Fuzzy TSK Approximation Using Type-2 Fuzzy Logic Systems and Its Application to Modeling a Photovoltaic Array

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Abstract—This paper presents a simple approach to the fuzzy approximation of input-output data with a type-2 fuzzy model. The approach allows fuzzy type-2 modeling using data with input uncertainty, output uncertainty, or input-output uncertainty. The approximation error associated with the fuzzy representation is specified by the modeler and is used in the procedure to obtain the fuzzy model. We provide an example where type-2 fuzzy models of the characteristics of a photovoltaic array are obtained from input-output data with input, output, and input-output uncertainty.

I. INTRODUCTION

TYPE-2 fuzzy logic systems (FLS) were originally introduced in 1975 by L. A. Zadeh [1], [2]. Operations on type-2 sets were developed by Mizumoto and Tanaka [3]. Fuzzy logic researchers devoted little effort to type-2 fuzzy logic systems until the late 1990s when they became an active research topic. Since then, many publications on type-2 systems have appeared (see for example, [4], [5]). An important development was the introduction of interval type-2 systems which simplify computation and are adequate for most practical applications [6]. With the publication of Mendel’s text [7], fuzzy type-2 systems became more accessible and attracted more attention. The monogram on fuzzy type-2 applications by Castillo and Melin demonstrated the applicability of type-2 systems [8].

The first step in using type-2 systems in any practical application is to obtain a type-2 model from input-output data. Mendel [7] provides the following classification of fuzzy modeling approaches using input-output data:

1. Select the shapes and parameters of the antecedent and consequent membership functions then determine the fuzzy rules using the data.
2. Select the shapes of all membership functions and the parameters of the antecedent membership functions then use the data to determine the parameters of the consequent membership functions.
3. Select the shapes and parameters of the antecedent and consequent membership functions then use the data to determine the parameters of all membership functions.

These approaches were first proposed for type-1 systems but can be extended to type-2 fuzzy logic systems [7]. The approaches are for Mamdani type models but similar approaches using least squares optimization or neural networks can be used to find Takagi-Sugeno Kang (TSK) models.

A simple alternative type-1 TSK modeling approach that guarantees prescribed approximation accuracy was presented in [9] and [10]. The approach uses input-output data to generate fuzzy rules as well as antecedent membership functions and consequents for TSK models. The designer prescribes the desired accuracy and the functional form of the consequents in the TSK model. In this paper, we extend the approach of [9] to obtain type-2 TSK models from input-output data. We show how the approach can be used in the presence of input uncertainty, output uncertainty, or input-output uncertainty and apply it to the fuzzy modeling of the characteristics of a PV array.

In recent years the need for renewable energy has become more pressing. Photovoltaic (PV) cell arrays are a promising source of renewable energy since solar energy is free, abundant and readily available in many locations. Although the current price of PV cells is high, technological innovations and mass production are expected to bring down the price to competitive levels, given the projected increases in the cost of nonrenewable energy.

PV arrays are governed by their current-voltage (I-V) and current-radiation characteristics. I-V characteristics are critical to utilizing PV arrays in different applications. In [11], the I-V characteristics are used for the regulation of PV arrays. In [12], the I-V characteristics are used to obtain an analytical solution for the maximum power point tracking (MPPT) problem. In fault detection, changes in the I-V characteristics are monitored to detect PV array faults [13]. The current-radiation characteristic is useful in applications where photovoltaic output voltage varies in a relatively small range, such as grid-connected applications [14]. We show how our modeling methodology can be used to represent I-V and current-radiation characteristics with type-2 fuzzy TSK models.

The organization of the remainder of the paper follows. Section II reviews the type-1 fuzzy modeling approach of [9]. We extend the approach to type-2 fuzzy modeling in Section III. Section IV presents the application of PV array modeling with input uncertainty, output uncertainty,
II. TYPE-I MODELING WITH PRESCRIBED ACCURACY

In this section, we review several upper bounds on the approximation error for fuzzy systems that were derived in [9], [10]. The results are summarized here without proof except for Theorem 4, which is new. We begin with some basic definitions.

Definition 1: TSK Fuzzy System
The TSK fuzzy systems considered in this paper consist of four principal components [15]:
1. A fuzzifier that maps to normal, complete and consistent trapezoidal fuzzy sets.
2. A complete fuzzy rule base of the form:
   \[ R_{i_1,...,i_n} : \text{IF} \bigwedge_{j=1}^{n} x_j \text{ THEN} y = \sum_{i=1}^{n} \sum_{k=0}^{l} a_{i_1,...,i_n,k} x_j^k \]  
   \( (1) \)
   where \( x_j \) are the input variables of the fuzzy system, \( i_j = 1,2,...,N_j \) is the number of the membership functions of the \( j \)-th input, the trapezoidal fuzzy set \( A_j^{i_j} \subset U_j \) is a linguistic term characterized by the fuzzy membership function \( \mu_{A_j^{i_j}}(x_j) \), \( y \) is its output variable, and \( a_{i_1,...,i_n,k} \subset V \) are constants corresponding to rule \( R_{i_1,...,i_n} \).
4. A center-average defuzzifier.

A function is \( C^i \) if it is continuous with all its partial derivatives continuous up to order \( i \). We assume that the function \( g(x) \) to be approximated is \( C^2 \) on \( U = \{ x \in \mathbb{R}^n : a_0 \leq x_1 \leq b_0, \ldots, a_n \leq x_n \leq b_n \} \subset \mathbb{R}^n \) and define

\[ g^{(l)}(x_i) = \max_{j=1,2,...,N_i} \left| \frac{\partial g^{(l)}(x_1,...,x_n)}{\partial x_i} \right| \]  
\( (2) \)
\[ a_j = x_j^0 < x_j^1 < \ldots < x_j^{N_j-1} = b_j \]
and

\[ e_i^{l_j} = \min_{x_i^l \leq a_i} \max_{x_i^l \leq x_i^{l+1} \leq b_i} \left| g^{(l)}(x_i) - g^{(l)}(a_i) \right| \]  
\( (3) \)
\( i = 1,2,...,n, \quad j = 1,2,...,N_i-1, \quad a_i^j \in [x_i^j, x_i^{j+1}] \)

We define the \( \infty \)-norm for a function \( g(x) \) in \( U \) to be

\[ \|g(x)\|_{\infty} = \max_{x \in U} |g(x)| \]  
\( (4) \)

We now review the approximation error bounds for MISO systems derived in [9]. The first bound holds true for all continuous functions and yields fuzzy rules of the form (1) with constant consequents, i.e. \( m_i = 0 \) for all \( i = 1,2,...,n \). The second bound holds true for all \( C^i \) functions and also yields fuzzy rules of the form (1) with \( m_i = 0 \) for all \( i = 1,2,...,n \). Finally, the third bound holds true for all \( C^i \) functions and yields fuzzy rules of the form (1) with \( m_i = 0 \) for all \( i = 1,2,...,n \).

**Theorem 1:** Assume that \( \forall x \in U \) (i) \( g(x) \) is \( C^i \), (ii) \( f(x) \) is a fuzzy TSK approximation of \( g(x) \) as in Definition 1 with \( m_i = 0 \) for all \( i = 1,2,...,n \), and (iii) \( g(x^*) = f(x^*) \), \( x^* = (x_1^1, x_2^1, \ldots, x_n^1) \) where \( k_i = 1, i = 1,2,\ldots,n \), and \( j_i = 1,2,\ldots,N_i-1 \), then

\[ |g(x) - f(x)| \leq \frac{1}{2} \sum_{i=1}^{n} h_i^j \left( \max_{x_i^l} \left| g^{(l)}(x_i) \right| + e_i^{l_j} \right) \]  
\( (5) \)

**Corollary 1:** For any approximation error \( \delta > 0 \) \( g(x) - f(x) \leq \delta \) holds if

\[ \left\| \frac{\partial g(x)}{\partial x_i} \right\|_{\infty} < \frac{2\delta}{\|g^{(2)}(x)\|_{\infty}} \]  
\( (6) \)

The corollary follows by substituting (8) into (7).

**Theorem 2:** Assume that \( \forall x \in U \) (i) \( g(x) \) is \( C^i \), (ii) \( f(x) \) is a TSK fuzzy approximation of \( g(x) \) comprising as in Definition 1 with \( m_i = 0 \) for all \( i = 1,2,...,n \), and (iii) \( g(x^*) = f(x^*) \), \( x^* = (x_1^1, x_2^1, \ldots, x_n^1) \) where \( k_i = 1, i = 1,2,\ldots,n \), and \( j_i = 1,2,\ldots,N_i-1 \), then

\[ |g(x) - f(x)| \leq \frac{1}{8} \sum_{i=1}^{n} h_i^j \left\| g^{(2)}(x_i) \right\|_{\infty} \left( \max_{x_i^l} \left| g^{(l)}(x_i) \right| + e_i^{l_j} \right) \]  
\( (7) \)

**Corollary 2:** For any approximation error \( \delta > 0 \) \( g(x) - f(x) \leq \delta \) holds if

\[ h_i^j = \left\| \frac{\partial^2 g(x_i)}{\partial x_i^2} \right\|_{\infty} \]  
\( (8) \)

The corollary follows by substituting (11) into (10).

**Theorem 3:** Assume that \( \forall x \in U \) (i) \( g(x) \) is continuous and \( \frac{\partial g^{(m)}}{\partial x_i} \) exist for all \( i = 1,2,...,n \), (ii) \( f(x) \) is a TSK fuzzy approximation of \( g(x) \) as in Definition 1. (iii) \( g(x^*) = f(x^*) \), \( x^* = (x_1^1, x_2^1, \ldots, x_n^1) \) where \( k_i = 1, i = 1,2,\ldots,n \), and \( j_i = 1,2,\ldots,N_i-1 \), and \( x_i^j = x_i^{j+1} < \ldots < x_i^1 \). Then
Next, we present an approach to approximate an unknown function with measurable inputs and outputs with known error bounds. Such data are common in engineering applications and cannot be analyzed using conventional control methodologies. The following theorem gives approximation error bounds for data with known error bounds.

**Theorem 4:** Given a set of input output data representing an unknown function \( g_a(x) \) approximated by a polynomial \( g(x) \) of order \( n \) in the variable \( x \) with bounded error of magnitude \( \delta_x/2 \). Assume that \( \forall x \in U \) (i) \( f(x) \) is a TSK fuzzy approximation of \( g(x) \) as in Definition 1. (ii) \( g(x^*) = f(x^*), x^* = (x_{1,k_1}^*, x_{2,k_2}^*, \ldots, x_{n,k_n}^*) \) where \( k_i = 0,1, \ldots, m_i, i = 1,2, \ldots, n \), \( j_i = 1,2,\ldots,N_i-1 \), and \( x_{i,j_i}^* = x_{i,j_i}^* < \ldots < x_{i,m_i+1}^* = x_i^* + 1 \). Then

\[
\left| g_a(x) - f(x) \right| \leq \delta_x \sum_{i=1}^{n} \frac{h_i^{m_i+1}}{2^{m_i+1}(m_i + 1)!} \left( g_i^{(m_i+1)}(x_i^*, \alpha_i^{j_i}) + \varepsilon_i^{j_i} \right) \tag{10}
\]

\( h_i^j = x_i^{j+1} - x_i^j, \quad \alpha_i^{j} \in [x_i^j, x_i^{j+1}] \)

\[
g_i^{(m_i+1)}(x_i^*, \alpha_i^{j}) = \sum_{j_{m_i+1}}^{j_1} \frac{j!}{(j-m_i-1)!} a_j(x_i^*) \alpha_i^{j-m_i-1} \tag{11}
\]

where \( x_i^* \) denotes vector with the exclusion of \( x_i \).

**Proof:**

The theorem follows from Theorem 3 and the inequality

\[
\left| g_a(x) - f(x) \right| = \left| g_a(x) - g(x) + g(x) - f(x) \right| \\
\leq \left| g_a(x) - g(x) \right| + \left| g(x) - f(x) \right| \tag{12}
\]

If the unknown function is obtained using data with bounded error, then we use a polynomial approximation of the data with a known error bound. The polynomial approximation can be differentiated to obtain the derivatives needed for Theorems 1-3. The error bounds for this case are given by the following theorem. To design a fuzzy approximator for an unknown function, we follow the following steps:

1. Specify each controller input and output variables.
2. Obtain the input-output data pairs of a successful trajectory and fit the data with a polynomial.
3. Obtain the partial derivatives of the polynomial fit of the data with respect to all the input variables.
4. Use Theorem 4 of Section III to design the membership functions of the inputs.

5. Construct the rule base from the set of measurements corresponding to input values determined from the membership functions of Step 3.

### III. TYPE-2 MODELING WITH PRESCRIBED ACCURACY

To model the input uncertainty, we use the modeling method presented in Section II, and change the singleton fuzzifier to a non-singleton fuzzifier. The fuzzifier is assumed to have a membership function based on input measurement error. The weight associated with each rule is:

\[
w'_i(x_{\text{max}}) = \prod_{i=1}^{n} \left[ \sup_{x_i} \mu_i(x_i) \right] \tag{13}
\]

\[
x_{\text{max}} = \left[ x_{1,\text{max}}, x_{2,\text{max}}, \ldots, x_{p,\text{max}} \right]^T \tag{14}
\]

\[
x_{i,\text{max}} = \arg \left[ \sup_{x_i} \mu_i(x_i) \right] \tag{15}
\]

where \( \ast \) denotes any t-norm. The output of the TSK system is

\[
y = \frac{\sum_{i=1}^{M} w'_i(x_{\text{max}})y'_i}{\sum_{i=1}^{M} w'_i(x_{\text{max}})} \tag{16}
\]

For modeling output uncertainty, we need the following theorem.

**Theorem 5:** For a TSK model, adding a constant \( a_o \) to all the consequents results in adding \( a_o \) to its output.

**Proof:** Consider a TSK model with rules:

\[
\begin{align*}
\text{R}_i^1 : & \text{IF } x_i \text{ is } A_i \text{ and ... and } x_n \text{ is } A_n \\
& \quad \quad \text{THEN } y^1 = f_i(x), i = 1, \ldots, L
\end{align*}
\]

The output of the model is:

\[
y = \frac{\sum_{i=1}^{L} \mu_i f^i(x)}{\sum_{i=1}^{L} \mu_i} \tag{17}
\]

The output of the model with a constant \( a_0 \) added to all the consequents is:

\[
y_u = \frac{\sum_{i=1}^{L} \left[ \mu_i f^i(x) + \mu_i a_o \right]}{\sum_{i=1}^{L} \mu_i} = \frac{\sum_{i=1}^{L} \mu_i f^i(x) + \sum_{i=1}^{L} \mu_i a_o}{\sum_{i=1}^{L} \mu_i} + a_0 = y + a_0 \tag{18}
\]

Based on Theorem 5, a TSK rule with consequent

\[
y^i = f_i(x) + a_i, a_i \in A_i = \left\{ a_i \in \mathbb{R}, a_i = \frac{a_o}{\mu_i(x)} \right\}
\]

models uncertainty in the output. The following procedures give an approximate type-2 model including model uncertainty using experimental data.

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Procedure 1: Type-2 Approximation of data with input uncertainty

1. Obtain a type-1 fuzzy system using experimental data and the approach of [9].
2. Use a non-singleton fuzzifier with fuzzification based on the input measurement error.

Procedure 2: Type-2 Approximation of data with output uncertainty

1. Obtain a type-1 fuzzy system using experimental data and the approach of [9].
2. Add the type-1 uncertainty based on the output measurement error to the consequent of each type-1 fuzzy rule obtained in Step 1.
3. Use type-reduction methods to transform the type-1 output to a crisp output [7].

Procedure 3: Type-2 Approximation of data with input-output uncertainty

1. Obtain a type-1 fuzzy system using experimental data and the approach of [9].
2. Add the type-1 uncertainty based on the output measurement errors to the consequent of each type-1 fuzzy rule obtained in Step 1.
3. Use a non-singleton fuzzifier with fuzzification based on the input measurement errors.
4. Use type-reduction to transform the type-1 output to a crisp output [7].

IV. EXAMPLE

The mathematical equation relating the array current to the array voltage for solar cells arranged in \( N_p \) parallel sets of \( N_s \) series cells is: \([17][16]\)

\[
I_{sa} = N_p I_{ph}
\]

\[
- N_p I_{sa} \left\{\exp\left[\frac{q}{AKT} \left( \frac{V_{sa}}{N_s} + \frac{I_{sa}R_s}{N_p} \right) \right] - 1 \right\}
\]

(20)

\[
- N_p \left( \frac{V_{sa}}{N_s} + \frac{I_{sa}R_s}{N_p} \right)
\]

where \( I_{sa} \) is the solar-array output current (A), \( V_{sa} \) is the solar array output voltage (V), \( I_{ph} \) is the light-generated current (A) representing radiation, \( I_{sa} \) is the array reverse saturation current (A), \( q \) is the electronic charge (C), \( A \) is the dimensionless deviation factor from the ideal p-n junction diode, \( k \) is Boltzmann's constant, \( T \) is the cell temperature (K), \( R_s \) is the series resistance (\( \Omega \)), and \( R_{sh} \) is the shunt resistance (\( \Omega \)). The values of all the parameters are given in Table 1.

Due to the randomness of radiation patterns and temperature changes, as well as the presence of dust or other material on the array, the characteristics vary significantly and deviate from the nominal equations (20).

In addition, the mathematical model does not cover the behavior of the array over the entire available voltage range. Thus, there is a need to provide a more general mathematical description of PV arrays that includes modeling uncertainty. In \([16][17]\), the authors used neural networks to provide such a description. Here, we use our proposed type-2 modeling approach to develop a fuzzy TSK model of the PV array including input, output, or input-output uncertainty.

To develop our fuzzy model, we use measured data from a solar array [13], together with noisy data points from its mathematical model when possible, to model the solar array I-V characteristic. We develop fuzzy type-2 models to represent both an I-V characteristic for a fixed radiation level and a current-radiation characteristic for a fixed voltage of 25 V. Radiation is the main source of uncertainty in PV arrays.

\[ a) \quad \text{Input Uncertainty} \]

The power generated by a PV array is a stochastic variable dependent upon the level of solar radiation. In grid-connected arrays \([16]\), one can assume that the voltage is constant (20 volts here) and that the current depends on the radiation level. Current also depends on temperature, but the effect of temperature change is usually negligible \([18]\).

Solar radiation is typically represented in the literature using an equivalent light-generated current (\( I_{ph} \)). In sunny regions summer solar radiation can be modeled as a Gaussian random variable. To model the photovoltaic array under such conditions, the data for the current-radiation characteristic is used with uncertainty in the solar radiation as input.

For modeling input uncertainty, we use Procedure 1. The procedure uses a nonsingleton fuzzifier to represent the input uncertainty. The random input uncertainty is considered to be Gaussian with constant variance (0.01 here) and with mean equal to the nominal measurement value.

![Figure 1. Membership functions of fuzzy type-1 approximator determined using differentiation of current-radiation characteristic with input uncertainty.](image-url)
generated current ranges each of which can be associated with a fuzzy set and given a linguistic label such as very small, small, and so on.

The data points are shown along with the output of the type-1 fuzzy model in Figure 2. Note that in this case the output of the fuzzy model is crisp and there is no need for type reduction. The input uncertainty is handled through nonsingleton fuzzification on the input side.

![Figure 2. Current-radiation characteristic modeled with uncertainty in input measurement.](image)

**b) Output Uncertainty**

For uncertain output measurements, the system must allow for output current variation at each voltage value. Our noisy data is assumed to be polluted with uniformly distributed noise in the interval $[-0.1, 0.1]$. **Procedure 2** provides the required model for this case. Figure 3 shows the type-1 membership functions used to represent the output uncertainty based on the noisy data. The membership functions correspond to different voltage ranges each of which can be associated with a fuzzy set and given a linguistic label such as very small, small, and so on.

As shown in Section III, the output of the model for a system with output uncertainty is a type-1 fuzzy set. To obtain a scalar output one can use any type reduction approach [7]. Here, we use center-of-sets type reduction to obtain the results shown in Figure 4.

![Figure 3. Membership functions of fuzzy type-1 approximator determined using differentiation of I-V characteristic.](image)

**c) Input-output Uncertainty**

We now consider the current-radiation data with uncertainty in both the input (radiation) and output (current) measurements. The uncertainty in the measured output is again considered to be uniformly distributed in the interval $[-0.1, 0.1]$.

![Figure 4. I-V characteristic modeled with uncertainty in output measurement.](image)

![Figure 5. Membership functions of fuzzy type-1 approximator determined using differentiation of current-radiation characteristic with input and output uncertainty.](image)

![Figure 6. Current-radiation characteristic modeled with uncertainty in output and input measurement.](image)

Using **Procedure 3**, we obtain the membership functions of Figure 5. The output of the fuzzy type-2
model is shown in Figure 6 together with the data used in modeling.

We observe that the input membership functions are similar but not identical to those of Figure 1. In addition, the output membership functions are similar but not identical to those of Figure 3. The differences are due to the use of different data samples in each case.

V. CONCLUSION

This paper provides a new modeling approach for type-2 FLS. The approach is an extension of the type-1 approach of [9], [10]. It allows the modeler to develop type-2 models including input uncertainty, output uncertainty, or both input and output uncertainty. The main advantages of the new approach are its simplicity and its ability to provide models including characterization of the model uncertainty. We applied the new approach to modeling the characteristics of a photovoltaic array. The results show that the approach captures the behavior of the photovoltaic array including modeling uncertainties. Future work will apply the approach to experimental data and investigate its use to develop fuzzy controllers for systems with significant model uncertainty.

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REFERENCES


TABLE 1.

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