Tuning of fractional PI controllers for fractional order system models with and without time delays

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Abstract—In this paper, a servo control strategy for tuning of fractional order PI (FO-PI) controllers is proposed for fractional order system models with and without time delays. The proposed strategy is based on a reference model, whose open-loop transfer function is given by Bode’s ideal transfer function, which provides an infinite gain margin and constant phase margin. To come up with satisfactory tuning parameters of the controller, an iterative optimization method is suggested, based on minimization of a quadratic cost function. This cost function is defined as weighted sum of squares of control input moves and sum of the integral squared error (ISE) between the time response of the reference model and the fractional system with the FO-PI controller. The resulting closed-loop system is shown to exhibit features of robustness to process gain variations and the step responses exhibit iso-damping property. Simulation results are presented and analyzed here to illustrate the effectiveness of the proposed algorithm.

I. INTRODUCTION

The idea of fractional calculus (FC), which is a generalization of classical integer order calculus to non-integer orders, is a very old topic in mathematics. Before the 19th century, the theory of fractional calculus developed mainly as a pure theoretical field of mathematics useful only for mathematicians. A significant amount of discussions aimed at this subject has been presented by [14] and [16]. It is only in the last few decades, that there has been an increasing amount of interest in fractional-order systems (also called fractal systems). This is because it was observed that there are many physical systems whose behavior could be better and more compactly represented using fractional system models rather than using classical integer order models. Several applications of fractional calculus can be found in [1]. In the area of automatic control systems, the application of FC can be found in [16], and

PID controllers are the most commonly used control algorithms in industry. So, a search for new algorithms for better design of these controllers has never been an unreasonable quest for the research community. The idea of fractional-order algorithms for the control of dynamic systems was first introduced by [15] where he also demonstrated the superior performance of the CRONE (French abbreviation for Commande Robuste d’Ordre Non Entier) method over the classical PID controller. Later, [10] proposed a generalization of the PID controller, namely the fractional order PID (FO-PID or PI\(^{\lambda}\)D\(^{\mu}\)) controller, involving an integrator of order \(\lambda\) and differentiator of order \(\mu\) (the orders \(\lambda\) and \(\mu\) may assume real non-integer values). The authors also demonstrated better performance of this type of controllers, in comparison with the classical PID controllers, when used for the control of fractional-order systems.

In the last decade many tuning rules have been proposed for designing FO-PID controllers. Some of these techniques are based on the extension of the classical PID control theory. There are several analytical ways to tune them [6]; [7]. [11] proposed the technique to tune fractional PID controllers by requiring them to satisfy certain conditions such as phase margin, gain crossover frequency and sensitivity function. The tuning parameters for the FO-PID is then obtained by solving the linear numerical optimization problem. In another paper by [12] the authors proposed a scheme to tune FO-PI controllers in order to fulfill three different robustness design specifications for the compensated system; an optimization method based on nonlinear minimization function subject to nonlinear constraints is used to tune the controller. [3] proposed a generalized MIGO (\(M_s\) based integral optimization) based controller tuning method called F-MIGO, to handle the FO-PI case given the fractional order of the system. Also, [17] proposed two sets of tuning rules for FO-PID with the proposed rules bearing similarities to the rules proposed by Ziegler and Nichols for integer PID, and made use of the same plant time response data. [19] proposes FO-PID tuning rule for a class of fractional order system. A fractional-order control strategy known as fractional sliding mode control has also been successfully applied in the control of a power electronic buck converter [5]. The recent paper by [13] summarizes many of the recent advances and applications for tuning of fractional order controllers in detail.

In many of these papers, the focus has been to design fractional order controllers for integer order plants to enhance robustness or the closed loop system performance. However, it is intuitively true, as also argued in [16], that the fractional order models require much more than classical PID controllers to achieve good closed loop performance. Also, [8] argued that fractional order control is ubiquitous when the dynamic system is of distributed parameter nature. In this paper, a tuning strategy for fractional order PI controllers is proposed for fractional order system models. The proposed strategy is based on a reference model, whose open-loop transfer function is given by Bode’s ideal transfer function. To obtain the tuning parameters of the controller, an iterative
optimization method is used, based on minimization of a quadratic cost function. This cost function is defined as weighted sum of square of control input moves and sum of the integral square error (ISE) between the time response of the reference model and the fractional system with the FO-PI controller. Thus we want to match the closed loop response to that of a reference system and also penalize the control input moves. This work extends the tuning strategy proposed by [2] for tuning integer order PID controllers for integer order models to designing FO-PI controllers for the fractal systems. The resultant closed-loop system (with the FO-PI controller) exhibits the features of robustness to gain variations and the step responses exhibit the iso-damping property.

This paper is organized as follows. Section II presents a brief theory of fractional calculus with an introduction to fractional order controllers. The proposed tuning strategy is presented in Section III. To study the efficacy of the proposed strategy developed in Section III, some examples of fractional order model systems are studied in Section IV to illustrate its applicability. Concluding remarks appear in Section V.

II. FRACTIONAL CALCULUS THEORY

Fractional calculus is a generalization of integration and differentiation to non-integer orders. The two most popular definitions used for the general fractional differintegral are the Grünwald-Letnikov (GL) discrete form of the definition and the Riemann-Liouville (RL) definition ([14]). The GL definition for a function $f(t)$ is given as

$$D^\lambda f(t) = \lim_{h\to\infty} \frac{1}{h^\lambda} \sum_{i=0}^{\infty} \left[ (-1)^i \binom{\lambda}{i} f(t - ih) \right]$$

where

$$\binom{\lambda}{i} = \frac{\Gamma(\lambda+1)}{\Gamma(i+1)\Gamma(\lambda-i+1)}$$

and the operator $D^\lambda$ defines fractional differentiation or integration depending on the sign of $\lambda$, $\Gamma(\cdot)$ being the well known Euler’s Gamma function and $h$ is the finite sampling interval. This definition is particularly useful for digital implementation of fractional order controllers.

For convenience, the Laplace domain notation is usually used to describe fractional differ-integral operation. When the initializations are assumed to be zero, we have

$$L\{D^\lambda f(t)\} = s^\lambda F(s) \quad (\lambda \in \mathbb{R})$$

The generic single-input single-output (SISO) fractional order system representation in the Laplace domain is given as

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_0 s^{\beta_0} + b_1 s^{\beta_1} + \ldots + b_m s^{\beta_m}}{1 + a_1 s^{\alpha_1} + \ldots + a_n s^{\alpha_n}}$$

where $b_0, b_1, \ldots, b_m$ and $\alpha_1, \alpha_2, \ldots, \alpha_n$ are constant model parameters or model coefficients, while $\beta_0 < \beta_1 < \ldots < \beta_m$ and $\alpha_1 < \alpha_2 < \ldots < \alpha_n$ are the fractional powers or fractional orders (real numbers). The transfer function given by equation (4) can be classified as either a commensurate transfer function or a non-commensurate transfer function. The transfer function (4) is called commensurate when $\beta_j, \alpha_i$ are integral multiple of a single real (fractional) number and it is called non-commensurate when $\beta_j, \alpha_i$ can take any arbitrary values.

A fractional-order PI$^\lambda$D$^\mu$ controller is considered as the generalization of the conventional PID controllers involving an integrator of order $\lambda$ and a differentiator of order $\mu$. The structure of PI$^\lambda$D$^\mu$ controller with the transfer function $(C(s))$ is given as

$$C(s) = K_c + \frac{K_i}{s^\lambda} + K_d s^\mu$$

where $K_c$, $K_i$ and $K_d$ are the proportional gain, integral gain and derivative gain, respectively of the fractional order controller. The main advantage of using fractional-order PID controllers for a linear control system is that we have more degrees of freedom in the controller design using the additional parameters of the integral and differential orders and, as a consequence, it is expected that the use of FO-PID controllers can enhance the feedback control loop performance over the integer-order controllers. In this work, we study the problem of designing a fractional order proportional-integral controller ($\mu = 0$) of the form

$$C(s) = K_c + \frac{K_i}{s^\lambda}$$  (6)

For digital implementation of the fractional order operator, the key step is numerical evaluation or discretization of the operator. Power series expansion and continued fraction expansion (CFE) of the Euler’s, Tustin and Al-Alaoui operators give different discrete approximations of the fractional order operator. These methods for discretization are suitable for realization and implementation of fractional-order controllers. The details for many discretization schemes can be found in [18]. However, sometimes frequency domain fitting in the continuous time domain of this fractional order operator is done first and then discretization of this transfer function is done to get the discrete approximation. One of the good continuous approximation for this fractional order operator is the Oustaloup continuous approximation [15] where it makes use of a recursive distribution of poles and zeroes. We will be using the approximation for GL definition(1) for the simulation of fractional order controllers as well as the closed loop systems. Note that the proposed method is independent from the way fractional differentiation and integration are simulated in the time-domain.

III. ALGORITHM FOR FO-PI CONTROLLER TUNING

A. Bode’s ideal transfer function and design of a FO-PI controller

Bode in his study on design of feedback amplifiers ([4]), suggested an ideal shape of the open-loop transfer function of the form:

$$G_{ref}(s) = \frac{K_r}{s^\gamma} \quad (1 < \gamma < 2)$$

This open loop transfer function with gain $K_r$, and fractional order $\gamma$ shows very interesting properties as listed in Table I.
If we consider a feedback system with Bode’s ideal transfer function inserted in the forward path, then based on the frequency domain analysis, this closed loop system exhibits important properties such as infinite gain margin and constant phase margin (dependent only on $\gamma$). These properties are listed in Table II.

$$G_{refCL}(s) = \frac{K_r}{s^\gamma + K_r}$$ (8)

Thus, this closed-loop system (equation 15) is robust to process gain variations and the step response exhibits iso-damping property. As for this reference system, the order $\gamma$ and the gain $K_r$ establish the overshoot and the speed of the output response, respectively.

### Table II

**Properties of feedback system with Bode’s ideal transfer function**

<table>
<thead>
<tr>
<th>Property</th>
<th>Relation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gain margin, $A_m$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>Phase margin, $\Phi_m$</td>
<td>$\pi(1 - \frac{\gamma}{2})$</td>
</tr>
<tr>
<td>Overshoot, $M_p$</td>
<td>$\approx 0.8(\gamma - 1)(\gamma - 0.75)$</td>
</tr>
</tbody>
</table>

The motivation for using Bode’s ideal transfer function can be demonstrated with the following examples for fractal systems. If $C(s)$ is the controller transfer function and $G(s)$ is the plant transfer function, then for the case when we have the open loop transfer function ($G_{OL}(s) = G(s)C(s)$) close to the $G_{ref}$, the closed loop response of this system will also behave like the closed loop response of the reference system giving us the important properties of the reference system. Thus, if $G_{OL}(s)$ is close to $G_{ref}$ then

$$G(s)C(s) \sim G_{ref}(s)$$ (9)

For the fractals systems with the transfer function model

$$G(s) = \frac{K}{\tau s^\alpha + 1}$$ (10)

then

$$C(s) \sim \frac{K_r}{K} \left( \frac{\tau}{s^{\gamma-\alpha}} + \frac{1}{s^\gamma} \right) \forall \gamma > \alpha$$ (11)

which is a FO-PI controller with no proportional gain. Now, for the systems represented by the transfer function model

$$G(s) = \frac{K}{a_2.s^\beta + a_1.s^\alpha + a_0}$$ (12)

then

$$C(s) \sim \frac{K_r}{K} \left( \frac{a_2}{s^{\gamma-\beta}} + \frac{a_1}{s^{\gamma-\alpha}} + \frac{a_0}{s^\gamma} \right) \forall \gamma > \beta > \alpha$$ (13)

which is again a FO-PI controller with three fractional integrators. The purpose of demonstrating these examples is that we will always get a fractional order controller for fractal systems if we want the open loop transfer function close to Bode’s ideal transfer function. For our case if we fix the structure of the controller as the FO-PI controller with the transfer function given by

$$C(s) = K_c + \frac{K_i}{s^\lambda}$$ (14)

and we are interested in finding controller parameters such that for the case when there is a set point change, we get a closed loop response which resembles the closed loop response of the reference system. Thus, fixing $\gamma$ and $K_r$ fixes the gain and phase margins for the reference system and if the closed loop system (with FO-PI controller) closely resembles the reference system, then it also means fixing the phase and gain margins of the required closed loop system implicitly.

Here we extend the tuning strategy proposed by [2] for tuning integer order PID controllers for integer order models to designing FO-PI controllers for the fractal systems. We are interested in a tuning strategy for a set point tracking scenario. For the step change in the set point, the tuning parameters of FO-PI controller ($K_c, K_i, \lambda$) are obtained by minimizing a quadratic cost function. This cost function is defined as weighted sum of square of control input moves and sum of the integral square error (ISE) between the time response of the reference model and the fractional system with the FO-PI controller. Iterative non-linear optimization algorithm is used to simultaneously estimate all parameters of the FO-PI controller. The feedback loop of Figure 1 shows the details of this proposed scheme. The algorithm has also been discussed in detail in section C.
or process measurements. We need to modify the control structure to tune the controller for models with time delays. We can either change the structure of the reference model or use smith predictor type control structure to overcome the time delays that characterize these systems. If the time delays are known with a good degree of precision, we can incorporate this knowledge into our reference system and modify the structure of \( G_{ref.CL} \) as

\[
G_{ref.CL}(s) = \frac{K_r}{s^3 + K_r} e^{-Ls}
\]

(15)

Then the parameters are tuned according to this modified reference model. Smith predictor control formulation using this algorithm for tuning fractional order controllers has not been discussed in this paper due to lack of space.

B. Imposing constraints

Additional constraints can also be introduced to the above optimization problem. The constraints could be limits on the sensitivity function\((S)\) and the complimentary sensitivity function\((T)\):

\[
S = \frac{1}{1 + C(s)G(s)}
\]

(16)

\[
T = \frac{C(s)G(s)}{1 + C(s)G(s)}
\]

(17)

It may be required at some specified frequency \((\omega_h \text{ for } T, \omega_l \text{ for } S)\) that their magnitude be less than some specified gain:

\[
\frac{C(j\omega_h)G(j\omega_h)}{1 + C(j\omega_h)G(j\omega_h)} < H
\]

(18)

\[
\frac{1}{1 + C(j\omega_l)G(j\omega_l)} < L
\]

(19)

Please note that imposing these constraints in frequency domain where the objective function is defined in the time domain doesn’t affect the optimization algorithm.

C. Algorithm for the proposed method

The algorithm for finding the parameters for FO-PI controller \((\theta = [K_c, K_i, \lambda]^T)\) by imposing the constraints is given below. The two weighting matrices in the objective function are diagonal matrices and defined as \(W_y\) and \(W_u\) respectively.

**Step 1.** Fix \(K_r\) and \(\gamma\).

**Step 2.** Initialization: Initialize the algorithm with some initial value for \(\theta\). We can also start with tuning parameters for PI controller based on integer order approximation of the fractal system.

**Step 3.** Iterate on \(\theta\) based on the minimization of this constraint optimization problem

\[
\hat{\theta} = \arg\min_{\theta} \left[ \sum_{k=1}^{\infty} w_y(k)[Y(k, \theta) - Y_{ref}(k)]^2 + \sum_{k=1}^{\infty} w_u(k)[\Delta u(k, \theta)]^2 \right]
\]

subject to

\[
0 < \lambda < 2 \quad |T(j\omega_h)| < H \quad \text{and} \quad |S(j\omega_l)| < L
\]

where \(Y(t, \theta)\) and \(Y_{ref}(t)\) are the closed loop step responses of the system with FO-PI controller (with tuning parameter vector \(\theta\)) and the reference system, respectively. \(w_y\) and \(w_u\) are weights or penalties for the two terms.

IV. Simulation study

The efficacy of the proposed tuning method is demonstrated by carrying out FO-PI controller design on three different FO models. The truncation length of 50 is used for the GL approximation and time delays are modeled using first order pade’s approximation. Note that, we can design a classical PI controller for these processes only if an integer order approximation is available for these fractional models. It is not entirely fair to compare the closed loop performance of the designed FO-PI controller with some of the classical PI controller settings as we use an extra parameter for controller design and the comparison is shown only to justify the need to design a fractional order controller for these fractional system models. For regulatory control, the step change in input type disturbance is given at 1 sec time instant.

A. Example 1

First, we consider the simplest fractional order transfer function given as

\[
G_{FO_1}(s) = \frac{K}{s^{0.5} + 1}
\]

(20)

with \(K = 1\). The design specifications for the reference system are chosen as

- \(\gamma = 7/6\), or phase margin \(\sim 75^\circ\).
- \(K_r = 1.5\) or \(\omega_c = 1.41 rad/s\).

As we are using the discrete approximation, the sampling time is chosen as 0.01sec. We also imposed constraints on the closed loop system as

- \(|S(j\omega_l)| < -20dB\) at \(\omega_l = 0.01 rad/s\) and \(|T(j\omega_l)| < -10dB\) at \(\omega_l = 10 rad/s\).

Also, equal weighting matrices (equal to identity matrix) is used here in the objective function. For these specifications, the estimated tuning parameters of the FO-PI controller are

\[
C_{FO_1}(s) = 0.191 + \frac{2.813}{s^{0.97}}
\]

Next we examined the robustness property of this closed loop system by varying the process gain \((K)\) by +40% to -40% i.e. \(K = \{0.6, 0.8, 1, 1.2, 1.4\}\). The closed loop step responses and the Bode diagrams for the open loop systems are illustrated in Figures 2 and 3 respectively. The step responses show that the response maintains a constant overshoot to plant gain variations i.e. it has the iso-damping property. In the open-loop Bode plots it is seen that the phase curve is flat around the gain crossover frequency \(\omega_c\) and that the system has a phase margin of approximately 75°. These observations lead to the conclusion that the FO-PI controller, tuned by the proposed method makes the closed loop system robust against process gain variations and also exhibits the iso-damping property.
B. Example 2

This example of a heating furnace as considered in [10], can be modeled by the integer as well as fractional order differential equations. The fractional model is given as

$$G_{FO_2}(s) = \frac{1}{14994s^{1.31} + 6009.5s^{0.97} + 1.69}$$ \hspace{1cm} (22)

and its integer approximation as first order plus time delay model is given as

$$G_{IO_2}(s) = \frac{0.518}{2520.26s + 1}e^{-14.97s}$$ \hspace{1cm} (23)

The design specifications for the reference system are chosen as

- $\gamma = 4/3$, or phase margin $\sim 60^\circ$.
- $K_r = 0.0167$ or $\omega_c = 0.046$ rad/s.
- $|S(j\omega_l)| < -20dB$ at $\omega_l = 0.01$ rad/s and $|T(j\omega_l)| < -10dB$ at $\omega_l = 1$ rad/s.

The sampling time is chosen as 0.1 sec. For this process, we kept $W_i = I$ and $W_u = 0$. For these specifications, FO-PI controller obtained based on the proposed algorithm is,

$$C_{FO_2}(s) = 428.68 + \frac{41.89}{s^{0.638}}$$ \hspace{1cm} (24)

The closed loop step responses and the Bode plots for the open loop systems are illustrated in Figures 4 and 5 respectively. These plots shows that the closed loop system with FO-PI controller tuned by the proposed method, is robust against process gain variations and also exhibits the iso-damping property. As we have a integer order model for this process, we can also design a classical PI controller here and compare the closed loop performance for these controllers.

IMC PI controller tuning (using closed loop time constant as 60sec) and Hagglund and Astrom ([9]) PI controller settings are used for comparison. The tuning parameters are derived from the approximate integer order model for this system but is implemented on the real fractional system. The servo and regulatory performance from the three controllers is shown in Figure 6. It can be seen that the FO-PI controllers designed
order systems and provides both good servo and regulatory control.

C. Example 3

The FO system model and its integer model approximation are

$$G_{FO_i}(s) = \frac{K}{s^{0.5} + 1} e^{-0.5s}$$

$$G_{IO_i}(s) = \frac{K}{1.5s + 1} e^{-0.1s}$$

with $K = 1$. The design specifications for the reference system are chosen as

- $\gamma = 9/8$,
- $K_r = 1.492$ or $\omega_c = 1.43rad/s$,
- $|S(j\omega)| < -20dB$ at $\omega_l = 0.01 rad/s$

Again, equal weighing matrices (equal to identity matrix) is used here in the objective function. The sampling time is chosen as 0.01sec. For these specifications, FO-PI controller obtained based on the proposed algorithm is

$$C_{FO_i}(s) = 0.353 + \frac{1.505}{s^{0.905}}$$

We can compare the performance of this controller with results from standard tuning rules like IMC and Hagglund and Astrom (H&A) PI settings. Figure 7 shows the closed loop response from these three settings, H&A tuned controller has not been shown as it gave unstable response. The closed loop time constant for IMC is chosen as 0.67 sec. As can be seen, performance is worst with designed integer order controllers using integer approximations for these fractional system models.

![V. CONCLUSIONS](image)

In this paper, a fractional order PI controller design method is proposed for fractional order models. The proposed strategy is based on a reference model, whose open-loop transfer function is given by Bode’s ideal transfer function. The parameters of the controller are estimated by formulating a constrained non-linear optimization problem. The performance of the fractional order PI controller designed based on the proposed method has been demonstrated through three fractional order dynamic models. The resultant closed-loop system is robust to process gain variations and the step response exhibits iso-damping property. The proposed technique appears to have promise for the control of fractional order systems instead of designing a integer order counterpart. In the future, an interesting perspective would be designing PI$^\lambda$D$^\mu$ controllers for integer order models and also to apply them to control of some real experimental setup.

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REFERENCES


