A Control Strategy for a Class of Cascade Systems Including Saturation Elements

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Abstract. We are considering the problem of controlling systems including, in cascade, two dynamic linear subsystems and two static nonlinear elements of the saturation type. Furthermore, the second dynamic subsystem is allowed to be of type one (i.e. include an integrator). The problem is dealt with designing a cascade control strategy consisting of two main control loops. The control signal generated by the outer loop acts as reference input for the inner loop which in turn generates the actual control signal applied to the system. It is formally shown that proposed cascade controller stabilizes the resulting closed-loop system and ensures perfect tracking of step-like output reference signals that are compatible with the system saturating elements.

I. INTRODUCTION

Saturation elements are unavoidable in control systems. They generally arise from those technological limitations introduced for system operation safety (e.g. limits of valve opening, stops in mechanical systems, voltage limits and saturations in power amplifiers ...). In some control problems, limitations are deliberately introduced as part of the control strategy, in the form of input or state constraints, in order to meet performance requirements (e.g. keeping a system around an operation point so that e.g. linear approximation can be considered). As long as the control input is concerned, soft saturations (i.e. those constraints introduced as control design features) can be interpreted as hard saturations (originating from actuator limitations) and vice-versa. This is not the case when internal signals are considered. For instance, a linear system remains linear even when some of its internal state variables are subject to saturation constraints. Contrarily, the presence of internal hard (i.e. technological) saturation elements in a system makes this nonlinear even if the rest of its components are linear.

Most of previous works on control system involving saturation have been focusing on the case of linear systems subject to actuator saturation, see e.g. the textbooks of Glattfelder and Schaufelberger (2003), Hu and Lin (2001), Saberi et al (1999). Then, the main issue is that feedback is broken whenever the actuator saturates causing performance deterioration. One possible way to deal with such issue consists in finding an invariant subset of the state space where the considered state feedback controller never saturates. Using LQ or GPC design techniques, it is possible to find controllers that ensure the positive invariance of the closed-loop linear region or a subset of it including all admissible initial conditions. Such a positive invariant set is then a local region of stability. The survey of Blanchini (1999) gives a comprehensive description of this approach and recent related results can be found in e.g. (Zhao and Lin, 2008), (Wang et al, 2008). The second approach to handle input saturation constraint is one that allows the control signal to saturate. It is clear that, during the time intervals where the control signal is saturating, the closed-loop system is no longer linear. Global signal boundedness is then only possible for linear systems with all poles in the closed stability region, provided discontinuous or nonlinear regulators are used (Sussmann et al, 1994), (Suarez, 1997).

If one further seeks asymptotic stabilization then the controlled system must be marginally-stable e.g. type one (Chaoui et al., 2001).

In this paper, the focus is made on systems with NLNL structure of Fig. 1. Such structure involves, in addition to input actuator saturation, an inner hard saturation element, flanked by two dynamic linear subsystems. Furthermore, the second linear subsystem is allowed to be of type 1. In fact, the NLNL system structure may capture quite different practical situations. For instance, this may be viewed as a cascade of two physical systems, namely $N_1L_1$ and $N_2L_2$, each one being composed of one actuator modeled by a simple saturation element ($N$) and one dynamic linear subsystem ($L$). A different situation is that where only one physical system is involved that is composed of one actuator and one dynamic linear subsystem. Specifically, the actuator is represented by the cascade $N_1L_1N_2$ (see Fig. 1) and the dynamic subsystem is represented by the type 1 dynamic subsystem $L_2$. This actuator modelling is more accurate than the standard way reducing it to a simple saturation ($N$). Actually, all actuators involve dynamic components and output saturation behavior and a minimal model should be $L_1N_2$. The additional input saturation ($N_i$) in Fig. 1 then may account for a possible (soft) input constraint. It is clear that the actuator model reduces to a single saturation element ($N_i$) whenever their dynamics (represented by $L_i$) are much more rapid than the system dynamics ($L_2$). Otherwise, the more accurate model $N_1L_1N_2$ must be based upon. Finally, it is worth pointing out that the control approach and results that will be later for systems with $NLNL$ structure can be easily adapted to simpler structures such as $LN$, $NLAN$ or $LNL$. Note that the $LN$ structure is suitable for systems with saturating sensor and the $NLN$ one is convenient for systems with both actuator and sensor saturations (provided actuator and sensor dynamics are much more rapid than the system dynamics).
To stabilize the NLNL system of Fig. 1 and enforce its output to perfectly track any step-like reference signal, a cascade control strategy is resorted. This involves two saturated regulators designed in such a way that a suitable closed-loop pole placement is met in case the control action stops saturating. It is formally shown, using input-output stability tools, that such pole placement objective is actually met asymptotically implying the achievement of closed-loop asymptotic stability and output reference asymptotic tracking. A crucial step to reach this achievement is to prove asymptotic stability and output reference asymptotic met asymptotically implying the achievement of closed-loop output to perfectly track any step-like reference signal, a stability tools, that such pole placement objective is actually closed-loop pole placement is met in case the control action saturated regulators designed in such a way that a suitable cascade control strategy is resorted. This involves two

Fig. 2: Connected NL systems interpretation

class of controlled systems

A. Class of Controlled Systems

We are interested in NLNL cascade systems (Fig. 1) where the different elements are described, in discrete-time, as follows:

\[ N_1: \quad u_1 = \text{sat}(v_1, u_{1\text{max}}) = \min\{v_1, u_{1\text{max}}\} \quad \text{sign}(v_1) \]  

\[ L_1: \quad y_1(t) = \frac{B_1(q^{-1})}{A_1(q^{-1})} u_1(t) \]  

\[ N_2: \quad u_2 = \text{sat}(y_1, u_{2\text{max}}) = \min\{y_1, u_{2\text{max}}\} \quad \text{sign}(y_1) \]  

\[ L_2: \quad y_2(t) = \frac{B_2(q^{-1})}{A_2(q^{-1})} u_2(t) + w_2(t) \]

where the real constants \(u_{1\text{max}}\) and \(u_{2\text{max}}\) define the maximal amplitudes imposed by the saturating elements on \(u_1(t)\) and \(u_2(t)\), respectively. \(w(t)\) is a bounded sequence accounting for external disturbance (and possibly modeling errors). \(A_i(q^{-1}), B_i(q^{-1}), A_2(q^{-1}), B_2(q^{-1})\) are polynomials in the backward shift operator \(q^{-1}\), of the form:

\[ A_i(q^{-1}) = 1 + a_{i1}q^{-1} + a_{i2}q^{-2} + \ldots + a_{i\text{max}}q^{-i\text{max}} \]  

\[ B_i(q^{-1}) = q^{-\text{d}}(b_{i0} + b_{i1}q^{-1} + b_{i2}q^{-2} + \ldots + b_{i\text{max}}q^{-i\text{max}}) \]  

\[ A_2(q^{-1}) = (1 - q^{-1})(1 + a_{21}q^{-1} + a_{22}q^{-2} + \ldots + a_{2\text{max}}q^{-2\text{max}}) \]  

\[ \Delta(q^{-1}) = A_2(q^{-1}) \]  

\[ B_2(q^{-1}) = q^{-\text{d}}(b_{20} + b_{21}q^{-1} + b_{22}q^{-2} + \ldots + b_{2\text{max}}q^{-2\text{max}}) \]

with

\[ \Delta(q^{-1}) = 1 - q^{-1} \]

All model parameters i.e. the integers \(na_i, nb_i, d_i\) \((i = 1, 2)\) and the real coefficients \(a_{ij}, b_{ij}\) are constant and known. The structure parameters \(na, nb, d_i\) \((i = 1, 2)\) and \(A_2(q^{-1}), B_2(q^{-1})\) are coprime.

Remarks 1.

a) Assumption A1 is justified by the fact that we are seeking global stabilisation of the system using saturated linear state feedback. Then, unstable systems must be discarded.

b) Assumption A2 ensures the controllability of \(B_i(q^{-1})/A_i(q^{-1})\) and \(B_2(q^{-1})/A_2(q^{-1})\). This could be slightly relaxed letting \(A_i(q^{-1})\) and \(B_i(q^{-1})\) \((\text{resp.} A_2(q^{-1})\) and \(B_2(q^{-1})\)\), have common zeros in the open unit disk. Such common zeros can always be cancelled in the model \((1a-c)\) leading to an exponentially vanishing terms in \((1b)\) and \((1c)\). These eventual terms are accounted for through the disturbance \(w(t)\).

c) As mentioned in the introduction, many different situations may be captured through the model \((1a-d)\). For instance, the model may be viewed as a cascade of two NL systems each one composed of an actuator (reduced to a hard saturation) and a linear dynamic subsystem (Fig. 2). A different situation, illustrated by Fig. 3, is that where the whole model describes a single system composed of an actuator (represented by the block \(L_1N_2\)) and a linear dynamic block \((L_2)\). Then, the element \((N_1)\) is not a hard saturation, it accounts in the model for control input constraint imposed by the user. As pointed out in the introduction the interpretation of Fig. 3 makes a sense when the actuator dynamics \((L_1)\) are not too rapid compared to the remaining dynamics \((L_2)\). Obviously, the signal \(y_1(t)\) is not accessible to measurement in this case.
B. Control Objective

We aim at designing, for the class of systems described in the previous subsection, a controller able to stabilize globally and asymptotically the overall closed loop and enforce the output \( y_2(t) \) to track, as closely as possible, any reference signal \( y_{r2}(t) \).

The point is that perfect tracking is impossible to reach, even in ideal situations (no disturbance and no saturation), for arbitrary shape reference sequences and disturbances, except if the controlled system is minimum phase. As the system is presently not necessarily minimum phase, the above ambitious tracking objective must be abandoned. Then, the focus is made on a narrower, but still wide and practical, class of references and disturbances. Specifically, we seek the tracking of references that are slowly varying in the mean. To formalize this objective, we make use of the following definition of smallness in the mean.

**Definition 1** ([11]). 1) Let \( \alpha \) be any real number and \( s \) any real sequence. \( \alpha \) is said to be \( \alpha \)-small in the mean (briefly \( \alpha \)-SM), if:

\[
\limsup_{h \to \infty} \frac{1}{h} \sum_{k=h}^{h+k} s(t) \leq \alpha \quad \text{(for all } h \in \mathbb{N}).
\]

For a given \( \alpha \), the set of all \( \alpha \)-SM sequences is denoted \( \text{SM}(\alpha) \). 2) The mean size of a bounded sequence \( s \) is the smallest real \( \alpha \) such that \( \alpha \in \text{SM}(\alpha) \). Its mean rate is the smallest real \( \mu \) such that the increment sequence is \( \text{SM}(\mu) \).

Let \( \mu_{yr} \) and \( \mu_w \) denotes the reference and disturbance mean rates. These are the smallest real numbers such that:

\[
\begin{align*}
\left\{ (y_{r2}(t) - y_{r2}(t-1))^2 \right\} & \in \text{SM}(\mu_{yr}) \\
\left\{ (w(t) - w(t-1))^2 \right\} & \in \text{SM}(\mu_w)
\end{align*}
\]

It is more convenient to consider the maximal rate:

\[
\mu = \max(\mu_w, \mu_{yr})
\]

With these definitions, the control objective can be reformulated as follows: find a controller such that there exist real constants, say \( \mu^* > 0 \) and \( \mu^* > 0 \), so that if \( 0 \leq \mu \leq \mu^* \) then one gets \( \left\{ y_2(t) - y_{r2}(t) \right\} \in \text{SM}(\mu^*) \). That is the smaller the mean rate of the external signals (reference and disturbance) the better the mean tracking quality.

III. INNER LOOP DESIGN AND ANALYSIS

To achieve the above control objective, a cascade control strategy will now be developed. This consists of two control loops (Fig. 4): (i) an inner loop that generate the control action \( v_1(t) \) using (an estimate of) the intermediate output \( y_1(t) \); (ii) an outer loop generating the reference input of the inner regulator, using the system measured output \( y_2(t) \).

A. Inner Loop Design

As \( A_1(q^{-1}) \) is Hurwitz, equation (1b) suggest the following estimator of \( y_1(t) \):

\[
A_1(q^{-1})y_1(t) = B_1(q^{-1})u_1(t)
\]

Combining equations (1b) and (4a), one gets:

\[
A_2(q^{-1})e_1(t) = 0 \quad \text{with } e_1(t) = \hat{y}_1(t) - y_1(t)
\]

This implies that \( e_1(t) \) vanishes exponentially because \( A_2(q^{-1}) \) is Hurwitz.

Now, let \( P_1(q^{-1}) \) be any Hurwitz polynomial of the form:

\[
P_1(q^{-1}) = 1 + p_1q^{-1} + p_2q^{-2} + \ldots + p_{np1}q^{-np_1}
\]

with \( np_1 \leq 2n_{1-1} \) and \( n_{10} = \max\{np_{10} + 1, nb_{10} + d_1\} \). As \( A(q^{-1})A_1(q^{-1}) \) and \( B_1(q^{-1}) \) are coprime (by assumption A3), there are unique polynomials \( R_1(q^{-1}) \) and \( S_1(q^{-1}) \):

\[
R_1(q^{-1}) = 1 + r_1q^{-1} + r_2q^{-2} + \ldots + r_{n1-1}q^{-n1+1}
\]

\[
S_1(q^{-1}) = s_1 + s_1q^{-1} + s_2q^{-2} + \ldots + s_{n1-1}q^{-n1+1}
\]

that verify the Bezout equation:

\[
R_1(q^{-1})A(q^{-1}) + S_1(q^{-1})B(q^{-1}) = R_1(q^{-1})
\]

(8)

Let \( y_{r1}(t) \) denote the reference sequence of the internal loop to be generated by the outer loop (Fig. 4). The inner loop must act so that the inner output \( y_1(t) \) follows as closely as possible its reference \( y_{r1}(t) \). This tracking problem is converted into a regulation problem involving the inner tracking error \( \hat{y}_1(t) = y_1(t) - y_{r1}(t) \). Operating \( A_1(q^{-1}) \Delta(q^{-1}) \) on \( \hat{y}_1(t) \) and using equation (1b), yields:

\[
A_2(q^{-1})\Delta(q^{-1})\hat{y}_1(t) = B_1(q^{-1})\Delta(q^{-1})u_1(t) - A_1(q^{-1})\Delta(q^{-1})y_{r1}(t)
\]

(9)
Equation (9) is seen as a virtual system with input $\Delta(q^{-1})u_1(t)$ and output $\tilde{y}_1(t)$. The term $A_1(q^{-1})\Delta(q^{-1})y_{r1}(t)$ stands as a disturbance. To steer $\tilde{y}_1(t)$ to zero, we propose the following saturated regulator:

$$v_1(t) = (1 - R_1(q^{-1})\Delta(q^{-1}))u_1(t) - S_1(q^{-1})\tilde{y}_1(t) - y_{r1}(t)$$  \hspace{0.5cm} (10)

where:

$$u_1(t) = \text{sat}(v_1(t), u_{1\max})$$  \hspace{0.5cm} (11)

due to (1a). Equations (10-11) define the internal regulator.

**Remark 2.** In case the above regulator stops saturating it becomes:

$$R_1(q^{-1})\Delta(q^{-1})u_1(t) + S_1(q^{-1})(\tilde{y}_1(t) - y_{r1}(t)) = 0$$  \hspace{0.5cm} (12)

This is a unit linear feedback with integral action of pole placement type (due to (8)). where $v_1(t)$ is the action to apply at the input of the upstream saturation element.

**B. Internal Loop Analysis**

Recall that the internal loop is expected to enforce the inner output $y_1$ to track as closely as possible the reference $y_{r1}$, yet to be generated. The point is that the inner output is obtained from a constrained input ($|u_1| \leq u_{1\max}$) through the linear system $L$. Therefore, the mentioned tracking objective is only achievable if the reference $y_{r1}$ is compatible with the constraint. This is formally described in the following proposition where the tracking quality is characterized using the square mean rate of $y_{r1}$; this is the smallest real $\mu_{y_{r1}} \geq 0$ such that:

$$\left\{ |y_{r1}(t) - y_{r1}(t-1)|^2 \right\} \in SM(\mu_{y_{r1}})$$  \hspace{0.5cm} (13)

**Proposition 1.** Consider the closed loop system consisting of: (i) the subsystem $N_1L_1$ described by (1a-b) and subject to assumptions (A1-A2); (ii) the saturated regulator (10)-(11). Suppose the reference $y_{r1}$, that has yet to be generated, is bounded and compatible with the constraint $|u_1| \leq u_{1\max}$ in the sense that:

$$\sup_{0 < t < \infty} |y_{r1}(t)| < \left( \frac{B_1(1)}{A_1(1)} \right) u_{1\max}$$  \hspace{0.5cm} (14)

Then one has the following properties:

a) All sequences of the internal loop remain bounded.

b) If further

$$\inf_{0 \leq \omega < 2\pi} |\text{Re} \left( \frac{P_1(e^{-j\omega})}{A_1(e^{-j\omega})} \right)| > 0$$  \hspace{0.5cm} (15)

then there exist real constants $K_{y_{r1}}^*$ and $K_{y_{r1}}^2$, independent on $\mu_{y_{r1}}$, such that:

$$\left\{ |\Delta(q^{-1})\tilde{y}_1(t)|^2 \right\} \in S(K_{y_{r1}}^*)$$

$$\left\{ |y_1(t)|^2 \right\} \in S(K_{y_{r1}}^2)$$  \hspace{0.5cm} (16)

c) If $\left\{ |\Delta(q^{-1})y_{r1}(t)| \right\} \in l_2$ then $\left\{ |\Delta(q^{-1})\tilde{y}_1(t)| \right\} \in l_2$ and, consequently, $|\tilde{y}_1(t)|$ converges to 0.

This proposition is established using tools from the theory of input-output stability, [1],[2]. The proof follows similar arguments as theorem 1 to come. For space limitation, the proof of this proposition is not given.

**IV. OUTER LOOP DESIGN AND ANALYSIS**

**A. Outer Loop Design**

The outer loop aims at generating the inner loop reference $y_{r1}$ so that the global system output $y_2$ tracks as closely as possible its reference $y_{r2}$. It is now designed using a specific closed loop pole placement. Consider any Hurwitz polynomial of the form:

$$P_{20}(q^{-1}) = 1 - p q^{-1}$$  \hspace{0.5cm} (17)

As $A_{20}(q^{-1})$ and $B_{21}(q^{-1})$ are coprime (by assumption A2), the Bezout equation:

$$A_{20}(q^{-1})\Delta(q^{-1})R_2(q_{21}) + B_{21}(q^{-1})S_2(q_{21}) = A_{20}(q^{-1})P_{20}(q^{-1})$$  \hspace{0.5cm} (18)

has a unique solution of the form:

$$R_2(q_{21}) = 1 + r_2 q_{21}^{-1} + r_2 q_{21}^{-2} + K + r_{n21} q_{21}^{-n_{211}}$$

$$S_2(q_{21}) = s_0 + s_{21} q_{21}^{-1} + s_{22} q_{21}^{-2} + K + s_{n21} q_{21}^{-n_{211}}$$  \hspace{0.5cm} (19)

Then, the outer loop is defined by the following control law:

$$y_2(t) = (1 - R_2(q^{-1}))u_2(t) - S_2(q^{-1})\tilde{y}_2(t)$$  \hspace{0.5cm} (20a)

$$\tilde{y}_2(t) = y_2(t) - y_{r2}(t)$$  \hspace{0.5cm} (20b)

$$y_{r2}(t) = \text{sat}(y_2(t), y_{r2\max})$$  \hspace{0.5cm} (20c)

with

$$y_{r2\max} = \min \left( \frac{B_1(1)}{A_1(1)}, u_{1\max}, u_{2\max} \right)$$  \hspace{0.5cm} (21)

**Remarks 3.**

a) In case the above control law does not saturate, it simplifies to the following unitary linear feedback:

$$R_2(q^{-1})u_2(t) + S_2(q^{-1})\tilde{y}_2(t) = 0$$  \hspace{0.5cm} (22)

Then, it can be checked that the closed loop system undergoes the following equations:

$$\tilde{y}_2(t) = \frac{-R_2(q^{-1})A_{20}(q^{-1})}{P_{20}(q^{-1})A_{20}(q^{-1})}\Delta(q^{-1})(y_{r2}(t) - w(t))$$  \hspace{0.5cm} (23)

$$u_2(t) = \frac{S_2(q^{-1})A_{20}(q^{-1})}{P_{20}(q^{-1})A_{20}(q^{-1})}\Delta(q^{-1})(y_{r2}(t) - w(t))$$  \hspace{0.5cm} (24)

These show that if the external inputs ($y_{r2}$ and $w$) are
constant, then the tracking error \( \tilde{y}_2 \) vanishes asymptotically.

b) Although it is commonly known that using the inverse system dynamics as a controller could result in a non-robust feedback controller, let us notice that the condition (15) made upon the inner closed-loop characteristic polynomial shows that a robust behavior is here ensured as this condition actually leads to an internal control which is known guaranteeing more robustness to the control system.

**B. Global Closed Loop Analysis**

**Theorem 1.** Consider the closed loop system consisting of:
(i) the system \( N_1L_1N_2L_2 \) described by (1a-d) and subject to assumptions (A1-A2); (ii) the inner regulator (10-11) and the outer regulator (20a-c). Suppose the reference sequence \( y_{r_2}(t) \) is compatible with the constraint \( |\tilde{r}_2| \leq u_{2\max} \) in the sense that:

\[
\sup_{0 \leq t < \infty} |y_{r_2}(t)| < \left( \frac{B_2(t)}{A_2(t)} \right) u_{2\max} \tag{25}
\]

1) All the sequences of the closed loop remain bounded, whatever the initial conditions.

2) There exists a real \( K^* \), independent on \( \mu \), such that:

- \( \{ |u_2(t)| \} \in S(K^* \mu) \), \( \{ |y_2(t)| \} \in S(K^* \mu) \) and
- \( \{ |\tilde{y}_2(t)| \} \in S(K^* \mu) \)

3) If further the external inputs are asymptotically slowly varying i.e.

\[
\limsup_{t \to \infty} |y_{r_2}(t) - y_{r_2}(t-l)| \leq \mu
\]

and

\[
\limsup_{t \to \infty} |w(t) - w(t-l)| \leq \mu
\]

Then, one has:

\[
\limsup_{t \to \infty} |u_2(t)| \leq K^{**} \mu, \limsup_{t \to \infty} |v_2(t)| \leq K^{**} \mu
\]

and

\[
\limsup_{t \to \infty} |y_{r_2}(t) - y_{r_2}(t)| \leq K^{**} \mu
\]

for some real constant \( K^{**} > 0 \), independent on \( \mu \).

This theorem is also established using flat tools from the theory of input-output stability (see i.e. [1],[2]) and type 1 systems properties. The proof follows similar arguments but it is too long. Thus, for space limitation, its proof is not given. Nevertheless, the authors welcome any reader of this paper who expresses his wish to receive it.

**Remarks 4.**

a) The proof of Theorem 1 is a bit long and technical. In the case the reader is interested in, he can just ask the authors for. It is omitted due to space limitation. The major part of this proof concerns in fact Part 1. The positioning of the saturation element \( N_2 \) just above the subsystem \( L_1 \) makes it by no mean obvious to establish the boundedness of all closed-loop sequences, due to the type-1 nature of \( L_1 \).

b) Part 2 and 3 of the theorem show that the quality of output-reference tracking depends on the rates of the closed-loop external inputs i.e. \( y_{r_2}, w \). The slower these inputs are varying (in the mean or asymptotically), the best the tracking quality.

c) Theorem 1 relies on Proposition 1 and both hold provided that the positive real condition (15) is satisfied. In fact, these define a neighbourhood of the open-loop poles (i.e. zeros of \( A_i(q^{-1}) \)) within which should be placed the closed-loop poles (i.e. zeros of \( P_i(q^{-1}) \)).

**V. SIMULATION**

The controlled system is described by (1a-d)-(2a-e) with:

\[
A_i(q^{-1}) = 1 + 0.65q^{-1}; \quad B_i(q^{-1}) = q^{-2}(1 + 3q^{-1})
\]

\[
A_2(q^{-1}) = (1 - q^{-1})(1 - 0.9q^{-1}); \quad B_2(q^{-1}) = q^{-3}[1 - 4q^{-1}]
\]

These polynomials satisfy assumptions A1-A2. Also, let us notice that \( B_1(q^{-1}) \) and \( B_2(q^{-1}) \) are not Hurwitz polynomials i.e. the system is not minimum phase. The saturation parameters \( u_{1\max}, u_{2\max} \) will be defined latter. For space limitation, the focus is made on the tracking capability of the cascade controller (developed in Sections III and IV). Then the output disturbance \( w \) is null. Following the above controller design method, the polynomials \( P_1(q^{-1}) \) and \( P_{20}(q^{-1}) \) have to be chosen bearing in mind conditions (15) and (26), on one hand, and the fact that the closed loop system should be as more rapid as possible, compared to the (stable part of the) controlled system. Accordingly, the following choices are made:

\[
P_1(q^{-1}) = 1; \quad P_{20}(q^{-1}) = 1 - 0.88q^{-1}
\]

Then, the Bezout equations (8) and (18) are solved with respect to \( R_i(q^{-1}), S_1(q^{-1}), R_2(q^{-1}) \) and \( S_2(q^{-1}) \). The cascade controller thus obtained is implemented, together with the controlled system, using Matlab software.

**A. Controller Tracking Performances**

The reference \( y_{r_2}(t) \) is a square sequence with period 500, switching between 1 and -1. The saturation elements are characterized by \( u_{1\max} = 0.01, u_{2\max} = 0.02 \).

The resulting controller performances are illustrated by Fig. 5. It is seen (Fig. 5a) that the system output \( y_2(t) \) tracks well its varying reference \( y_{r_2}(t) \), despite the saturation effect on the inner output \( y_1(t) \) (Fig. 5b) and the inner sequence \( u_2(t) \) and the control signal \( u_i(t) \).
Importance of the Real Positivity Conditions

To illustrate the importance of conditions (15) and (26), let the polynomials $P_1(q^{-1})$ and $P_{20}(q^{-1})$ be chosen as follows:

$$P_1(q^{-1}) = 1, \quad P_2(q^{-1}) = 1 - 0.97q^{-1} + 0.0792q^{-2}$$

It is easy to verify that (26) is not satisfied. A cascade controller is obtained from the above polynomials, using the design method of Sections III and IV. The reference sequence is similar to part B. The resulting performances are illustrated by Figs 6. Clearly, the tracking performances deteriorate drastically.

VI. CONCLUSION

The problem of controlling $N_1L_1N_2L_2$ systems has been addressed. The nonlinear elements are of saturation types. While $N_2$ is a hard saturation, $N_1$ may be hard or soft. Several practical situations can be captured through this model structure. Interestingly, the linear subsystem $L_2$ is allowed to be type-1. The control problem is coped with using a cascade controller involving two loops. Both loops involve a saturated linear regulator designed by the pole placement technique. The real positive condition (15) shows where the poles of the closed-loop system must assigned to. It is formally shown that the cascade controller globally stabilizes the controlled system and guarantees quite interesting output-reference tracking performances, provided the reference sequence $y_{r_2}$ is compatible with the limitations imposed by the saturation elements.

REFERENCES