite dimension approximation of the system is chosen such that it includes the modes in question.

E. Optimization problem formulation

The objective of this work is to identify $K$ actuator locations $\xi_j \in \Omega_\alpha$, $j = 1, \ldots, K$ that

1) maximize the spatial $\mathcal{H}_2^f$ norm and
2) minimize the spillover to the higher modes
3) while they maintain high authority over the specific modes that are most affected by the disturbance,
4) and maintain a reasonable level of controller authority over each slow mode.

The constrained optimization problem for actuator group is then formulated as:

$$\xi_{a, opt} = \arg \max_{\xi_a \in \Omega_\alpha} \left\{ \mathbf{H}^2(\xi_a) - \omega_s S^2(\xi_a) - \omega_d D \right\}$$

s.t.

$$\mathbf{D} = \sum_{i=1}^{N} R(d_i^2 - f_i(\xi_a))^2, \quad (\text{Pc-I})$$

$$\kappa(\mathcal{M}(\xi_a)) \leq \beta_c,$$

$$\mathcal{M}_i(\xi_a) \geq \beta_m, \quad \forall i = 1, 2, \ldots, N.$$  

The above formulation maximizes a weighted spatial $\mathcal{H}_2^f$ norm, where the $i$th modal norm $f_i$ is weighted by the $i$th modal disturbance norm. The choice of parameter gamma in the above formulation signifies our considerations of the control authority over the spatial domain in an average sense. For $\gamma = 0$, formulation focuses on the expunction of the disturbance effects on the process, while for $\gamma = 1$, the spatial $\mathcal{H}_2^f$ norm and in extension the control authority over the spatial domain is also taken into consideration.

The formulated optimization problems can be solved using standard search algorithms such as Newton-based, interior-point or direct-search methods [32]. Due to the nonlinear nature of the objective function and the inequality constraints, global optimization methods are preferable, including αBB [33], particle swarm optimization [34] and simulated annealing. Another issue is the “rugged” topology of the objective function; algorithms that efficiently handle such objective functions have been developed, including funneling [35], derivative free and direct search [32] optimization algorithms. In the event the set $\Omega_\alpha$ is disjoint, branch and bound algorithms [32] and genetic algorithms [12] can also be used to obtain the optimal actuator locations. In the
current work a 2-level optimization scheme was employed, comprising of an inner sequential quadratic programming algorithm (guaranteeing local optimality), and a simulated annealing outer structure, to identify the optimal location.

Remark 2: During the solution of the optimization problem, assigning values to constraint parameters $\beta$ and $\beta$, may become an issue. This issue can be relieved by initially solving an optimization problem for the process in the absence of disturbances (i.e., $W = 0$) to obtain the best case scenario values of $H^t(\xi_a)$, $S^t(\xi_a)$. The constraints can then be evaluated based on the optimal non-disturbance values.

Remark 3: An issue which often arises during the search of optimal actuator locations using modal methods is the weight in the objective function that should be assigned to each mode. In the current formulation the weight that is assigned to each mode is dependent upon the eigenvalue of the specific eigenmode. As a result, modes that are less stable have a greater contribution to $H^t(\xi_a)$ and as a consequence the actuator placement search will gravitate towards locations that assign greater force on the specific modes.

Remark 4: Even though the problem formulation necessitates the computationally intensive step of calculating the singular values of $M(\xi_a)$ at each iteration, for a typical problem under consideration, the computational requirements of the search and the numerical algebra algorithms are well within the capabilities of current processors.

Remark 5: In the above optimization formulations, the use of component modal controllability constraints $\max_{j=1,...,k} M_{i,j}(\xi_a) \geq \beta_m$, $\forall i = 1,2,...,N$ over total modal controllability constraints $M_i(\xi_a) \geq \beta_m$, $\forall i = 1,2,...,N$ may be preferred. Such explicit constraints will account for possible solutions where not one actuator has authority over the specific mode (i.e., a mode with low component modal controllability values) in which case the revised optimization problem is infeasible. Furthermore, for a large number of actuators, the computation of $M_i$ can become much more challenging than the computation of $M_{i,j}$ due to the necessity to solve an optimization problem to find $\max f_i$.

IV. NUMERICAL EXPERIMENT

To present the proposed control actuator placement we consider a representative spatially distributed process, mathematically described by the following linear one-dimensional PDE

$$\frac{\partial}{\partial t} x(t, \xi) = 0.01 \frac{\partial^2}{\partial \xi^2} x(t, \xi) + b(\xi; \xi_a) u(t) + w(\xi) \theta(t),$$

subject to Neumann boundary conditions:

$$\frac{\partial x}{\partial \xi} |_{\xi = 0} = 0, \quad \frac{\partial x}{\partial \xi} |_{\xi = L} = 0,$$

where the spatial domain is defined as $\Omega = [0, L]$, with $L = 2$. We assume that we have one actuating devise available. The spatial distribution of this actuating device placed at location $\xi_a$ is a gauss distribution $b(\xi; \xi_a) = 4 \exp\left(-2(\xi - \xi_a)^2\right)$. The permissible actuator domain is defined as $\Omega_a = [0 + \varepsilon/2, L - \varepsilon/2]$ with the span of the actuating device $\varepsilon = L/50$. The disturbance to the process is located at the boundary with an exponential decay distribution term $W = 3 \exp(\xi)$ (depicted in Figure IVc).

For the specific system, the associated eigenvalues and eigenfunctions (normalized) can be computed analytically and are of the form

$$\lambda_i = -\left(\frac{\pi}{L}\right)^2 \text{ and } \phi_i(\xi) = \sqrt{\frac{2}{L}} \cos\left(\frac{i\pi\xi}{L}\right), \quad i = 1,\ldots,\infty.$$

To compute optimal actuator locations, we assume that the first five modes are important for the calculation of the objective function $S^N(\xi)$, (i.e. $N = 5$). To reduce the effects of spillover the next ten higher modes are considered and thus $M = 15$. The associated constrained optimization problem of Pc-II then takes the specific form

$$\xi_{opt} = \arg \max_{\xi \in \Omega_a} \left\{ \sum_{i=1}^{5} d_i f_i(\xi_a) \right\} \quad \text{s.t.} \quad H(\xi_a) \geq \beta, \quad S(\xi_a) \leq \beta, \quad M(\xi_a) \geq 0.6, \quad \forall i = 1,2,\ldots,5.$$

Figures 1(a) and 1(b) depict the modal controllabilities of the first five modes. We observe that for every value of $\xi$ in the set of admissible locations there is at least one modal controllability above the limit $\beta = 60\%$. Considering thus the inequality constraints of controllability level of $60\%$ in the optimization program of Pc-III, the set of admissible locations remains unchanged (the associated inequality constraints will always be inactive during the search); this is not the case in general (e.g., for $\beta \geq 75\%$ they become active during the search). If we do not consider spillover effects (i.e., $\gamma = 100\%$), the solution to the optimization problem of Pc-III for $c = 10$ is computed to be the location at the edge of the permissible domain $\xi_{opt} = 1.98$. This value maximizes the objective function as can be deduced from Figures 1(a) and 1(b). The issue with the chosen location though is that it also maximizes the spill-over effects.

Solving the optimization problem of Pc-III, for the nominal optimization parameters, the optimal actuator location was computed to be $\xi_{opt} = 1.73$. This is shown in Figure 1(c) which depicts the spatial distribution of the actuator $b(\xi; \xi_a)$ as a function of the spatial distance $\xi$.

REFERENCES


