Price of Anarchy and Price of Information in $N$-Person Linear-Quadratic Differential Games

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Abstract—The price of anarchy (PoA) has been widely used in static games to quantify the loss of efficiency due to noncooperation. Here, we extend this concept to a general differential games framework. In addition, we introduce the price of information (PoI) to compare game performances under different information structures. We further characterize these two relative measures of performance for a class of scalar linear quadratic differential games. We obtain bounds on the PoI and the PoA for feedback differential games. We also find their approximations in a large population regime.

I. INTRODUCTION

Recently, game-theoretical methods have found many applications involving resource allocation in communication networks [1]–[3], [6]. The solution concept that has been widely used to characterize the outcome of a game is Nash equilibrium, which is based on the assumption that the players in the game are rational and individually self-optimizing. As a result of the selfish competitive behavior, the Nash equilibrium generally leads to situations which is not socially optimal, that is, not efficient [4]. Notwithstanding this inefficiency, it has been of long-term interest to researchers to know whether Nash equilibrium can be a reasonable approximation to the optimum when global optimum is not a reasonable expectation because of lack of sufficient information and incentive to cooperate.

Introduced in [5], the price of anarchy (PoA) is proposed as a utility ratio between the worst possible Nash solution and the social optimum, and it serves to quantify the loss of efficiency due to competition. It has been shown that in routing games and resource allocation games [5], [6] PoA is bounded by a constant, allowing agents to achieve some level of efficiency despite being suboptimal.

The novel idea of quantifying the gap between social optimality and game equilibrium solutions has sparked many follow-up work with a similar purpose. In [7], price of simplicity has been introduced for a pricing game in communication networks as the ratio between the revenue collected from a flat pricing rule and the maximum possible revenue. In [8], price of uncertainty has been introduced to measure the relative payoff of an expert user of a security game under complete information to the one under incomplete information. In [9], price of leadership has been proposed as a measure of comparison of utilities in a power control game between Nash equilibria and Stackelberg solutions.

In this paper, we extend the notion of PoA to differential games (DGs). Heretofore work on PoA has been primarily limited to static continuous kernel games. No similar study exists for dynamic games, in particular DGs where agents (players) aim to minimize their long-term costs subject to a differential equation constraint representing the evolution of the state under the controls of the players. We first define PoA for general DGs, and then characterize it for a class of scalar linear-quadratic (LQ) DGs. We quantify the efficiency loss in the long run when the players behave noncooperatively, under the Nash equilibrium concept.

One of the main differences between static and dynamic (differential) games is the existence and richness of information structures (ISs) in the latter. Different ISs yield different equilibrium solutions, and hence IS is a crucial factor in the investigation for PoA. In this regard, we also propose another index, the price of information (PoI), which is a result of the comparison of the equilibrium utilities or costs under different ISs. We show that for the class of scalar LQ DGs with closed-loop feedback IS, the PoA has some appealing upper bounds, which can further be approximated when the number of players is sufficiently large; whereas, under the open-loop IS, we can obtain an expression for the PoA in closed form. In addition, the PoI between the feedback and open-loop ISs is shown to be bounded from below by $\sqrt{2}/2$ and from above by $\sqrt{2}$ in the large population regime.

The structure of the paper is as follows. In Section II, we define the indices, PoA and PoI, in a general DG framework. In Section III, we investigate the PoA for a class of scalar LQ feedback DGs. In Section IV, we study the LQ DGs under open-loop IS, and in Section V we establish bounds on the PoI. We conclude and identify future work in Section VI.

II. GENERAL PROBLEM FORMULATION

In this section, we introduce the notion of a general PoA for DGs, which is dependent naturally on the IS, game horizon and the number of players in the game. Let $\mathcal{N} = \{1, 2, \ldots, N\}$ be the set of players. At each time instant $t \in [0, T)$, each player, say Player $i$, chooses an $m_i$-dimensional control value (action) $u_i(t)$ from his set of feasible controls $U_i \subset \mathbb{R}^{m_i}$, where we also make the standard assumption that as a function of $t$ the control function $u_i(\cdot)$ is piecewise continuous on $[0, T)$. The $n$-dimensional vector state $x(t) \in \mathbb{R}^n$ is the state of the game, which we assume to be piecewise continuously differentiable on $[0, T)$.

The “$\cdot$” notation is introduced to capture two cases: finite horizon when $T$ is finite and infinite horizon when $T$ is infinite.
state \( x(\cdot) \) evolves according to the differential equation
\[
\dot{x}(t) = f(x(t), u_1(t), \ldots, u_N(t), t), \quad x(0) = x_0,
\]
where \( x_0 \in \mathbb{R} \) is the initial value of the state and the system dynamics \( f(\cdot) : \Omega \to \mathbb{R}^n \) is defined on the set
\[
\Omega = \{(x_1, u_1, \ldots, x_N, t) | x \in \mathbb{R}^n, t \in [0, T), u_i \in U_i, i \in \mathcal{N}\},
\]
as a jointly piecewise continuous function which is also Lipschitz in \( x \). Each player \( i \in \mathcal{N} \) seeks to minimize his objective functional
\[
J_i(u) = \int_0^T F_i(x(t), u_1(t), \ldots, u_N(t), t) dt + S_i(x(T))
\]
when \( T < \infty \), and
\[
J_i(u) = \int_0^\infty F_i(x(t), u_1(t), \ldots, u_N(t), t) dt
\]
when \( T = \infty \), where \( u = \{u_1, \ldots, u_N\} \), which he cannot of course do independently of other players’ controls, which necessitates the introduction of an equilibrium solution concept, to be done shortly. In the expression above, for each \( i \in \mathcal{N} \), the function \( F_i : \Omega \to \mathbb{R} \) is Player \( i \)’s instantaneous cost function, and in the first expression \( S_i : \mathbb{R}^n \to \mathbb{R} \) is the terminal value function.

What we have formulated above is a differential game with open-loop information structure (OL IS), because the controls depend on time only (and the initial value of the state, which we have suppressed in the formulation). To accommodate other ISs, we have to introduce policy variables for the players, say \( \gamma_i \in \Gamma_i \), for Player \( i \), which is a mapping from the set of information available to the player to his control (action) set. We will concentrate in this paper on two specific ISs: Open loop (OL), as discussed above, where \( u_i(t) = \gamma_i(t); x_0 \), and Feedback (FB), where \( u_i(t) = \gamma_i(t); x(t) \). Note that in the latter case, we have to require each \( \gamma_i(t) \) to be Lipschitz in \( x \), in addition to being jointly piecewise continuous in its arguments, and further that \( f \) be Lipschitz not only in \( x \) but also in \( \{u_1, \ldots, u_N\} \), so that the differential equation generating the state,
\[
\dot{x}(t) = f(x(t), \gamma_1(t); x(t)), \ldots, \gamma_N(t); x(t), t), \quad x(0) = x_0,
\]
admits a unique piecewise continuously differentiable solution for each \( \gamma_i \in \Gamma_i, i \in \mathcal{N} \). To capture the possibility of other ISs, we will write \( \gamma_i \) for \( \gamma_i \in \Gamma_i \) as \( \gamma_i^\eta \), where \( \eta \) stands for the underlying IS (which could also be different for different players). Furthermore, we write the objective functionals in terms of the policy variables (by simply replacing \( u_i \)’s with \( \gamma_i \)’s as we have done in the state equation), \( J_i(\gamma^\eta) \), where \( \gamma^\eta := \{\gamma_1^\eta, \ldots, \gamma_N^\eta\} \); we will occasionally drop the superscript \( \eta \) when the IS is clear from context.

Let \( \gamma_i^\eta \) denote the collection of policies of all players except Player \( i \), i.e., \( \gamma_i^\eta = \{\gamma_1^\eta, \ldots, \gamma_{i-1}^\eta, \gamma_{i+1}^\eta, \ldots, \gamma_N^\eta\} \), in a game with IS \( \eta \). If \( \gamma_i^\eta \) is fixed as \( \gamma_i^\eta = \gamma_i^\eta \), Player \( i \) is faced with the dynamic optimization (optimal control) problem: 2
\[
(OC(\iota)) \min_{\gamma_i \in \Gamma_i^\eta} J_i(\gamma_i, \gamma_i^\eta_i)
\]
\[
\begin{equation}
\begin{aligned}
& := \int_0^T F_i(x, \gamma_i(\eta), \gamma_i^\eta(\eta), t) dt + S_i(x(T)) \\
& \text{s.t.} \dot{x}(t) = f(x, \gamma_i(\eta), \gamma_i^\eta(\eta), t), \quad x(0) = x_0.
\end{aligned}
\end{equation}
\]
In the case of infinite horizon, the problem remains the same with \( S_i(T) = 0 \) and \( T = \infty \). If we denote the solution to \( OC(\iota) \) by \( \gamma_i^{\eta^*} \), and carry out the optimization for each \( i \), then what we have is a Nash equilibrium compatible with the IS that defines the DG. This is made precise below.

**Definition 2.1: [\eta-Nash equilibrium]** For a DG with IS \( \eta \), the policy \( N \)-tuple \( \{\gamma_i^{\eta^*}, i \in \mathcal{N}\} := \gamma^{\eta^*} \) is an \( \eta \)-Nash equilibrium if, for each \( i \in \mathcal{N} \), \( \gamma_i^{\eta^*} \) solves the optimal control problem \( (OC(\iota)) \). Let \( \Gamma^\eta \) be the set of all \( \eta \)-Nash equilibria, as a subset of \( \Gamma^\eta \).

Let \( J_i^{\eta^*}, i \in \mathcal{N} \), denote the achieved values of the objective functions of the players under a particular \( \eta \)-Nash equilibrium \( \gamma^{\eta^*} \), and a corresponding total cost achieved (as a convex combination of the individual costs) be given by
\[
J^{\eta^*} = \sum_{i \in \mathcal{N}} \mu_i J_i^{\eta^*},
\]
where \( \mu_i \) is a positive weighting factor on Player \( i \)’s cost, satisfying the normalization condition \( \sum_{i \in \mathcal{N}} \mu_i = 1 \). Assume, without any loss of generality, that \( J_i^{\eta^*} > 0 \) for all \( i \in \mathcal{N} \), and hence a fortiori \( J^{\eta^*} > 0 \).

The game framework is reasonable when players optimize their payoffs strategically and independently without any centralized information. On the other hand, with central coordination, the players can form a team and achieve an optimal social objective under a centralized control. The corresponding underlying optimization problem is the optimal control problem: 3
\[
(COC) \min_{\gamma \in \mathcal{F}} \sum_{i=1}^N \mu_i \left\{ \int_0^T F_i(x(t), \gamma(\eta), t) dt + S_i(x(T)) \right\}.
\]
\[
\text{s.t.} \dot{x}(t) = f(x, \gamma(\eta), t), \quad x(0) = x_0,
\]
where the optimization could also be carried out with respect to controls, \( u \), that is, in an open-loop fashion, since the problem is deterministic. Hence, the optimal value of this optimal control problem is independent of the IS, which we denote by \( J^{\eta^*}_\mu \) and the corresponding (open-loop) optimal control by \( u^{\eta^*} = \{u_1^{\eta^*}, \ldots, u_N^{\eta^*}\} \). Note that we necessarily have \( 0 < J^{\eta^*}_\mu \leq J^{\eta^*} \), where \( J^{\eta^*}_\mu \) is under any NE out of \( \Gamma^\eta \).

**Definition 2.2 (Price of Anarchy):** Consider an \( N \)-person DG as above and its associated optimal control problem (COC) with \( J^{\eta^*}_\mu > 0 \). The price of anarchy for the DG is 4
\[
P^{\eta^*}_{N, \mu; T} = \max_{\gamma \in \Gamma^\eta} J^{\eta^*}_\mu / J^{\eta^*}_\mu
\]
the worst-case ratio of the total game cost to the optimum social cost.

2We use “OC(\iota)” to denote Player \( i \)’s individual optimal control problem.
3The acronym “COC” stands for “Centralized Optimal Control”.
4If the maximum below does not exist, then it is replaced by supremum in the definition of PoA.
In addition to its dependence on the cost functions, PoA depends on the number of players in the game, the IS, the weights on individual players and the time horizon. Note that the PoA as defined in (2) is lower-bounded by 1.

**Definition 2.3 (Price of Information (PoI))**: Let \( \eta_1 \) and \( \eta_2 \) be two ISs. Consider two \( N \)-person DGs which differ only in terms of their ISs, with game 1 having IS \( \eta_1 \), and game 2 having \( \eta_2 \). Let the values of a particular \( \mu \) convex combination of the objective functions be \( J_{\mu}^{\eta_1} \) and \( J_{\mu}^{\eta_2} \), respectively, achieved under the Nash equilibria \( \gamma_{\eta_1}^{\mu} \) and \( \gamma_{\eta_2}^{\mu} \). The price of information between the two ISs (under cost minimization) is given by

\[
\chi_{\eta_1}^{\eta_2}(\mu) = \frac{\max_{\gamma_{\eta_1}^{\mu}} J_{\mu}^{\eta_1}}{\max_{\gamma_{\eta_2}^{\mu}} J_{\mu}^{\eta_2}}.
\]

The PoI compares the worst-case costs under two different ISs for the same convex combination, and quantifies the relative loss or gain when the DG is played under a different IS. Clearly, when \( \chi_{\eta_1}^{\eta_2}(\mu) < 1 \), the IS \( \eta_2 \) is superior to its counterpart \( \eta_1 \). The connection between PoI and PoA can be captured by \( \chi_{\eta_1}^{\eta_2}(\mu) = p_{N,\mu,T}^{\eta_1} / p_{N,\mu,T}^{\eta_2} \).

**III. SCALAR LQ FEEDBACK DIFFERENTIAL GAMES**

The analysis of price of anarchy is complex for general DGs as there often exist more than one Nash equilibrium, which show strong dependence on the underlying IS. For specific game structures, however, its analysis may be tractable. One such class is scalar linear quadratic DGs with state feedback IS, which is what we focus on in this section. These games also enjoy wide applications in economics and communication networks; see, [10], [1]. We first state our model and review some important results on LQ feedback DGs; for details see [11], [12].

**A. Game Model**

Consider an infinite horizon scalar \( N \)-person LQ DG in which each player attempts to minimize his cost function

\[
J_i = \int_0^\infty \left( q_i x^2(t) + r_i u_i^2(t) \right) dt, \quad i \in \mathcal{N},
\]

where \( u \in \mathbb{R}, r_i \in \mathbb{R}_{++}, q_i \in \mathbb{R}_{++} \) and \( x_i(t) \in \mathbb{R} \) evolves according to linear system dynamics

\[
\dot{x}(t) = ax(t) + \sum_{i=1}^{N} b_i u_i(t), \quad x(0) = x_0
\]

where \( x_0 \in \mathbb{R} \) is the initial value of the state, \( a \in \mathbb{R} \) and \( b = [b_1, \ldots, b_N] \in \mathbb{R}^N \), where we assume without any loss of generality that \( b_i \neq 0, \forall i \in \mathcal{N} \). We are interested in Nash equilibrium (NE) under the feedback IS, that is players have access to the current value of the state, \( x \). We further require that the NE policies be stationary (that is time invariant) and that the equilibrium solution be strongly time consistent, that is it be a NE of any time-truncated version of the game, starting at any time \( t \) and with any initial state \( x(t) \) at \( t \); such a solution is known as feedback Nash equilibrium [11].

**Theorem 3.1**: [Feedback NE, [11], [12]] Let \( \{k_i, i \in \mathcal{N}\} \) solve the coupled set of algebraic Riccati equations

\[
2 \left( a - \sum_{i=1}^{N} s_i k_i \right) k_i + q_i + s_i k_i^2 = 0, \quad i \in \mathcal{N}
\]

satisfying the stability condition \( a - \sum_{i=1}^{N} s_i k_i < 0, \) where \( s_i := b_i^2 / r_i \). Then, the \( N \)-tuple of policies \( \gamma_i^*(x) = -b_i x_k x_i, i \in \mathcal{N}, \) constitutes a feedback NE, with the corresponding cost for Player \( i \) being \( J_i = k_i x_0^2. \) Furthermore, the positively weighed total cost is \( J_\mu = k x_0^2, \) where \( k = \sum_{i=1}^{N} k_i k_i. \)

If the coupled algebraic Riccati equations do not admit a solution which is also stabilizing, then the DG does not have a feedback NE.

The main challenge in computing the feedback NE solution for this DG is that equation (6) is a nonlinear coupled system of equations. The fact that we have a scalar problem alleviates the difficulty somewhat, since it is possible to turn it into a linear problem through a change of variables, as outlined in [13], [14]. Let \( \sigma_i = s_i q_i, \sigma_{\text{max}} = \max_i \sigma_i, p_i = s_i k_i, i = 1, \ldots, N, \) and

\[
\lambda = \sum_{i=1}^{N} p_i - a.
\]

Multiplying (6) by \( s_i \), we rewrite it as

\[
p_i^2 - 2\lambda p_i + \sigma_i = 0, \quad i = 1, \ldots, N.
\]

Let \( \Omega \subset \mathcal{N} \) be an index set, \( \Omega_{-1} = \Omega \setminus \{i\} \), and \( n_\Omega = |\Omega| \). For every \( \Omega \neq \emptyset \), we have (after some manipulations)

\[
\prod_{j \in \Omega} p_j \lambda = \frac{1}{2 n_\Omega - 1} \left( \sum_{i \in \Omega} \sigma_i \prod_{j \in \Omega, -1} p_j - n_\Omega \prod_{j \in \Omega} p_j + a \prod_{j \in \Omega} p_j \right).
\]

When \( \Omega = \emptyset \), we define

\[
\prod_{j \in \Omega} p_j \lambda := \lambda = \sum_{j=1}^{N} p_j - a.
\]

Hence, for every \( \Omega \), we have an equation in the form of either (9) or (10). Let \( p = [p_1, p_2, \ldots, p_N, p_1 p_2, \ldots, p_1 p_N, p_2 p_3, \ldots, p_{N-1} p_N, \ldots, \prod_{i=1}^{N} p_i]^T \).

We can write (9) and (10) into

\[
\tilde{\mathbf{M}} \mathbf{p} = \lambda \mathbf{p}.
\]

Let \( \mathbf{p} := [k_1, k_2, \ldots, k_N] \) and \( \mathbf{D} = \text{diag}(1, s_1, s_2, \ldots, s_N, s_1 s_2, \ldots, s_1 s_N, s_2 s_3, \ldots, s_{N-1} s_N, \ldots, \prod_{i=1}^{N} s_i) \). Hence, we can rewrite \( \mathbf{p} = \mathbf{D} \mathbf{k} \) and (11) into

\[
\tilde{\mathbf{M}} = \lambda \mathbf{M}, \quad \mathbf{M} := \mathbf{D}^{-1} \tilde{\mathbf{M}} \mathbf{D}.
\]

Equation (12) is an eigenvalue problem with each index set \( \Omega \) corresponding to a row enumerated starting from the empty set. It has maximum \( 2^N \) distinct eigenvalues and \( 2^N \) eigenvectors. The vector formed by the second entry to the \( N+1 \)-st entry of the eigenvectors yields the solution to (6)
when the first entry of the vector is normalized to 1 and they satisfy the stability condition of Thm. 3.1. This leads to:

**Theorem 3.2:** [Feedback NE Computation, [12]] Suppose $M$ is a nondefective matrix with distinct eigenvalues. Let $(\lambda, k)$ be an eigenvalue-eigenvector pair such that $\lambda \in \mathbb{R}_+$ and $\lambda > \sigma_{\text{max}}$. Then, a feedback NE $\gamma_i^\ast(x) = -\frac{b_i}{\mu_i r_i} k_i x, i \in N$, is yielded by $k^* = 1^T k$ provided that the resulting solution is stabilizing, where $1 = [0, 1, \ldots, 1, 0, \ldots, 0]^T$ is a vector whose 2nd to $N+1$-st entries are 1’s.

**Theorem 3.3:** [Uniqueness of Feedback NE] Let $\bar{p} := \sum_{j \in N, j \neq i} p_j, i := \sum_{j \in N, j \neq i} p_j$. There exists a unique feedback NE for the LQ DG described by (4) and (5) under either of the following two conditions: 
(i) $N$ is sufficiently large such that $\rho_{-i} > 1, \forall i$. (ii) $a = 0$. Furthermore, the feedback NE is of the following forms under the corresponding conditions above.

*(s-i) $p_i = (\bar{p} - a) - \sqrt{(\bar{p} - a)^2 - \sigma_i}$ :  

*(s-ii) $p_i = \bar{p} - \sqrt{\bar{p}^2 - \sigma_i}$, where  

\[ \bar{p} - a = \frac{1}{N-1} \left( \sum_{i=1}^{N} \sqrt{(\bar{p} - a)^2 - \sigma_i} + a \right). \]  

**Proof:** From (8), we obtain

\[ p_i^2 + 2(p_{-i} - a)p_i - \sigma_i = 0, \]  

which admits the solutions:

\[ p_i = (a - p_{-i}) \pm \sqrt{(a - p_{-i})^2 + \sigma_i}. \]  

Since we need $p_i > 0$, we retain the one with “+” sign. By rearranging the positive solution of (15), we arrive at

\[ (\bar{p} - a)^2 = (p_{-i} - a)^2 + \sigma_i, \]  

and, therefore, in terms of $\bar{p}$, we have

\[ p_i = (\bar{p} - a) \pm \sqrt{(\bar{p} - a)^2 - \sigma_i}. \]  

Under condition (i), we have $p_i - \bar{p} + a < 0$, hence we obtain a unique solution (s-i). Under scenario (ii), (17) reduces to $p_i = \bar{p} + \sqrt{\bar{p}^2 - \sigma_i}$. Since, $p_i < \bar{p}$, we again obtain the unique solution (s-ii).

By summing over (17), we have a fixed point equation (13). Let $\bar{P}(\bar{p}) = \frac{1}{N-1} \left( \sum_{i=1}^{N} \sqrt{(\bar{p} - a)^2 - \sigma_i} + a \right) - (\bar{p} - a)$. Its derivative is given by

\[ \frac{d\bar{P}}{d\bar{p}} = -1 + \frac{\bar{p} - a}{N-1} \left( \frac{1}{\sqrt{(a - \bar{p})^2 - \sigma_i}} \right). \]

Since $\sigma_i \geq 0$ and $\bar{p} > a > 0$, it follows that

\[ \frac{d\bar{P}}{d\bar{p}} \geq -1 + \frac{\bar{p} - a}{N-1} \left( \frac{N}{\sqrt{(a - \bar{p})^2}} \right) = \frac{1}{N-1} > 0, \text{ for } N \geq 2. \]

This says that $\bar{P}$ is a monotonically increasing function, and hence the solution to $\bar{P} = 0$ is unique. Hence, under (i) or (ii), there exists a unique feedback NE.

**B. Team Model**

When players form a specific team to achieve an optimal social objective, a specific total cost is minimized. Let $\bar{q}_i = \sum_{j=1}^{N} \mu_j q_i, \bar{R}_i = \text{diag}\{\mu_1 r_1, \ldots, \mu_N r_N\}$, and consider

\[ \text{FOC} \min_{u(t)} \int_0^\infty (\bar{q}_i x^2(t) + u^T(t) \bar{R}_i u(t)) \, dt \]

s.t. $\dot{x}(t) = a x(t) + \sum_{i=1}^{N} b_i u_i(t)$.

The solution to this optimal control problem is standard, and is given below for future reference (where we suppress the dependence of $\bar{q}$ and $\bar{R}$ on $\mu$).

**Theorem 3.4:** [Centralized Optimization] The optimal control problem (FOC) admits a unique feedback solution which is further stabilizing. The optimal policies are

\[ \gamma_i^\ast(x) = -\frac{b_i}{\mu_i r_i} k_i x, \quad \hat{k}_i := \frac{a + \sqrt{a^2 + \frac{b^2}{\mu_i}}} {b}, \]  

with $b := \sum_{i=1}^{N} (\frac{b^2}{\mu_i})$, and minimum cost is $J^\ast = \hat{k}_i x_0^2$. The optimal control can also be expressed in open loop:

\[ u_i^\ast = -\frac{b_i}{\mu_i r_i} \hat{k}_i \Phi(t,0) x_0, \]

where $\Phi(t,0)$ is the unique solution to

\[ \Phi(t,0) = \left( a - \sum_{i=1}^{N} \frac{b_i^2}{\mu_i r_i} \hat{k}_i \right) \Phi(t,0), \quad \Phi(0,0) = 1. \]

**C. Price of Anarchy (PoA)**

Here, we provide a closed-form expression for the PoA in the feedback LQ DG, where we make the natural assumption that $x_0 \neq 0$, as otherwise the costs are all zero.

**Theorem 3.5:** The PoA of the LQ feedback DG described by (4) and (5) is characterized by the following:

(i) Given a weight vector $\mu$, the PoA $\rho_{\mu}^{FB}$ is equal to

\[ \rho_{\mu}^{FB} = \max_{k \in \mathcal{K}} \left[ \mu^T k / \hat{k} \right], \]

where $\mu = [0, \mu, \mu^T, 0, \ldots, 0]^T$ and $\mathcal{K}$ is the set of all eigenvectors of the matrix $M$.

(ii) Suppose $\mu_i = \hat{\mu}_i := s_i / \sum_{j=1}^{N} s_j, i \in N$. Then,

\[ \rho_{\mu}^{FB} \leq \frac{\rho(\mu) + a}{\sum_{i=1}^{N} s_i \hat{k}}, \]

where $\rho(M)$ is the spectral radius of $M$.

(iii) Let $\rho_{\mu}^{FB} = \max_{i \in N} \mu_i / s_i$. Given a weight vector $\mu$ that satisfies $\sum_{i=1}^{N} \mu_i = 1$, the PoA is bounded by

\[ \rho_{\mu}^{FB} \leq \rho_{\mu}^{\ast \text{max}} (\rho(M) + a) / \hat{k}. \]

**Proof:** The proof is a direct application of the results in Theorem 3.1 and Theorem 3.4. PoA is the worst-case ratio of the game cost under feedback NE to the optimum social cost as defined in (2). Under the feedback IS, an LQ DG has

\[ \rho_{\mu}^{FB} = \max_{k} \frac{\sum_{i=1}^{N} h_i k_i (x_0)^2}{k(x_0)^2} = \max_{k} \frac{\mu^T k}{\hat{k}}. \]
This leads to statement (i). The price of anarchy under \( \bar{\mu} \) is

\[
\rho_{FB}^{\bar{\mu}} = \max_k \frac{\sum_{i=1}^N \bar{\mu}_i k_i}{k} = \max_k \frac{\sum_{i=1}^N s_i k_i}{k} = \max_{\lambda} \frac{\lambda + a}{\sum_{i=1}^N s_i k_i}.
\]

The last equality is due to (7). Hence, by taking the largest eigenvalue, we obtain (ii). The equality is achieved when \( \bar{\mu} = \bar{\mu}_i \) is reduced to

\[
k_i^{\star} \sim \frac{1}{\sqrt{\bar{\sigma}}} \left( q_i^2 + \sigma_i q_i^2 \bar{\sigma} \right).
\]

IV. OPEN-LOOP LQ DIFFERENTIAL GAMES

In this section, we look at the DGs described in (4) and (5) with open-loop information. Each player only knows the initial state of the system. Since the cost runs from zero to infinity, we assume players are interested in controls that yield finite costs. We restrict the controls to be in the set \( \mathcal{U}(x_0) = \{ u \in L_2 \mid J_i(x_0, u) \text{ exists in } \mathbb{R} \cup \{-\infty, \infty\} \} \), where \( L_2 \) is the space of square-integrable functions.

**Theorem 4.1:** [Open-Loop Nash Equilibrium, \[12\], \[11\]] Consider the \( N \)-person LQ DG in (4) and (5), and assume that there exists a unique solution \( \xi^* \) to the set of equations

\[
0 = 2a \xi_i + q_i - \xi_i \left( \sum_{j=1}^N s_j \xi_j \right),
\]

such that \( a - \sum_{j=1}^N s_j \xi_j^* < 0 \). Then the game admits an open-loop Nash equilibrium for every initial state, given by

\[
u_i^*(t) = -\frac{b_i}{r_i} \xi_i^* \exp \left[ \left( a - \sum_{j=1}^N s_j \xi_j^* \right) t \right] x_0.
\]

The optimal cost to player \( i \) using \( u_i^* \) is \( J_i^* = k_i^* x_0 \), where \( k_i^* \) is the unique solution to

\[
2 \left( a - \sum_{j=1}^N s_j \xi_j^* \right) k_i + q_i + s_i (\xi_i^*)^2 = 0.
\]

The quantities in Thm. 4.1 can further be obtained in analytic form. By a slight abuse of notation, we use \( p_i = s_i \xi_i \) as in the feedback information case. We multiply (27) and (28) by \( s_i \) and obtain

\[
0 = 2a p_i + \sigma_i - p_i \bar{p}, \quad \text{and } 0 = 2s_i k_i (a - \bar{p}) + \sigma_i + p_i^2,
\]

where \( \bar{p} = \sum_{i=1}^N p_i \), and hence we can solve for \( p_i, k_i \) and obtain

\[
p_i = \sigma_i / (\bar{p} - 2a) \quad \text{(29)}
\]

\[
k_i = \sigma_i + p_i^2 / (2s_i (\bar{p} - a)). \quad \text{(30)}
\]

To obtain \( \bar{p} \), we sum (29) over \( i \) and obtain \( \bar{p} = \frac{\sigma}{\bar{p} - 2a} \). Thus,

\[
\bar{p} = \sqrt{a^2 + \sigma + a}. \quad \text{(31)}
\]

Note that we have left out the other solution to the quadratic equation, because the negative solution is not relevant to the problem. It should also be pointed out that since the relevant \( \bar{p} \) is unique, we have a unique open-loop NE. Using (31), we can find \( \xi_i^* \) and thus the optimal control, given by

\[
\xi_i^* = \frac{q_i}{\sqrt{a^2 + \sigma}}. \quad \text{(32)}
\]

Using (31) and (29) in (30), we find a closed-form expression

\[
k_i^* = \frac{1}{\sqrt{a^2 + \sigma} \left( \frac{q_i}{2} + \frac{\sigma_i q_i}{2(\sqrt{a^2 + \sigma} - a)^2} \right)}. \quad \text{(33)}
\]

When \( a = 0 \), \( k_i^* \) is reduced to

\[
k_i^* = \frac{1}{\sqrt{a^2} \left( \frac{q_i}{2} + \frac{\sigma_i q_i}{2a} \right)}. \quad \text{(34)}
\]
Given weighting $\mu$, the open-loop NE yields a total cost of
\[ J^*_\mu = \sum_{i=1}^{N} \mu_i J^*_i = \sum_{i=1}^{N} \mu_i k^*_i(x_0)^2 =: k^*_\mu. \]
Since the open-loop NE solution is unique, the PoA under open loop IS can thus be easily found to be: $\rho^*_{\mu} = k^*_\mu / k^*_\mu$.

V. PRICE OF INFORMATION (POI)

In the previous sections, we have introduced the concept of PoA as a measure of efficiency with respect to the centralized and decentralized framework. Here, we study the price of information (PoI) as a measure of efficiency with respect to the ISs within the game framework. Following Definition 2.3, PoI between open-loop and feedback ISs is defined by
\[ \chi_{FB} = \max_{k} J^{FB}_{*} / \max_{k} J^{OL}_{*}. \]
When the feedback NE is unique, we have the expression $\chi_{FB} = \rho^*_{\mu} / \rho^*_{\mu}$. Using Thm. 3.5, we can obtain a bound on PoI:
\[ \chi_{FB} \geq \frac{k^*}{\mu^* \max(\theta(M) + a)}. \]

The following theorem further characterizes PoI.

**Theorem 5.1:** Suppose $a = 0$, and the number of players is large so that $N$ satisfies (C-i), (C-ii), and (C-iii). Then, the PoI is bounded from above and below by two constants:
\[ \sqrt{2}/2 \leq \chi_{FB} \leq \sqrt{2}. \]  

**Proof:** Under conditions (C-i), (C-ii), and (C-iii), we have a unique feedback NE that can be approximated as in statement (iv) of Thm. 3.7. Hence, from (31) we obtain
\[ \chi_{FB} = \frac{J^{FB}_{*}}{J^{OL}_{*}} = \frac{\sqrt{2}}{2} \left( 1 + \frac{1}{\max(\theta(M) + a)} \right) \leq \sqrt{2}, \]
where the last inequality is obtained by noting that
\[ \sum_{i=1}^{N} \mu_i q_i \sigma_i \geq \sum_{i=1}^{N} \mu_i q_i \sum_{j=1}^{N} \sigma_j. \]
The lower bound can be achieved by noting that $\sigma_i, q_i, \mu_i$ are nonnegative.

The upper bound in (36) is achievable if
\[ \sum_{i=1}^{N} \mu_i q_i \sigma_j = 0. \]  

Thm. 5.1 is useful in the design of games via access control or pricing mechanisms. Let $\bar{\chi} \in (\sqrt{2}/2, \sqrt{2})$ be some target PoI to achieve so that $\chi_{FB} \leq \bar{\chi}$. For example, when $\bar{\chi} = 1$, it means the game need to be designed so that the open-loop NE yields no larger cost than the feedback NE. Hence, a necessary condition to meet the design criterion is:
\[ \frac{\sum_{iN} \mu_i q_i \sigma_j}{\sigma} \leq \sqrt{2} \chi_{FB} - 1. \]  

An access control is to admit a set $N'$ of players so that (38) is satisfied when all the system and player parameters are given. When set $N'$ is fixed and not adjustable, we may use “pricing” mechanisms to control the parameters $r_i$ or $q_i$, which reflects the unit “price” of penalty on the control effort and the state, respectively. In the following corollary, we study the special case of homogeneous players.

**Corollary 5.2:** Suppose the DG satisfies the conditions in Theorem 5.1. In addition, let the players be symmetric so that $\sigma_i = \sigma, p_i = p, \forall i \in N$. When $N \geq 3$, the open-loop IS yields better total optimal cost; otherwise the FB information does better. In addition, as $N \rightarrow \infty$, $\lim_{N \rightarrow \infty} \chi^*_{FB} = \sqrt{2}^2$ at the rate of $O \left( \frac{1}{N} \right)$.

**Proof:** The proof directly follows from Thm. 5.1. The price of information under the additional assumptions becomes $\chi^*_{FB} = \frac{1}{\sqrt{2}} \left( 1 + \frac{1}{N} \right)$. It is independent of the parameters of the players and goes to $\sqrt{2}$ as $N \rightarrow \infty$. By letting $\chi^*_{FB} \leq 1$, we obtain $N \geq 1 / (\sqrt{2} - 1)$. Hence, since $N$ is an integer, the open-loop NE does better than the feedback NE when there are 3 or more players.

VI. CONCLUSION AND FUTURE WORK

In this paper, we have studied the price of anarchy and the price of information for a class of scalar LQ differential games. We have obtained bounds and approximations on these two indices. In the large population regime, there exist computable upper and lower bounds on the price of information. As future work, one possibility is to extend the results to non-scalar cases. The challenge of this extension is to find appropriate ways to tackle coupled matrix algebraic Riccati equations. We also intend to apply the results from this paper to problems in communications and economics.

REFERENCES