Abstract—Adaptive control using multiple identification models is currently well established. Switching, tuning, and switching and tuning in such systems have been extensively investigated, and their stability properties have been established [1]. If the number of models is sufficiently large, and is uniformly distributed in parameter space, the method results in fast and accurate response.

In practice, even for reasonably complex dynamical systems, the number of models needed for improved performance may be quite large. The paper attempts to propose a general methodology for achieving comparable response using significantly smaller number of models. The principal idea is to either redistribute available fixed and adaptive models even as the system is in operation, or utilize adaptive models more efficiently. Several different approaches are described and in each case the theoretical and practical questions that arise are discussed. Simulation results are included to demonstrate the improvement in performance using the methods proposed.

I. INTRODUCTION

Adaptive control using multiple models (MMAC) was introduced in [1] to cope with large parametric uncertainty. Both fixed models and adaptive models are used to identify an unknown plant and based on an index of performance, the control input is chosen corresponding to the model that is considered to be the “best”. A qualitative explanation is that the index of performance would, in general, choose the model closest to the plant in some sense and, consequently, adaptation would start from the corresponding model, resulting in improved performance. Switching (to the closest model) results in fast response, and tuning (from the model) improves the identification and control errors on a slower time scale. In the above method, both fixed models and adaptive models are seen to play important roles. The fixed models are needed to determine the approximate location of the plant, and the adaptive model initialized from one of the fixed models results in faster and more accurate performance. Extensive simulations were carried out in the 1990s [1], [2] to verify empirically that the above statements are indeed valid. More recently in [3], [4], [5], considerably more detailed simulations were carried out to study the adaptation mechanism in greater detail. The ideas presented in this paper have their origin in these simulation studies.

In adaptive control it is generally assumed that the region of uncertainty in parameter space is bounded and is known. When the number of models used is large, and uniformly distributed in $D$, the plant will contain one or more models in its neighborhood (suitably defined). Since adaptive control is known to perform well when parametric errors are small, it is not surprising that the procedure described above results in substantial improvement in performance. These facts have been verified in [3], [4], [5] for low order systems with only a few parameters.

In more realistic problems, where the dimension of the plant and the dimension of the unknown parameter vector are large, the number of models needed to satisfy the conditions described earlier becomes prohibitively large. Assuming that $N$ is the number of models that the designer would consider “reasonable” and also that this number is not adequate to achieve satisfactory response when the parameters are unknown, the question naturally arises as to how the resources (i.e. models) can be used more effectively. This gives rise to the problem discussed in this paper.

Making the resources more effective calls for a judicious choice of the location of both fixed and adaptive models and a higher level of adaptation in which these locations are themselves adjusted on the basis of their performance, even as the system is in operation. Various methods with different theoretical and practical advantages are proposed in this paper.

The results given in the paper merely represent a first step in the evolution of a new approach to adaptive control. More complex plants in higher dimensions (and greater uncertainty) need to be considered before the approach can be considered to be practically viable.

Qualitatively, we can pose the problem as follows:

If $N_1 \leq N$ models are located in $D$ in the parameter space, either on the boundary or on a lattice, is it possible by observing the responses of the different models over a finite period of time to estimate the approximate location of the plant and distribute $(N - N_1)$ models (or redistribute all $N$ models) such that one of the models is closer to the plant and consequently results in faster and better performance? The questions that arise in this context are described in the following sections.

On the other hand, if the models are not to be redistributed, the question is whether it is possible to combine the information provided by each model in an efficient fashion such that a more accurate estimate of the location of the plant is generated leading to better performance. This problem is
also addressed in this paper.

Before proceeding to introduce the proposed schemes, we first describe the standard identification and control of a linear time-invariant system to provide a framework for our discussions.

II. PROBLEM FORMULATION

The identification and control of a linear dynamical system using a single model is first described in Section II-A and II-B. Extensions of this to the multiple model case are treated in Section II-C and II-D.

A. Case I: Plant in Companion Form (Identification Problem)

A stable LTI plant \( \Sigma_p \) is described by the state equations

\[
\Sigma_p : \dot{x}_p(t) = A_p x_p(t) + b u(t)
\]

where \( x_p(t) \in \mathbb{R}^n \), \( u(t) \in \mathbb{R} \) and \( A_p \) and \( b \) are in companion form and defined as

\[
A_p = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
a_{p_1} & a_{p_2} & a_{p_3} & \cdots & a_{p_n}
\end{bmatrix},
\]

\[
b = \begin{bmatrix}
0 \\
0 \\
\vdots \\
0 \\
1
\end{bmatrix}. \tag{2}
\]

The parameters \( a_{p_i} \) (\( i = 1, 2, \ldots, n \)) are unknown and have to be estimated from observations on \( x_p(t) \) and \( u(t) \). Let \( \theta = [a_{p_1}, a_{p_2}, a_{p_3}, \ldots, a_{p_n}]^T \) be the unknown parameter vector. A single model \( \Sigma \) is set up described by the equation

\[
\Sigma : \dot{x}(t) = A_m \dot{x}(t) + [\dot{A}(t) - A_m] x_p(t) + b u(t) \tag{3}
\]

where \( A_m \) is a stable matrix in companion form with parameters \( a_{m_i} \) (\( i = 1, 2, \ldots, n \)) and \( A(t) \) is a matrix in companion form with the last row consisting of adjustable parameters \( \hat{a}_i(t) \).

Defining \( \hat{a}_i(t) \) \( - \) \( a_{p_i} = \hat{a}_i(t), \dot{\hat{x}}(t) - x_p(t) = e(t) \) we obtain the error equations

\[
\dot{e}(t) = A_m e(t) + b \hat{\theta}^T x_p(t) \tag{4}
\]

where \( \hat{\theta}^T = [\hat{a}_1(t), \hat{a}_2(t), \ldots, \hat{a}_n(t)] \).

Using the Lyapunov function candidate

\[
V(e, \hat{\theta}) = e^T P e + \hat{\theta}^T \hat{\theta} \tag{5}
\]

it follows that

\[
\dot{V}(e, \hat{\theta}) = -e^T Q e + 2e^T P \hat{\theta}^T x_p + 2\hat{\theta}^T \dot{\hat{\theta}} \tag{6}
\]

so that if the adaptive law

\[
\dot{\hat{\theta}}(t) = \hat{\theta}(t) - e^T P b x_p \tag{7}
\]

is used, then \( \dot{V}(e, \hat{\theta}) = -e^T Q e \leq 0 \). This implies that \( e \), \( \theta \) and \( x_p \) are bounded and using standard arguments of adaptive control theory it follows that \( e(t) \) tends to zero asymptotically with time. Further, it is well known that \( \dot{\theta}(t) \) tends to zero if \( u(t) \) is persistently exciting, so that \( \lim_{t \to \infty} \hat{a}_i(t) = a_{p_i} \).

B. Case II: Plant in Companion Form (Control Problem)

In Case I, the plant was assumed to be stable. If the plant is unstable, direct adaptive feedback control can be used to stabilize the system and achieve exact model following with the output of a reference model defined by

\[
\Sigma_m : \dot{x}_m(t) = A_m x_m(t) + b r(t) \tag{8}
\]

where \( A_m \) is stable and in companion form.

By using an input

\[
u(t) = k^T(t) x_p(t) + r(t) \tag{9}
\]

if follows (using the same arguments as in Case I) that adaptive laws for adjusting \( k(t) \) can be derived as

\[
\dot{k}(t) = -e^T(t) P b x_p(t), \tag{10}
\]

where \( e(t) \) is the control error. [Alternatively, if the indirect approach is used, the unknown plant parameters can be first identified and \( k(t) \) adjusted algebraically].

C. Case III: Multiple Models (Identification Problem)

In this section we describe the extension of the above approach when multiple models are used.

Let \( \Sigma_1, \Sigma_2, \ldots, \Sigma_N \) be \( N \) identification models described by the state equations (corresponding to equation (3)):

\[
\Sigma_i : \dot{x}_i(t) = A_m x_i(t) + [A_i(t) - A_m] x_p(t) + b u(t). \tag{11}
\]

We then obtain \( N \) error equations and the corresponding adaptive laws as follows (see equations (4) and (7))

\[
\dot{e}_i(t) = A_m e_i(t) + b \hat{\theta}_i^T x_p(t), \quad e_i(0) = 0, \tag{12}
\]

and

\[
\dot{\hat{\theta}}_i(t) = -e_i^T(t) P b x_p(t), \tag{13}
\]

where \( e_i(t) \) are the identification errors. All the equations describing the models are identical, and only the initial conditions of the model parameters (i.e. \( \hat{\theta}_i(0) \)) are different.

As stated above, the problem is simplified substantially since the state vector \( x_p(t) \) of the plant is assumed to be accessible. Hence, all the models use the same adaptive laws but with different initial conditions.

D. Case IV: Multiple Models (Control Problem)

As stated earlier, when the plant is unstable, it becomes mandatory that feedback control be used to stabilize it. Further, while \( N \) models can be simultaneously updated for identification (giving \( N \) different estimates), only one controller can be used at any instant of time. As was done in earlier work, the output identification errors \( e_i \) may be
used to generate performance indices

\[ J_i(t) = \int_0^t e_i^2(\tau) d\tau \quad (i = 1, 2, \ldots, N). \]

and the model corresponding to \( \min_i \{ J_i(t) \} \), for choosing the controller. (Many different performance indices have been suggested and used in the literature).

III. Redistribution of Models

As stated in the introduction, when the number of models is small, second level adaptation may be needed to use the models efficiently. In this section we consider methods in which the location of the models are changed after observing the system over a finite interval.

A. Algebraic Method Using Fixed Models

If fixed models with unknown parameter vectors \( \theta_i \) are used to estimate the plant parameter vector \( \theta_p \), it was shown (equation (12)) that the output identification errors \( e_i(t) \) and parameter error vectors \( \tilde{\theta}_i \) are related by the differential equation

\[ \dot{e}_i = A_m e_i + b \tilde{\theta}_i^T x_p. \]

A vast number of procedures for estimating \( \tilde{\theta}_i \) have been reported in the past. For the following discussion, we merely express \( e_i(t) \) as

\[ e_i(t) = H(t) \tilde{\theta}_i, \]

where \( H(t) = \int_0^t e_m^A(\tau-t)bx_p^T(\tau)d\tau \) so that \( e_i(t) \) and \( \tilde{\theta}_i \) are linearly related. Since \( A_m, b \) are known, and \( x_p(t) \) is accessible, \( H(t) \) can be computed theoretically at every instant of time. This implies that the integral square error \( J_i(t) \), of the \( i^{th} \) model

\[ J_i(t) = \int_0^t e_i^2(\tau) d\tau \]

\[ = \tilde{\theta}_i^T \int_0^t H(\tau) H(\tau) d\tau \tilde{\theta}_i \]

\[ = \tilde{\theta}_i^T W(t) \tilde{\theta}_i \]

is a quadratic function of the unknown parameter vector \( \tilde{\theta}_i \). Assuming that the symmetric matrix \( W(t) \) is positive definite for \( t \geq T \), it follows that the performance indices of all the models are merely points on a (time-varying) quadratic surface, whose minimum corresponds to the plant. This can consequently be used as a basis for the redistribution of the models.

Redistribution of Fixed Models:

It was earlier assumed that \( W(t) \) is positive definite for \( t \geq T \). It consequently follows that \( J_i(T) = \tilde{\theta}_i^T W(T) \tilde{\theta}_i \) is a quadratic function of \( \tilde{\theta}_i \). At time \( t = T \), the values \( J_i(T) \) represent points on a surface \( \tilde{\theta}_i^T W(T) \tilde{\theta}_i \), and the objective is to determine the location of the minimum of the function using the observed data. Several heuristic approaches were considered but most of them required explicit knowledge of the symmetric matrix \( W(T) \) (which can be theoretically computed). The following method, which merely assumes the quadratic nature of the surface, is simple, and scales readily to higher dimensions was chosen as the most satisfactory one.

If \( \theta_i \) (corresponding to model \( M_i \)) has the minimum index of performance (i.e. \( J_i = \min_j \{ J_j \} \)), all the other \( (N - 1) \) models \( M_j \) \( (j \neq i) \) are redistributed along the lines in parameter space connecting \( \theta_j \) to \( \theta_i \) with the new location determined by

\[ \tilde{\theta}_j = \sqrt{J_j} \theta_i + \sqrt{J_i} \theta_j. \]

In Figure 1 \( M_1 \) represents the model with the minimum index of performance, \( P \) represents the plant, and \( M_2 \) to \( M_4 \) represent 3 additional models which are redistributed. The performance indices corresponding to the new model positions are smaller, and the model closest to the plant has a smaller index than \( M_1 \).

![Fig. 1: Performance Index Function Contour](image)

Before proceeding to discuss how the redistribution of the fixed models at time \( T \) can be used for identification and control purposes, some comments concerning fixed and adaptive models are in order.

B. Fixed and Adaptive Models For Adaptive Control

The study of adaptive control based on multiple models has revealed that fixed models and adaptive models enjoy advantages that are complementary. Fixed models are computationally simple. By their very nature, they provide information regarding the behavior of a model at one point in parameter space. As shown earlier, they yield output errors which are linearly related to their parametric errors. In contrast to them, adaptive models are computationally intensive. They also provide information over a segment of a trajectory in parameter space. This accounts for the fact that relatively more fixed models can be used for identification and control.

Adaptive models are nonlinear which makes their analysis substantially more complex. However, they assure global
stability and asymptotic tracking of the desired output (i.e. the output of the reference model), or equivalently, the output error tending to zero. This implies that if adaptive models rather than fixed models are used in the redistribution schemes discussed in this section, the results will not be catastrophic, even if the performance of the system is not improved.

In view of the above considerations, it appears desirable to combine adaptive and fixed models to realize their individual advantages. It is also not entirely accidental that the classical adaptive control approach suggested in [1], as mentioned in the introduction, used \( N \) fixed models and 2 adaptive models. The fixed models provided information concerning the neighborhood of the plant from which adaptive models could be initiated. In the following discussion we shall refer to this as the \( A_{N,2} \) scheme and use it for comparison purposes. As in the \( A_{N,2} \) scheme, we will also use fixed models to determine where in parameter space the adaptive models should be initiated.

Returning to the redistribution of the fixed models considered earlier, we can either

(i) initiate \( N \) adaptive models from the computed values of the fixed models,

or (ii) initiate a single adaptive model from the computed value of the most likely position of the plant,

or (iii) use the \( A_{N,2} \) scheme with the modified distribution of fixed models.

C. Determination of Plant Location In a Subregion

If \( D \) is the region of uncertainty, and if, after observing the response of a set of fixed models over an interval of length \( T \) it can be concluded that the plant lies in a subregion \( D^1 \subset D \), then all the models can be redistributed in \( D^1 \). While several heuristic approaches were considered, very few combined mathematical tractability, simplicity, and scalability (to higher dimensions). One of the latter is briefly described below.

![Fig. 2: Two Dimensional Lattice](image_url)

\( D \) is partitioned into a uniform lattice with the \( N \) models provided. Using fixed models the performance index \( J_i \) \((i = 1, 2, \ldots, N)\) is evaluated at the vertices of the lattice. Using an appropriate criterion the location of the plant in a subregion is concluded. One such scheme computes \( J_i \) corresponding to the \( i^{th} \) subregion, where \( J_i \) is the sum of the \( J_i \) at its vertices. However, a theoretical justification for this is still under investigation. A two dimensional lattice in \( D \) with 4 subregions, and the subregion \( D^1 \) in \( D \) is shown in Figure 2. \( J \) corresponding to \( D^1 \) is \( J_5 + J_6 + J_8 + J_9 \).

**Comment:** In the schemes discussed thus far, the responses of fixed models over an interval \( T \) are used (i) to redistribute the models (III-B) or (ii) to determine the subregion \( D^1 \) containing the plant. \( T \) is consequently a critical adaptive parameter in both cases.

IV. A USEFUL PROPERTY OF ADAPTIVE MODELS

The redistribution of models described in Section III is a discontinuous process in parameter space. In this section we describe an important property of adaptive models which permits adjustment of control parameters in a continuous fashion.

**Theorem 1:** If \( N \) adaptive identification models described in Section II are adjusted using adaptive laws (13), with initial conditions \( \theta_i(0) \) and initial states \( x_i(0) = x_p(0) \), and if the plant parameter vector \( \theta \) lies in the convex hull of \( \theta_i(0) \) \((i = 1, 2, \ldots, N)\), then \( \theta \) lies in the convex hull of \( \theta_i(t) \) \((i = 1, 2, \ldots, N)\) for all \( t \geq 0 \).

**Proof:** Since \( \theta \) lies in the convex hull of \( \theta_i(0) \), positive constants \( \alpha_i \) exists with \( \sum_{i=1}^{N} \alpha_i = 1 \) such that

\[
\theta = \sum_{i=1}^{N} \alpha_i \theta_i(0).
\]

From the error equation (12), with \( e_i(0) = 0 \) it follows that \( e_i(t) = \Phi(t,0) b \tilde{\Theta}(t) x_p(t) \). From the adaptive laws (13) we have

\[
\dot{\hat{\theta}}_i(t) = -e_i^T(t) P b x_p(t) = -\tilde{A}(t) \hat{\theta}_i(t) \quad (18)
\]

where \( \tilde{A}(t) = \begin{bmatrix} b^T \Phi(t,0) P b \end{bmatrix} x_p(t) x_p^T(t) \) which is independent of the model used.

Since (18) is a linear time-varying differential equation \( \hat{\theta}_i(t) \) can be expressed in terms of the transition matrix \( \Psi(t,0) \) of (18) and the initial condition \( \theta_i(0) \) as

\[
\hat{\theta}_i(t) = \Psi(t,0) \hat{\theta}_i(0). \quad (19)
\]

It then follows that the plant parameter \( \theta \) can be expressed as \( \sum \alpha_i \theta_i(t) \), since

\[
\sum_{i=1}^{N} \alpha_i \hat{\theta}_i(t) = \Psi(t,0) \sum_{i=1}^{N} \alpha_i \hat{\theta}_i(0) = 0, \quad (20)
\]

or

\[
\sum_{i=1}^{N} \alpha_i \theta_i(t) = \theta. \quad (21)
\]

**Comment:** From Theorem 1 it follows that if \( \theta \) lies in the convex hull of \( \theta_i(0) \) \((i = 1, 2, \ldots, N)\), it also lies in the convex hull of \( \theta_i(t) \). If it lies outside the convex hull, it remains outside the convex hull for all time \( t \geq 0 \), and if it
lies on the boundary of the convex hull, it will stay on the boundary of the convex hull.

A crucial assumption in deriving the above results is that all the identification models have initial conditions \( x_i(0) = x_p(0) \) so that \( e_i(0) = 0 \).

**Example 1:** We illustrate the ideas discussed thus far using the following example. A stable dynamical system in \( \mathbb{R}^2 \) is described by the differential equation (1) with unknown parameter vector \( \theta = [-2, -2]^{T} \). Four identification models are initialized at \( \theta_1(0) = [5, 5]^{T} \), \( \theta_2(0) = [-5, 5]^{T} \), \( \theta_3(0) = [-5, -5]^{T} \), and \( \theta_4(0) = [5, -5]^{T} \) which are at the vertices of a rectangle, and it is known a priori that \( \theta \) lies within their convex hull. From Figure 3 it is seen that \( P \) also lies in the convex hull of \( \{ \theta_1(t), \theta_2(t), \theta_3(t), \text{and} \theta_4(t) \} \). However, due to the nature of the solutions of the equation (18), the convex hull at time \( t_1 \) is not nested within the convex hull at time \( t < t_1 \).

![Parameter Space Trajectories of Dynamical System](image)

**Fig. 3: Parameter Space Trajectories of Dynamical System**

From the results derived in [1] and [2], switching between the four adaptive models in Example 1 can be shown to be stable with the control error tending to zero. However, our interest in the new procedure is in determining better estimates of the plant parameter vector (which lies in the convex hull of \( \dot{\theta}_i(t) \) as described earlier) for control purposes. We also note that such an estimate is a linear combination of \( \dot{\theta}_i(t) \) (i.e. \( \dot{\theta}(t) = \sum_{i=1}^{N} \alpha_i(t) \dot{\theta}_i(t) \)) where the coefficients are time-varying and as stated earlier, are generated based on the performance indices \( J_i(t) \) (i = 1, 2, \ldots, N). In such cases, the stability of the overall system has to be established. The fact that all four models converge to the plant and hence the estimate \( \hat{\theta}(t) \) must also converge to the plant provides the rationale for assuming that the procedure will be stable. However, the extent to which this depends upon the persistent excitation of the input is not immediately evident (since convergence to the plant is not assured in such a case). This problem is currently being investigated.

**V. Simulation Results**

To determine how the different schemes proposed in Sections III and IV perform, the adaptive control of an unstable second order plant was simulated on the computer. The plant parameter vector was chosen as \( \theta = [5, 5]^{T} \) and the reference model parameter vector as \( \theta_m = [-6, -5]^{T} \). The region of uncertainty was assumed to be \( D = [-15, 15] \times [-15, 15] \). Four adaptive schemes were simulated. The first was the \( A_{N,2} \) scheme with 9 fixed models \( (N = 9) \). In the second scheme the fixed models were redistributed as described in Section III-B with \( T = 2 \), following which the \( A_{N,2} \) scheme was used. The third scheme chose a subregion in parameter space after time \( T = 0.5 \) and all 9 models were uniformly distributed in it. The last simulation used only 4 adaptive models with

\[
\alpha_i(t) = \frac{1}{\sum_{j=1}^{N} 1/j}. 
\]

The four different cases are shown in Figures 4(a) to 4(d). In each case the desired state variable \( x_{m1} \) (the first element of the state) and the corresponding state \( x_{p1} \) of the plant are plotted together, and the output error is plotted separately on an enlarged scale. The responses in the last three cases are seen to be significantly better than in the \( A_{9,2} \) scheme.

**VI. Comments, Conclusions, and Work in Progress**

The paper discusses a general approach for improving the performance of Multiple Model Adaptive Control (MMAC) systems, when the number of models \( N \) that can be used is limited by practical considerations. The methods suggested earlier [1], [2] result in satisfactory performance when one of the fixed models lies in the neighborhood of the plant in the domain \( D \) of uncertainty in parameter space. The methods suggested represent different ways of accomplishing this objective. Redistribution of fixed models after the system has been in operation for a finite interval \( T \) was suggested in Section III. Methods for determining whether or not the plant lies in a subregion \( D \subset D \) was also considered in that section, but no proofs were provided. In both cases the objective is to reduce the region of uncertainty.

In Section IV a “convex hull property” of multiple adaptive models was exploited. It was shown that the plant lies in the convex hull of the parameters of the different models, and this property was in turn used to define a virtual model of the unknown plant for controlling it in a stable fashion.

In all the methods proposed, it was assumed that the state variables of the plant are accessible so that models can be initialized to have the same initial condition as the plant. Work is currently in progress to relax this condition. In particular, how the “convex hull property” can be used when only the input and the output of the plant are accessible is currently being investigated.

The paper merely introduces a new idea for second level
adaptation when multiple models are used. Several different adaptive schemes were proposed and in each case a problem was suggested that needs further investigation. Among the different approaches proposed, the one based on the “convex hull property” appears to be the most attractive one from a theoretical standpoint. As stated in the paper it leads to a time-varying estimate of the plant parameter which converges to the desired value if the input is persistently exciting. If the estimate is used to control an unstable plant, the stability of the overall system needs to be established. Work is currently in progress in this area, particularly when the input is not time-varying. Extensive simulation studies are also being carried out in the presence of noise and disturbances to determine which of the methods proposed are practically viable and hence deserving of further theoretical study.

VII. ACKNOWLEDGEMENT

The authors would like to thank the National Science Foundation for supporting the work reported here and Dr. Jovan D. Bošković for many useful discussions.

VIII. APPENDIX [6]

Definition: A set $K$ in a linear space $L$ is called convex if the line segment $ab$ is contained in $K$ for any elements $a, b \in K$, i.e. $x_t = (1 - t)a + tb \in K$ for any $a, b \in K$ and any $t \in [0, 1]$.

Definition: Let $a_1, a_2, \ldots, a_m$ be arbitrary elements of $L$ and $\lambda_1, \lambda_2, \ldots, \lambda_m$ be any real numbers. The element $a = \lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_m a_m$ is called a linear combination of $a_1, a_2, \ldots, a_m$. If all numbers $\lambda_1, \lambda_2, \ldots, \lambda_m$ are nonnegative and $\lambda_1 + \lambda_2 + \cdots + \lambda_m = 1$, then the element $a$ is called a convex combination of $a_1, a_2, \ldots, a_m$.

Lemma: Let $K$ be a convex subset of $L$. Then every convex combination of $a_1, a_2, \ldots, a_m \in K$ is also an element of $K$.

Theorem: The closure of any convex set in a linear topological space is convex.

REFERENCES


Fig. 4: System Response.