Dilated LMI conditions for the robust analysis of uncertain parameter-dependent descriptor systems

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Abstract—This paper addresses the robust admissibility/$\mathcal{H}_2$ performance analysis for continuous-time uncertain parameter-dependent descriptor systems. In order to achieve less conservative results, the proposed approach uses parameter-dependent Lyapunov functions and slack variables. Our main contribution consists in proposing new necessary and sufficient conditions for the admissibility and $\mathcal{H}_2$ performance analysis of time-invariant singular systems. These conditions are formulated as a strict linear matrix inequality (LMI) solvability problem and represent a generalization to singular systems of some dilated LMI analysis results developed in the literature for state-space systems. Also, we have extended our results to the analysis of descriptor systems with time-varying parametric uncertainties. A numerical example shows the applicability of our approach.

Index Terms—Linear continuous-time descriptor systems, slack variables, dilated Linear Matrix Inequalities (LMIs).

I. INTRODUCTION

Due to their ability to describe many real-world systems (such as power systems, electrical networks, robotic systems, aircrafts, chemical processes and social economic systems [1]), descriptor systems have received considerable attention over the past two decades. The descriptor form provides a more general system representation than the classical state-space one as it allows to incorporate static (algebraic) constraints on physical variables [1], [2]. However, as pointed out in [3], the generalization of state-space systems theory to singular systems is a difficult problem. For instance, the stability/performance analysis is more complicated in the case of descriptor systems than it is when dealing with state-space ones. It requires to consider simultaneously the stability/performance robustness as well as the regularity and the absence of impulses (continuous-time systems) or the causality (discrete-time systems). In addition, as the Lyapunov matrix appearing in the study of singular systems is indefinite, the analysis results developed for state-space systems exploiting the symmetry and the positive-definiteness of the Lyapunov matrix cannot easily be generalized to descriptor systems. Despite these difficulties, many results developed for state-space systems have been extended to singular systems. In the case of uncertain descriptor systems, various problems have been addressed in the literature: robust stability (admissibility) [4]–[6] and root-clustering analyses [6], [7], robust stabilization [3], [8], [9], robust controllability/observability analysis [10] as well as $\mathcal{H}_2/\mathcal{H}_\infty$ norm characterizations and control [11], [12], positive real analysis and control [13]. Some of the proposed results make use of the linear matrix inequality (LMI) formulation incorporating either non-strict LMIs or equality constraints. Since LMI conditions with either non-strict LMIs or equality constraints may often be subject to numerical issues, the results presenting strict LMI conditions are more desirable.

It is well known that, when dealing with parameter-dependent systems, the use of parameter-dependent Lyapunov (PDL) functions reduces the conservatism of analysis approaches that are based on a single quadratic Lyapunov one [14]. Motivated by this fact, PDL functions (PDLF) have also been employed for the analysis of parameter-dependent descriptor systems [4], [5], [7], [11], [12], [15]. In general, PDLF-based analysis results are formulated as a LMI solvability problem. Exploiting these results for synthesis purposes leads to some solvability issues that are due to the coupling between the PDL matrix and the unknown synthesis matrices. The introduction of slack variables (leading to the so-called dilated LMI characterizations), in addition to increasing the degree of freedom of the design variables, allows to eliminate this coupling and replaces it with a coupling between a slack variable and the unknown synthesis matrices. Hence, the relaxation of the Lyapunov matrix leads to less restrictive analysis/synthesis conditions [16]–[19]. To our knowledge, the only results employing slack variables for the study of continuous-time parameter-dependent descriptor systems have been proposed in [12]. By considering time-invariant uncertainties, the authors have presented dilated LMI conditions for robust admissibility and $\mathcal{H}_2/\mathcal{H}_\infty$ performance analysis with an application to multiobjective state-feedback control.

In this paper, we address the robust admissibility/$\mathcal{H}_2$ performance analysis for continuous-time uncertain parameter-dependent descriptor systems by using PDL functions and slack variables. The uncertainties considered here are time-varying and include both polytopic and affine parameter-dependent uncertainties. Our contributions are two-fold. First, we propose new necessary and sufficient dilated LMI conditions for the admissibility and $\mathcal{H}_2$ performance analysis of linear time-invariant (LTI) descriptor systems. In addition to employing a slack variable, the proposed LMI conditions have the numerical advantage of being strict. They represent a generalization to singular systems of the dilated LMI stability/performance analysis results developed in [17] for state-space systems. Second, we extend these results to the robust admissibility/performance analysis of uncertain parameter-dependent descriptor systems and discuss the application of
the analysis results to robust state-feedback control design. Compared to the approach presented in [12], the class of systems considered in our paper is broader. Indeed, the uncertainties considered in this paper are time-varying and all the descriptor state-space matrices are parameter-dependent. The development of performance analysis conditions allowing all the system matrices to be parameter-dependent is necessary for the purpose of state-feedback control design. Indeed, descriptor systems whose state-space matrices are affine parameter-dependent can be represented as descriptor systems with only the state matrix being affine parameter-dependent. This can be achieved by appropriately choosing an augmented descriptor vector. This technique introduces some new state variables in the augmented descriptor vector that depend on uncertain parameters and/or the perturbation signal. Therefore, the augmented descriptor vector cannot be used anymore for state-feedback control.

This paper is structured as follows. The next section presents new LMI conditions for the admissibility and $H_2$ performance analysis of LTI singular systems. In §III, we extend these results to the class of uncertain time-varying parameter-dependent descriptor systems. A numerical example is presented in §IV and section §V concludes our work.

Notations: The relation $A > B$ ($A < B$) means the matrix $A-B$ is positive (negative) definite. The superscript $T$ stands for matrix transposition. The matrix $I_n$ stands for the identity matrix of dimension $n$. $(*)$ is used for the blocks induced by symmetry and $\mathcal{H}_2\{A\}$ means $A + A^T$. $Tr A$, $Ker A$ and $Range A$ represent the trace, the kernel and the range, respectively, of $A$. $\text{Bdiag}(A_1, \ldots, A_i)$ is the block-diagonal matrix having $A_1, \ldots, A_i$ on its main diagonal.

II. NEW LMI ANALYSIS CONDITIONS FOR LTI DESCRIPTOR SYSTEMS

Consider the class of continuous-time linear time invariant (LTI) descriptor systems defined by:

$$S: \begin{cases}
E \dot{x}(t) = Ax(t) + Bu(t) \\
z(t) = Cx(t) + Dw(t)
\end{cases}$$

(1)

where $x(t) \in \mathbb{R}^n$ is the descriptor vector, $w(t) \in \mathbb{R}^m$ is the exogenous input vector and $z(t) \in \mathbb{R}^p$ is the controlled output. $A$, $B$, $C$ and $D$ are constant matrices of appropriate dimensions. The matrix $E \in \mathbb{R}^{n \times n}$ may be singular with rank $E = r \leq n$.

In the following, we recall some basic definitions of descriptor systems [1], [20]. The system (1) or the pair $(E, A)$ is called regular if $\det(sE-A)$ is not identically zero which guarantees the existence and the uniqueness of system solutions. A regular system is impulse-free i.e. it doesn’t have impulsive modes if $\deg(\det(sE-A)) = \text{rank } E$. The finite modes (poles) of the system are the finite solutions of the characteristic polynomial $\det(sE-A) = 0$. The system is called admissible if it is regular, impulse-free and it has stable finite modes.

A. Admissibility analysis

As the matrix $E$ is of rank $r$, there exist invertible matrices $S$ and $T$ such that

$$SET = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}.$$  (2)

These matrices are not unique and can be obtained either by rank decomposition leading to the so-called second equivalent form [1] or by singular value decomposition (SVD) of $E$ leading to the so-called SVD coordinate system [21]. By using the coordinate transformation $x = TX$, the matrices of the equivalent SVD coordinate system are

$$SAT = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, SB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \text{ and } CT = \begin{bmatrix} C_1 & C_2 \end{bmatrix}.$$  (3)

where these matrices were partitioned accordingly to the partitioning of (2).

The following theorem presents our main contribution which consists in a new dilated strict LMI condition for the admissibility analysis of descriptor system $S$.

Theorem 2.1 (Admissibility): The following conditions are equivalent:

(i) The system (1) or the pair $(E, A)$ is admissible

(ii) There exist $P = P^T \in \mathbb{R}^{r \times r}$, $V \in \mathbb{R}^{r \times r}$, $X_1 \in \mathbb{R}^{(n-r) \times r}$ and $X_2 \in \mathbb{R}^{(n-r) \times (n-r)}$ such that

$$- (V + V^T) SAT \begin{bmatrix} V \\ X_1 \\ V \end{bmatrix} + \begin{bmatrix} P \\ 0 \\ 0 \end{bmatrix} \mathcal{H}_2 \{SAT \begin{bmatrix} 0 \\ 0 \\ X_2 \end{bmatrix} - \begin{bmatrix} P \\ 0 \\ 0 \end{bmatrix} \}^T \begin{bmatrix} P \\ 0 \\ -P \end{bmatrix} < 0.$$  (4)

(iii) There exist $P = P^T \in \mathbb{R}^{r \times r}$, $V \in \mathbb{R}^{r \times r}$, $X_1 \in \mathbb{R}^{(n-r) \times r}$ and $X_2 \in \mathbb{R}^{(n-r) \times (n-r)}$ such that

$$- (V + V^T) \begin{bmatrix} V^T \\ X^T \end{bmatrix} SAT + \begin{bmatrix} P \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} P \\ 0 \\ 0 \end{bmatrix} V^T \begin{bmatrix} 0 \\ 0 \\ -P \end{bmatrix} < 0.$$  (5)

Proof: Using the SVD form, the system (1) or the pair $(E, A)$ is admissible if and only if $A_{22}$ is invertible and $(A_{11} - A_{12}A_{22}^{-1} A_{21})$ is stable. Indeed, the invertibility of $A_{22}$ guarantees the regularity of the system and the absence of infinite modes while the second condition guarantees the stability of finite modes [21], [22]. According to [17], the matrix $(A_{11} - A_{12}A_{22}^{-1} A_{21})$ is stable if and only if there exist matrices $P = P^T$ and $V$ of appropriate dimensions such that:

$$\begin{bmatrix} -(V + V^T) \begin{bmatrix} (A_{11} - A_{12}A_{22}^{-1} A_{21})V + P \\ V \end{bmatrix} \end{bmatrix} < 0.$$  (6)
The previous inequality can be rewritten as
\[
\begin{align*}
LMI_1 = \mathcal{H}_e \left\{ \begin{pmatrix} -V & 0 & 0 \\ (A_{11} - A_{12}A_{22}^{-1}A_{21})V + P & -P/2 & 0 \\ V & 0 & -P/2 \end{pmatrix} \right\} < 0.
\end{align*}
\]
For any general matrix \( X_1 \) and \( X_2 \) invertible, the block \((2,1)\) of the previous inequality equals
\[
(A_{11}V + A_{12}X_1 + P) - (A_{12}X_2)(A_{22}X_2)^{-1}(A_{21}V + A_{22}X_1).
\]

Then, based on Lemma 1.1 and Lemma 1.2 in the appendix, the stability of \((A_{11} - A_{12}A_{22}^{-1}A_{21})\) and the invertibility of \( A_{22} \) are guaranteed whenever the following condition is satisfied
\[
\begin{align*}
\mathcal{H}_e \left\{ \begin{pmatrix} -V & 0 & 0 \\ A_{11}V + A_{12}X_1 + P & -P/2 & 0 \\ A_{21}V + A_{22}X_1 & 0 & 0 \end{pmatrix} \right\} < 0.
\end{align*}
\]

Multiplying this inequality by \( \mathbf{B} \text{diag}(I_r, I_r, \begin{bmatrix} 0 & I_{n-r} \\ I_r & 0 \end{bmatrix}) \) to the left and by its transpose to the right and using (3), we obtain (4).

(ii) \( \Rightarrow \) (ii) If \((E, A)\) is admissible then \( A_{22} \) is invertible and there exist matrices \( P = P^T \) and \( V \) such that (6) holds. Then there exists a sufficiently small scalar \( \epsilon > 0 \) such that
\[
\begin{align*}
\mathcal{LMI}_1 < -\epsilon \mathcal{H}_e(0, \frac{1}{2} A_{12}^T, 0).
\end{align*}
\]
Let \( X_2 = -\epsilon A_{22}^T \). As \( A_{22} \) is invertible, then
\[
\frac{\epsilon}{2} A_{12} A_{12}^T = -A_{12} X_2 (A_{22} X_2 + X_2^T A_{12}^T)^{-1} X_2^T A_{12}^T.
\]
As \( A_{22} X_2 + X_2^T A_{12}^T = -\epsilon A_{22} A_{22}^T < 0 \), the inequality (8) is equivalent by Schur complement to
\[
\begin{align*}
& \begin{pmatrix} -V + V^T & \ast & \ast \\ (A_{11} - A_{12}A_{22}^{-1}A_{21})V + P & -P & \ast \\ V & 0 & -P \end{pmatrix} < 0.
\end{align*}
\]

Let \( X_1 = -A_{12}^{-1}A_{21} V = 0 = A_{21} V + A_{22} X_1 \) and \((A_{11} - A_{12}^{-1} A_{21}) V + P = A_{11} V + A_{12} X_1 + P \). Hence, we obtain the inequality (7) which has been previously shown to be equivalent to (4).

(ii) \( \Leftrightarrow \) (iii) The inequality (iii) is the dual form of (ii) by the transformation \( SAT \to (SAT)^T \) and it is equivalent to (i) by similar arguments.

Remark 2.1: From condition (4), we can deduce that
\[
A_{22} X_2 + X_2^T A_{12}^T < 0 \quad \text{which, by Lemma 1.2 in the appendix, implies} \quad A_{22} \text{ invertible and therefore guarantees the regularity of the system as well as the absence of impulsive modes.}
\]

The proof of the same theorem also shows that the matrix \( P \) is a Lyapunov matrix ensuring the stability of the matrix \((A_{11} - A_{12}A_{22}^{-1}A_{21})\) i.e. of finite modes. Notice that the Lyapunov function ensuring the stability of the descriptor system (1) is
\[
V(x) = x^T E^T \bar{P} x = x^T E^T S^T \begin{bmatrix} P_{21} & 0 \\ 0 & P_{22} \end{bmatrix} T^{-1} x = \bar{x}_1^T \bar{P} \bar{x}_1.
\]
where \( \bar{x} = [\bar{x}_1^T \bar{x}_2^T]^T \).

Remark 2.2: When \( E = I_n \), the condition (4) and its dual form (5) of Theorem 2.1 reduce to diluted LMI stability conditions presented in [17].

B. \( \mathcal{H}_2 \) performance analysis

The following theorem presents a necessary and sufficient LMI condition for the \( \mathcal{H}_2 \) performance analysis of a descriptor system. In order to ensure the finiteness of the \( \mathcal{H}_2 \) norm [23], [24], we assume that the system (1) satisfies the condition:
\[
\text{Ker} E \subseteq \text{Ker} C.
\]

Without loss of generality, we consider that system (1) satisfies the strict properness assumption i.e. \( D = 0 \). If this is not the case then, by appropriately choosing an augmented descriptor variable, system (1) can be rewritten as an augmented system satisfying the strict properness assumption [20].

Theorem 2.2 (\( \mathcal{H}_2 \) Performance): For a given positive scalar \( \gamma_2 \), system (1) under assumption (9) is admissible and \( \| C(sE - A)^{-1} B \|_2 < \gamma_2 \) if and only if there exist matrices \( P = P^T \in \mathbb{R}^{r \times r}, V \in \mathbb{R}^{r \times x}, Z = Z^T \in \mathbb{R}^{p \times p}, X_1 \in \mathbb{R}^{(n-r) \times r} \) and \( X_2 \in \mathbb{R}^{(n-r) \times (n-r)} \) such that the LMI (10) is satisfied.

Proof: Based on the SVD form, system (1) is admissible if and only if \( A_{22} \) is invertible, \((A_{11} - A_{12}A_{22}^{-1}A_{21})\) is stable and the linear state-space system defined by the state matrices \( \bar{A} = A_{11} - A_{12}A_{22}^{-1}A_{21}, \bar{B} = B_{1} - A_{12}A_{22}^{-1}B_{2}, \bar{C} = C_{1} - C_{2}A_{22}^{-1}A_{21} \) and \( \bar{D} = D - C_{2}A_{22}^{-1}B_{2} \) satisfies the \( \mathcal{H}_2 \) performance criteria. Assumption (9) and the structure (2) of the matrix \( SAT \) imply \( C_2 = 0 \). This condition and the properness assumption of system (1) imply the strict properness property of the previous state-space system. The remaining of the proof follows by applying the \( \mathcal{H}_2 \) performance criteria developed in [17] to the state-space system \((A, B_{1}, C_{1}, 0)\) and by using similar arguments to the ones presented in the proof of Theorem 2.1.

By considering the dual system of (1), we can also obtain the dual form of Theorem 2.2 under the assumption [24]:
\[
\text{Range } B \subseteq \text{Range } E.
\]

Corollary 2.3 (\( \mathcal{H}_2 \) Performance - dual form): For a given positive scalar \( \gamma_2 \), system (1) under assumption (11) is admissible and satisfies \( \| (sE - A)^{-1} C \|_2 < \gamma_2 \) if and only if there exist matrices \( \bar{P} = \bar{P}^T \in \mathbb{R}^{r \times r}, V \in \mathbb{R}^{r \times x}, Z = Z^T \in \mathbb{R}^{m \times m}, X_1 \in \mathbb{R}^{(n-r) \times r} \) and \( X_2 \in \mathbb{R}^{(n-r) \times (n-r)} \) such that the LMIs (12) are satisfied.

III. DESCRIPTOR SYSTEMS WITH UNCERTAINTIES

In this section, we address the robust analysis and robust state-feedback control synthesis for uncertain descriptor sys-
tems. Let’s consider the class of uncertain descriptor systems:

\[ S_U : \begin{cases} E \dot{x}(t) = A(\theta)x(t) + B(\theta)w(t) \\ z(t) = C(\theta)x(t) \end{cases} \quad (13) \]

where \( E \in \mathbb{R}^{n \times n} \) is constant with \( \text{rank} \, E = r \leq n \), \( \theta = [\theta_1, \theta_2, \ldots, \theta_N]^T \in \mathbb{R}^N \) is the vector of uncertain, real and possibly time-varying parameters belonging to the uncertainty domain defined as follows:

- Each parameter \( \theta_i \) ranges between known extremal values which means that \( \theta \) is valued in a convex set whose vertices are defined by \( \mathcal{V} = \{v_i | i = 1, \ldots, M\} \).
- When the parameter vector is time-varying, we assume that the rate of variation of each parameter is bounded which means that the rate of variation of the parameter vector \( \dot{\theta} \) evolves in a convex set whose vertices are \( \mathcal{W} = \{w_i | i = 1, \ldots, M\} \).

The time-dependence of the parameters is omitted here for simplicity. The system matrices \( A(\theta), B(\theta) \) and \( C(\theta) \) are affine parameter-dependent i.e.

\[
\begin{bmatrix} A(\theta) & | & B(\theta) & | & C(\theta) \end{bmatrix} = \begin{bmatrix} A_0 & | & B_0 & | & C_0 \end{bmatrix} + \sum_{i=1}^{N} \theta_i \begin{bmatrix} A_i & | & B_i & | & C_i \end{bmatrix}.
\]

Therefore, given the uncertainty domain, these matrices belong to convex sets whose vertices are \( \{A(\theta)| v \in \mathcal{V}\}, \{B(\theta)| v \in \mathcal{V}\} \) and \( \{C(\theta)| v \in \mathcal{V}\} \), respectively.

**Remark 3.1**: Note that this definition of the uncertainty domain allows to gather time-invariant polytopic uncertainties as well as real, affine, time-varying parametric ones. Indeed, when dealing with time-invariant polytopic uncertainties, the parameters satisfy \( \theta_i \geq 0 \) and \( \sum_{i=1}^{N} \theta_i = 1 \). Then, the sets \( \mathcal{V} \) and \( \mathcal{W} \) have \( M = N \) vertices and reduce to

\[ \mathcal{V} = \{v_i = (0, 0, \ldots, 1, \ldots, 0) | \text{where 1 is on the i-th position and } i = 1, \ldots, N\} \text{ and } \mathcal{W} = \{0\}. \]

Moreover, \( A_0 = 0, B_0 = 0, C_0 = 0 \) and the vertices \( A(v), B(v) \) and \( C(v) \) are \( A_i, B_i \) and \( C_i \), respectively, for \( i = 1, \ldots, N \). When dealing with time-varying affine parametric uncertainties, \( \bar{\theta}_i \) and \( \Gamma_i \) evolve in hyper-rectangles with \( M = 2^N \) vertices given by:

\[ \mathcal{V} = \{(\omega_1, \omega_2, \ldots, \omega_N) | \omega_i \in \{\Gamma_i, \bar{\Gamma}_i\}\} \quad (14) \]

where \( \Gamma_i \) and \( \bar{\Gamma}_i \) are known extremal values of \( \theta_i \),

\[ \mathcal{W} = \{(\tau_1, \tau_2, \ldots, \tau_N) | \tau_i \in \{\Gamma_i, \bar{\Gamma}_i\}\} \quad (15) \]

where \( \Gamma_i \) and \( \bar{\Gamma}_i \) are known upper and lower bounds of \( \theta_i \).

In order to state robust analysis and synthesis results for both types of uncertainties, let’s define, in addition to the uncertainty domain, the indicator function \( q \):

\[ q = \begin{cases} 1 & \text{for time-varying affine parametric uncertainties} \\ 0 & \text{for time-invariant polytopic uncertainties} \end{cases} \]

Our robust analysis and synthesis results are based on parameter-dependent Lyapunov functions which are less restrictive than the classical ones based on a common Lyapunov function for the entire uncertainty domain. The parameter-dependent Lyapunov function candidate is

\[ V(x, \theta) = x^T \hat{P}(\theta)x \text{ where } \hat{P}(\theta) = q\hat{P}_0 + \sum_{i=1}^{N} \theta_i \hat{P}_i. \]

A. **Robust analysis**

In this section, we present the extension of our main results to the robust admissibility and \( \mathcal{H}_2 \) performance analysis.
Theorem 3.1 (Robust admissibility): The uncertain system (13) is robustly admissible if there exist symmetric matrices $P_0, P_1, ..., P_N \in \mathbb{R}^{r \times r}$ and general matrices $V \in \mathbb{R}^{r \times r}, X_1 \in \mathbb{R}^{(n-r) \times r}$ and $X_2 \in \mathbb{R}^{(n-r) \times (n-r)}$ such that the set of LMIs (16) are satisfied where $P(\theta) = qP_0 + \sum_{i=1}^{N} \theta_i P_i$ and $S, T$ are given by (2).

Proof: Based on the uncertain SVD form generated by (2), the uncertain system (13) is robustly admissible if and only if $A_{22}(\theta)$ is invertible and $A(\theta) = A_{11}(\theta) - A_{12}(\theta)A_{22}(\theta)^{-1}A_{21}(\theta)$ is robustly stable for the entire uncertainty domain where

$$SA(\theta)T = \begin{bmatrix} A_{11}(\theta) & A_{12}(\theta) \\ A_{21}(\theta) & A_{22}(\theta) \end{bmatrix} = SA_0T + \sum_{i=1}^{i=N} SA_iT \theta_i.$$  

(17)

Using similar arguments to the ones presented in the proof of Theorem 2.1 as well as the multi-convexity principle (an affine parameter-dependent inequality reduces to a finite set of inequalities evaluated on the vertices of the parameter domain), we obtain the inequalities (16).

Assuming that system (13) satisfies the condition:

$$\text{Range } B(\theta) \subseteq \text{Range } E$$

\forall \theta \text{ in the uncertainty domain}  

(18)

the next theorem presents LMI conditions for robust $\mathcal{H}_2$ performance analysis.

Theorem 3.2 (Robust $\mathcal{H}_2$ performance): Under assumption (18), system (13) is robustly admissible and has a $\mathcal{H}_2$ performance index $\gamma_2 > 0$ if there exist symmetric matrices $P_0, P_1, ..., P_N \in \mathbb{R}^{r \times r}$ and general matrices $V \in \mathbb{R}^{r \times r}, X_1 \in \mathbb{R}^{(n-r) \times r}$ and $X_2 \in \mathbb{R}^{(n-r) \times (n-r)}$ such that the set of LMIs (19) are satisfied where $P(\theta) = qP_0 + \sum_{i=1}^{i=N} \theta_i P_i$ and $S, T$ are given by (2).

Proof: The proof is based on the arguments presented in the proofs of Theorems 2.2 and 3.1.

Remark 3.2: The dual forms of the robust admissibility/$\mathcal{H}_2$ performance analysis conditions presented by Theorems 3.1 and 3.2 can easily be deduced by using the dual systems.

B. Robust state-feedback control

In this section, we apply the robust analysis results presented above to the robust state-feedback control. To this end, consider the class of forced uncertain descriptor systems:

$$\mathcal{S}_U : \begin{cases} e\dot{x}(t) = A(\theta)x(t) + B_u(\theta)u(t) + B_w(\theta)w(t) \\ z(t) = C(\theta)x(t) + D_u(\theta)u(t) + D_w(\theta)w(t) \end{cases}$$

(20)

where $\theta$ is the uncertain parameter vector belonging to the uncertainty domain previously defined. All state matrices are affine parameter-dependent and $u(t) \in \mathbb{R}^{m_u}$ is the control input. We assume that $\text{Range } B_w(\theta) \subseteq \text{Range } E$ for all $\theta$ in the uncertainty domain.

Our aim is to design a descriptor state-feedback control law $u(t) = Kx(t)$. Based on the analysis results developed in §III-A, the next theorem states sufficient LMI conditions for the synthesis of a robust descriptor state-feedback controller.

Theorem 3.3: Given a scalar $\gamma_2 > 0$, the closed-loop system is robustly admissible and has a $\mathcal{H}_2$ performance index $\gamma_2$ for all $\theta$ in the uncertainty domain if there exist symmetric matrices $P_i \in \mathbb{R}^{r \times r}$ for $i \in \{0, 1, \ldots, N\}$, $Z \in \mathbb{R}^{m \times m}$ and general matrices $V \in \mathbb{R}^{r \times r}, X_1 \in \mathbb{R}^{(n-r) \times r}$, $X_2 \in \mathbb{R}^{(n-r) \times (n-r)}$ and $Y \in \mathbb{R}^{m \times n}$ such that the set of LMIs (21) are satisfied where the blocks (2, 1) and (2, 2) are given by

$$(2, 1) = SA(v)T \begin{bmatrix} V \\ X_1 \end{bmatrix} + SB_u(v)Y \begin{bmatrix} I_r \\ 0 \end{bmatrix} + \begin{bmatrix} P(v) \\ 0 \end{bmatrix},$$

$$(2, 2) = He \left\{ SA(v)T \begin{bmatrix} 0 \\ 0 \end{bmatrix}, X_2 \right\} + He \left\{ SB_u(v)Y \begin{bmatrix} 0 \\ 0 \end{bmatrix}, I_n \right\} - \begin{bmatrix} P(v) - q(P(w)) - P_0 \\ 0 \end{bmatrix} 0 \end{bmatrix} 0 \end{bmatrix} < 0$$

and $P(\theta) = qP_0 + \sum_{i=1}^{i=N} \theta_i P_i$.

If $X_2$ is non-singular then the descriptor state-feedback gain is given by

$$K = Y \begin{bmatrix} V \\ X_1 \end{bmatrix} X_2^{-1} 0 \end{bmatrix}^{-1} T^{-1}.$$  

If $X_2$ is singular then there always exists an appropriate scalar $\kappa \in (0, 1)$ such that $X + \kappa I$ is invertible and (21a) holds. Then, the feedback gain is given by

$$K = Y \begin{bmatrix} V \\ X_1 \end{bmatrix} \begin{bmatrix} 0 \\ X_2 + \kappa I \end{bmatrix}^{-1} T^{-1}.$$
Proof: By making the change of variable \( x = T \bar{x} \) and using (2), the closed-loop system is equivalent to

\[
\begin{aligned}
\begin{bmatrix}
S A(v) T \\
C(v) T \\
B_w(v)^T S^T
\end{bmatrix}
\begin{bmatrix}
I_r \\
0
\end{bmatrix}
&> 0
\quad \forall v \in \mathcal{V}
\quad \text{and}
\quad T r Z < 1
\end{aligned}
\]

Applying Theorem 3.2 to the closed-loop system and making the changes of variable \( Y = K T \begin{bmatrix} V & 0 \end{bmatrix} \), we obtain the LMI conditions (21a) from (21a), we deduce that \( V + V^T > 0 \) which implies that \( V \) is invertible. When \( X_2 \) is invertible, the descriptor state-feedback gain \( K \) is computed based on the previous changes of variable. If \( X_2 \) is singular then we derive an invertible solution \( X_2 \) as follows. We use the solutions \( P_i, V, X_1, X_2 \) and \( Y \) provided by the LMI conditions (21) in order to obtain a scalar \( \kappa \) such that \( X_2 = X_2 + \kappa I \) is invertible and (21a) holds when \( X_2 \) is replaced by \( X_2 \). Note that such a scalar \( \kappa \) always exists. Hence, the descriptor state-feedback gain \( K \) is computed based on the previous change of variable and using \( X_2 = X_2 + \kappa I \).

Remark 3.3: Theorem 3.3 presents linear inequality conditions for designing a robust state-feedback controller. In fact, applying Theorem 3.2 or Theorem 3.1 for synthesizing a state-feedback controller does not raise any linearity problem since a simple change of variable can be employed, as shown in the proof of Theorem 3.3, in order to obtain linear design conditions. Note that the approach developed in [12] causes a linearity issue due to the presence of two couplings between the control matrix \( B_u(\theta) \) and two different slack variables. In order to design the state-feedback controller, the authors of [12] proposed an iterative procedure which linearizes the couplings by assigning a part of the multipliers or slack variables.

Remark 3.4: Note that Theorem 3.3 can be extended to affine linear parameter-varying (LPV) systems. Based on the parameters measurability, a parameter-dependent state-feedback controller can easily be synthesized by applying the analysis result presented in the previous section.

IV. NUMERICAL EXAMPLE

Consider the descriptor system with time-varying affine parametric uncertainties defined by the matrices:

\[
A(\theta) = \begin{bmatrix}
\theta_1 & 1 & -5 - 0.3 \theta_1 & 0 \\
1 & 2 \theta_1 & 0 & -1 - 1.5 \theta_1 \\
-\theta_1 & 0 & 0.8 \theta_1 & 2 + 0.4 \theta_1 \\
3 & 0 & \theta_1 & -1 + 0.2 \theta_1
\end{bmatrix}
\]

\[
B_u(\theta) = \begin{bmatrix}
\theta_3 & 0.2 \theta_2 & \theta_3 & \theta_2 \\
0 & -0.2 \theta_2 & 3 \theta_2 + 0.5 \theta_3 & 0 \\
0.1 \theta_3 & \theta_2 & 0.2 \theta_3 & -0.5 \theta_3 \\
\theta_3 & \theta_2 & -0.5 \theta_3 & 0.5 \theta_2 + 0.1 \theta_3
\end{bmatrix}
\]

\[
B_w(\theta) = \begin{bmatrix}
1 + 0.3 \theta_1 + 1.3 \theta_2 + \theta_3 \\
-0.7 + \theta_2 + 0.5 \theta_3 \\
0.1 \theta_1 - 0.1 \theta_3 \\
1 + 0.2 \theta_1 - 0.5 \theta_2
\end{bmatrix}
\]

\[
C(\theta) = [ -1 + 0.05 \theta_1 & 0.2 + 0.1 \theta_2 + 0.1 \theta_3 \\
0.1 & 0.3 + 0.1 \theta_1 - 0.3 \theta_2 + 0.2 \theta_3 ]
\]

\[
D_u(\theta) = 0.5 - \theta_1 + 0.2 \theta_2 \quad \text{and} \quad E = B \text{diag}(I_2, 0).
\]
The uncertainty domain is given by the lower and upper bounds of \( \theta_i \) and \( \dot{\theta}_i \) as follows: \( \theta_1 \in [-0.2, 0.1], \theta_2 \in [-0.3, 0.1], \theta_3 \in [-0.2, 0.1], \theta_4 \in [-0.2, 0.2], \theta_5 \in [-0.5, 0.5] \) and \( \theta_1 \in [-0.1, 0.1] \). As the uncertain parameters are time-varying and the matrix \( B_u(\theta) \) is parameter-dependent, the results presented in [12] cannot be applied.

The system is not admissible for the whole uncertainty domain regardless of the time-variance or invariance of the parameters since the system finite modes evolve between \(-700 \) and \( 3250 \). Using the LMI conditions of Theorem 3.3 leads to the state-feedback gain \( K = \begin{bmatrix} 6.8515 & 4.4006 & -2.5867 & -1.0554 \end{bmatrix} \) guaranteeing the admisssibility of the closed-loop system. The minimal \( H_2 \) norm achieved with this theorem is \( \gamma_2 = 0.7339 \).

V. CONCLUSION

In this paper, dilated LMI conditions for the admissibility and \( H_2 \) performance analysis of LTI descriptor systems have been presented. These new necessary and sufficient conditions represent a generalization to singular systems of the LMI analysis results developed for state-space systems in [17]. We have also presented an extension of our conditions to robust analysis and robust state-feedback control of uncertain parameter-dependent descriptor systems.

APPENDIX

Lemma 1.1: Let \( M \) be a general matrix partitioned as
\[
\begin{bmatrix} A & B \\ C & D \end{bmatrix}.
\]
If \( M \) is negative-definite (not necessarily symmetric) \( \text{i.e.} \), \( \mathcal{H}(M) < 0 \) then \( D \) is invertible and the Schur complement of \( M \) relative to \( D \) is also negative-definite (not necessarily symmetric) \( \text{i.e.} \), \( \mathcal{H}(A - BD^{-1}C) < 0 \).

Lemma 1.2: Let \( X \) and \( Y \) be real square matrices. If \( \mathcal{H}(XY) < 0 \) then \( X \) and \( Y \) are invertible.

REFERENCES