Output Feedback Adaptive Command Following for Nonminimum Phase Uncertain Dynamical Systems

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Abstract—In this paper, we develop an output feedback adaptive control framework for continuous-time nonminimum phase multivariable systems for output stabilization and command following. The approach is based on a nonminimal state space realization that generates an expanded set of states using the filtered inputs and filtered outputs of the original system. Specifically, a direct adaptive controller for the nonminimal state space model is constructed using the expanded states of the nonminimal realization and is shown to be effective for multi-input, multi-output nonminimum phase systems with unstable dynamics. The adaptive controller does not require any model information except for an expanded compatibility condition involving the nonminimal model which is far less restrictive than standard matching conditions for model reference adaptive control involving the actual system dynamics. Two illustrative numerical examples are provided to demonstrate the efficacy of the proposed approach.

I. INTRODUCTION

Mathematical models are critical in capturing and studying physical phenomena that undergo spatial and temporal evolution arising in most applications of science and engineering. These models are often based on first-principles of physics and are derived using fundamental physical laws. However, due to system complexity, nonlinearities, uncertainty, and disturbances, first-principle models are often based on simplifying approximations resulting in system modeling errors. For systems where the system model does not adequately capture the physical system due to idealized assumptions, model simplification, and model parameter uncertainty, adaptive control methods can be used to achieve system performance without excessive reliance on system models.

Direct adaptive controllers require less system modeling information than robust controllers and can address system uncertainties and system failures. These controllers adapt feedback gains in response to system variations without requiring a parameter estimation algorithm. This property distinguishes them from indirect adaptive controllers that employ an estimation algorithm to estimate the unknown system parameters and adapt the controller gains. Direct adaptive controllers can be classified as either full state feedback or output feedback designs.

Full state feedback designs assume knowledge of the state variables, and this assumption leads to computation-ally simpler adaptive controller algorithms as compared to output feedback algorithms. Output feedback direct adaptive controllers, however, are required for most applications that involve high-dimensional models such as active noise suppression, active control of flexible structures, fluid flow control systems, and combustion control processes. Models for these applications vary from (reasonably) accurate low frequency models in the case of structural control problems, to less accurate low-order models in the case of active control of noise, vibrations, flows, and combustion processes.

There has been a number of results in recent decades focused on output feedback direct adaptive controllers (see [1]–[7], and references therein). These results require an observer for unknown state variables, an observer for output tracking errors, an output predictor, and/or estimation of Markov parameters that lead to adaptive control algorithms with varying sets of assumptions. These assumptions include knowledge of the relative degree of the regulated system output and the dimension of the system, as well as the requirement that the system be minimum phase or passive.

Virtually all output feedback adaptive controllers are developed under a minimum-phase assumption. Notable expectations include [8]–[12]. In particular, to circumvent the system minimum phase assumption, the author in [8] developed a zero annihilation periodic law that uses lifting to move all system zeros to the origin. This framework, however, requires an open-loop operation during alternating data windows. Alternatively, the adaptive control algorithm in [9], [10] utilizes a surrogate cost function involving a quadratic optimization step. The proposed controller requires sign information of the high-frequency gain and knowledge of the nonminimum-phase zeroes of the system. A key limitation of this approach, however, is that the control law lacks a proof of stability. The approaches in [11] and [12] are limited to single-input, single-output systems and assume that the sign of the high-frequency gain is known. The underlying assumptions in [11] are that the relative degree of the regulated output variable is known, and that the zero-dynamics of the linear model for the plant captures the internal dynamics of the nonminimum-phase system to sufficient accuracy. Reference [12] employs a backstepping approach and assumes that the relative degree is known with a linearly bounded unmatched uncertainty.

In this paper, we develop an output feedback adaptive control framework for continuous-time, nonminimum phase multivariable systems for output stabilization and command following. The approach is based on a nonminimal state space realization that generates an expanded set of states using the filtered inputs and filtered outputs of the original system. Specifically, a direct adaptive controller for the nonminimal state space model is constructed using the expanded states of the nonminimal realization and is shown to be effective for multi-input, multi-output nonminimum phase systems with unstable dynamics. The adaptive controller does not require any model information except for an expanded compatibility condition involving the nonminimal model, which is far less restrictive than standard matching conditions for model reference adaptive control involving the actual system dynamics. In addition, the proposed adaptive controller does not require knowledge of the nonminimum phase system zeros.

The notation used in this paper is fairly standard. Specifically, $\mathbb{R}$ denotes the set of $n \times 1$ real column vectors, $\mathbb{R}^{n \times m}$
denotes the set of $n \times m$ real matrices, $(\cdot)^T$ denotes transpose, and $(\cdot)^{-1}$ denotes inverse. Furthermore, we write $\lambda_{\min}(M)$ (resp., $\lambda_{\max}(M)$) for the minimum (resp., maximum) eigenvalue of the Hermitian matrix $M$, $\| \cdot \|_2$ for the Euclidean norm, $\| \cdot \|_F$ for the Frobenius matrix norm, $\text{tr}(\cdot)$ for the trace operator, $\text{vec}(\cdot)$ for the column stacking operator, and $(\cdot)'$ for the Fréchet derivative.

II. NONMINIMAL STATE-SPACE REALIZATION FORMULATION

In this section, we present a nonminimal state-space realization architecture for continuous-time, linear multivariable uncertain dynamical systems. The nonminimal state space realization involves an expanded state space that consists entirely of the system filtered inputs and filtered outputs and their derivatives, which allows us to cast an output feedback control problem as a full-state feedback problem. Specifically, consider the controllable and observable linear uncertain dynamical system given by

$$\dot{x}_p(t) = A_p x_p(t) + B_p u(t), \quad x_p(0) = x_{p0}, \quad t \geq 0, \quad (1)$$

$$y(t) = C_p x_p(t), \quad (2)$$

where $x_p(t) \in \mathbb{R}^n$, $t \geq 0$, is the state vector, $u(t) \in \mathbb{R}^m$, $t \geq 0$, is the control input, $y(t) \in \mathbb{R}^p$, $t \geq 0$, is the system output, and $A_p \in \mathbb{R}^{n \times n}$, $B_p \in \mathbb{R}^{n \times m}$, and $C_p \in \mathbb{R}^{m \times n}$ are unknown system matrices. An input-output equivalent nonminimal observer canonical state space model of (1) and (2) for $p > 1$ is given by ([15])

$$\dot{x}_o(t) = A_o x_o(t) + B_o u(t), \quad x_o(0) = x_{o0}, \quad t \geq 0, \quad (3)$$

$$y(t) = C_o x_o(t), \quad (4)$$

where $x_o(t) \in \mathbb{R}^{pn}$, $t \geq 0$, is the state vector,

$$A_o = \begin{bmatrix} 0 & I_p & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & I_p \\ -a_0 I_p & -a_1 I_p & \cdots & -a_{n-1} I_p \end{bmatrix} \in \mathbb{R}^{pn \times pn}, \quad (5)$$

$$B_o = \begin{bmatrix} C_o B_p \\ C_o A_p B_p \\ \vdots \\ C_o A_p^{n-2} B_p \end{bmatrix} \in \mathbb{R}^{pn \times m}, \quad (6)$$

and

$$C_o = [I_p \; 0 \; \cdots \; 0] \in \mathbb{R}^{p \times pn}. \quad (7)$$

Note that $a_i$, $i = 0, 1, \ldots, n-1$, in (5) are the coefficients of the characteristic polynomial of the matrix $A_p$ in (1).

Next, let

$$\bar{B}_0 = C_o (a_1 I_{pn} + \cdots + a_{n-1} A_o^{n-2} + A_o^{n-1} B_o), \quad (8)$$

$$\bar{B}_1 = C_o (a_2 I_{pn} + \cdots + a_{n-1} A_o^{n-3} + A_o^{n-2} B_o), \quad (9)$$

$$\vdots$$

$$\bar{B}_{n-1} = C_o B_o. \quad (10)$$

Now, an alternative input-output equivalent nonminimal controllable state space realization of (1) and (2) is given by

$$\dot{x}_1(t) = A_1 x_1(t) + B_1 u(t), \quad x_1(0) = x_{10}, \quad t \geq 0, \quad (11)$$

$$y(t) = C_1 x_1(t), \quad (12)$$

where $x_i(t) \in \mathbb{R}^{(m+p)n}$, $t \geq 0$, is the known filtered expanded state vector given by

$$x_i(t) = \begin{bmatrix} y_1^T(t) \; \cdots \; y_{n-1}^T(t) \; u_1^T(t) \; \cdots \; u_{n-1}^T(t) \end{bmatrix}^T, \quad (13)$$

where $z^{(n)}(t) \triangleq d^n z(t)/dt^n$ and where $x_i(t)$ is obtained by filtering $u(t)$ and $y(t)$ through the filter $1/\Lambda(s)$, where

$$\Lambda(s) = (s + \lambda)^n = \sum_{k=0}^{n} \binom{n}{k} \lambda^{n-k}, \quad (14)$$

is a monic Hurwitz polynomial of degree $n$ with $\lambda > 0$.

$$A_f = \begin{bmatrix} 0 & I_p & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 \\ -a_0 I_p & -a_1 I_p & \cdots & -a_{n-1} I_p \end{bmatrix}, \quad (15)$$

$$B_f = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ I_m \end{bmatrix} \in \mathbb{R}^{(m+p)n \times (m+p)n}, \quad (16)$$

and

$$C_f = \begin{bmatrix} -a_0 I_p + \lambda I_p & \cdots & -a_{n-1} I_p + n \lambda I_p \\ \bar{B}_0 & \cdots & \bar{B}_{n-1} \end{bmatrix} \in \mathbb{R}^{p \times (m+p)n}. \quad (17)$$

**Theorem 2.1.** System (1) and (2) is input-output equivalent to system (11) and (12).

**Proof.** Using the input-output equivalence of (3) and (4) with (1) and (2), it follows that

$$a_0 y(t) = a_0 C_o x_o(t), \quad t \geq 0, \quad (18)$$

$$a_1 y(t) = a_1 [C_o A_o x_o(t) + C_o B_o u(t)], \quad (19)$$

$$\vdots$$

$$a_{n-1} y^{(n-1)}(t) = a_{n-1} [C_o A_o^{n-1} x_o(t) + C_o A_o^{n-2} B_o u(t) + \cdots + C_o B_o u^{(n-2)}(t)], \quad (20)$$

$$y^{(n)}(t) = C_o A_o^n x_o(t) + C_o A_o^{n-1} B_o u(t) + \cdots + C_o B_o u^{(n-1)}(t). \quad (21)$$
Now, adding the \( n + 1 \) equations in (18)−(21) we obtain
\[
y^{(n)}(t) = - \left[ a_0 I_p \ a_1 I_p \ \cdots \ a_{n-1} I_p \right] Y(t)
+ \left[ B_0 \ B_1 \ \cdots \ B_{n-1} \right] U(t)
+ C_n \left[ A_o^n + a_{n-1} A_o^{n-1} + \cdots + a_1 A_o + a_0 I_{pn} \right] x_o(t),
\]

where \( B_0, B_1, \ldots, B_{n-1} \) are given in (8)−(10) and
\[
Y(t), \quad t \geq 0, \quad \text{and} \quad U(t), \quad t \geq 0, \quad \text{are defined as}
\]
\[
Y(t) \triangleq \begin{bmatrix} y^T(t), \ y'^T(t), \ \cdots \ y^{(n-1)}(t) \end{bmatrix}^T,
\]
\[
U(t) \triangleq \begin{bmatrix} u^T(t), \ \dot{u}^T(t), \ \cdots \ u^{(n-1)}(t) \end{bmatrix}^T.
\]

Next, using the Cayley-Hamilton theorem [16], which states that every square matrix is a root of its characteristic polynomial, and noting that \( a_i, \ i = 0, 1, \ldots, n-1, \) are the coefficients of the characteristic polynomial of the matrix \( A_o \) in (3), it follows that
\[
A_o^n + a_{n-1} A_o^{n-1} + \cdots + a_1 A_o + a_0 I_{pn} = 0.
\]

Hence, (22) reduces to
\[
y^{(n)}(t) = - \left[ a_0 I_p \ a_1 I_p \ \cdots \ a_{n-1} I_p \right] Y(t)
+ \left[ B_0 \ B_1 \ \cdots \ B_{n-1} \right] U(t).
\]

Now, define the expanded state vector
\[
x_n(t) \triangleq \begin{bmatrix} Y^T(t), \ U^T(t) \end{bmatrix}^T
= \begin{bmatrix} y^T(t), \ \dot{y}^T(t), \ \cdots \ y^{(n-1)}(t),
\end{bmatrix}^T
\]
\[
= \begin{bmatrix} u^T(t), \ \dot{u}^T(t), \ \cdots \ u^{(n-1)}(t) \end{bmatrix}^T,
\]
so that (26) can be written as
\[
y^{(n)}(t) = \Phi x_n(t),
\]
where
\[
\Phi = \begin{bmatrix} -a_0 I_p \ a_1 I_p \ \cdots \ a_{n-1} I_p \ B_0 \ B_1 \ \cdots \ B_{n-1} \end{bmatrix} \in \mathbb{R}^{p \times (p+m)n}.
\]

Next, consider the \((m+p)n\)-th order nonminimal state space model given by
\[
\begin{align*}
\dot{x}_n(t) &= A_n x_n(t) + B_n u^{(n)}(t), \quad x_n(0) = x_{n0}, \quad t \geq 0, \quad (30) \\
y(t) &= C_n x_n(t),
\end{align*}
\]
where
\[
A_n = \begin{bmatrix}
0 & I_p & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \cdots & 0 \\
0 & 0 & I_p & 0 & 0 \\
0 & 0 & \cdots & \cdots & 0 \\
-a_0 I_p & \cdots & -a_{n-1} I_p & B_0 & 0 \\
\vdots & \cdots & \cdots & \cdots & \vdots \\
0 & \cdots & 0 & 0 & \cdots \\
\end{bmatrix}
\]
\[
B_n = \begin{bmatrix} 0 \\
\vdots \\
\vdots \\
0 \\
\end{bmatrix} \in \mathbb{R}^{(m+p)n \times m},
\]
\[
C_n = \begin{bmatrix} I & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{p \times (m+p)n}.
\]

To eliminate differentiating the actual input and output signals in (30), we filter the input signals in (30) and the output signals in (31) through the filter \( 1/A(s) \), where \( A(s) \) is defined by (14). In this case, the states \( x_n(t), \ t \geq 0, \) become \( x(t), \ t \geq 0, \) given by (13).

Now, let \( \lambda = [\lambda^n \ \cdots \ n \lambda]^T \), and note that the Laplace transform of the filtered input signal \( u^{(n)}_i(t), \ t \geq 0, \) can be written as
\[
\mathcal{L}\{u^{(n)}_i(t)\} = \frac{s^n}{(s+\lambda)^n} \mathcal{L}\{u(t)\}
= \frac{s^n - (s+\lambda)^n + (s+\lambda)^n}{(s+\lambda)^n} \mathcal{L}\{u(t)\}
= \frac{s^n - (s+\lambda)^n}{(s+\lambda)^n} \mathcal{L}\{u(t)\}
= \Omega_{n-1}(s) \mathcal{L}\{u(t)\} + \mathcal{L}\{u(t)\},
\]
where \( \mathcal{L}\{\cdot\} \) denotes the Laplace transform operator. Next, note that the inverse Laplace transform of the first term in the right-hand side of (36) is given by
\[
\mathcal{L}^{-1}\{\Omega_{n-1}(s)\} u(t)
= -\lambda^T [u_i^T(t) \ \dot{u}_i(t) \ \cdots \ u_i^{(n-1)}(t)]^T
= -\lambda^T U_i(t).
\]

Furthermore, the filtered version of (28) is given by
\[
y^{(n)}_i(t) = \mathcal{L}^{-1}\{\Omega_{n-1}(s)\} y(t) + y(t)
= -\lambda^T [y_i^T(t) \ \dot{y}_i(t) \ \cdots \ y_i^{(n-1)}(t)]^T
= -\lambda^T Y_i(t).
\]

Using (38) and (39), it follows that the actual system output is given by
\[
y(t) = y^{(n)}_i(t) + \lambda^T Y_i(t)
= \Phi x_i(t) + [\lambda^T, 0] x_i(t)
= \left( \Phi + [\lambda^T, 0] \right) x_i(t).
\]
Now,  filtering the signals in (30) and (31), and using (36) and (40), a nonminimal state space realization of (1) and (2) is given by (11) and (12), where  \( x_l(t) \in \mathbb{R}^{(m+p) n} \),  \( t \geq 0 \), is the known filtered expanded state vector given by (13), and  \( A_l \in \mathbb{R}^{(m+p) n \times (m+p) n} \),  \( B_l \in \mathbb{R}^{(m+p) n \times m} \), and  \( C_l \in \mathbb{R}^{p \times (m+p) n} \) are given by (15)–(17), respectively, with

\[
A_l = A_n - [0 \ B_n \bar{\lambda}_T], \quad (41)
\]
\[
B_l = B_n, \quad (42)
\]
\[
C_l = \Phi + \bar{\lambda}_T 0. \quad (43)
\]

Hence, (1) and (2) is input-output equivalent to (11) and (12).

\[ \square \]

**Remark 2.1.** The proof of Theorem 2.1 presents a construction of a nonminimal, albeit controllable, state space realization of (1) and (2) involving the expanded state  \( x_l(t) \),  \( t \geq 0 \), comprising of filtered versions of the inputs and outputs of the original system, without requiring differentiation of the actual input and output signals. It is important to note here that even though the original system is unknown, the expanded state vector  \( x_l(t) \),  \( t \geq 0 \), is known.

**Remark 2.2.** Since the controllable nonminimal state space realization of (11) and (12) is defined by a state that consists entirely of filtered inputs and outputs of the original system, an output feedback stabilization problem for (1) and (2) can be converted into a full-state feedback control design problem by equivalently considering (11) and (12). Furthermore, for an output feedback control design of (1) and (2) we require that \( (A_p, B_p) \) be controllable (or stabilizable) and \( (A_p, C_p) \) be observable (or detectable). In contrast, for a feedback control design using the input-output equivalent nonminimal state space model (11) and (12) we only require controllability (or stabilizability) of the pair \( (A_l, B_l) \), which is automatic. Finally, it is important to note that only the system matrix  \( A_l \) in (11) is unknown for full-state feedback control design, whereas the triple \( (A_p, B_p, C_p) \) is unknown in (1) and (2) for output feedback control.

**Remark 2.3.** For a nominal state space model the proposed nonminimal state space realization was first used in [13], [14] for active noise blocking and robust control.

### III. Adaptive Control for the Nonminimal State-Space Model

In this section, we introduce a direct adaptive state feedback control architecture for the nonminimal state space model (11) and (12) that guarantees adaptive output stabilization and command following for the original system (1) and (2). Specifically, consider the controlled linear multivariable uncertain system given by (11), where the known state vector  \( x_l(t) \),  \( t \geq 0 \), is given by (13), the unknown matrix  \( A_l \) is given by (15), and the known input matrix  \( B_l \) is given by (16).

Next, consider the reference model given by

\[
\dot{x}_m(t) = A_m x_m(t) + B_m r(t), \quad x_m(0) = x_{m0}, \quad t \geq 0, \quad (44)
\]

where  \( x_m(t) \in \mathbb{R}^{(m+p) n} \),  \( t \geq 0 \), is the model reference state vector,  \( r(t) \in \mathbb{R}^m \),  \( t \geq 0 \), is a bounded piecewise continuous reference input, and  \( A_m \in \mathbb{R}^{(m+p) n \times (m+p) n} \) and  \( B_m \in \mathbb{R}^{(m+p) n \times m} \) are given matrices with  \( A_m \) being Hurwitz. Since  \( A_m \) is Hurwitz, it follows from converse Lyapunov theory that there exist a scalar  \( \varepsilon_m > 0 \) and a positive-definite matrix  \( P_m \in \mathbb{R}^{(m+p) n \times (m+p) n} \) such that

\[
A_m^T P_m + P_m A_m \leq -\varepsilon_m I. \quad (45)
\]

Finally, note that since  \( r(t) \) is bounded for all  \( t \geq 0 \), then  \( x_m(t) \) is uniformly bounded for all  \( x_{m0} \in \mathbb{R}^{(m+p) n} \) and  \( t \geq 0 \).

For the next result, we assume that there exist a positive-definite matrix  \( Q^* \in \mathbb{R}^{p \times p} \) and a matrix  \( \Theta^* \in \mathbb{R}^{p \times (m+p) n} \) such that the expanded compatibility conditions

\[
B_l Q^* = B_m, \quad (46)
\]
\[
A_l + B_l \Theta^* = A_m, \quad (47)
\]
are satisfied.

**Theorem 3.1.** Consider the uncertain system given by (11) and the reference model given by (44), and assume the expanded compatibility conditions (46) and (47) hold. Then the adaptive feedback control law

\[
u(t) = \Theta(t) x_l(t) + Q(t) r(t), \quad t \geq 0, \quad (48)
\]

where  \( \Theta(t) \in \mathbb{R}^{p \times (m+p) n} \),  \( t \geq 0 \), and  \( Q(t) \in \mathbb{R}^{p \times p} \),  \( t \geq 0 \), with update laws

\[
\dot{\Theta}(t) = -B_m^T P_m e(t) x_l^T(t) \Gamma, \quad \Theta(0) = \Theta_0, \quad t \geq 0, \quad (49)
\]
\[
\dot{Q}(t) = -B_m^T P_m e(t) r(t) \Gamma, \quad Q(0) = Q_0, \quad (50)
\]

where  \( \Gamma_0 \in \mathbb{R}^{(m+p) n \times (m+p) n} \) and  \( \Gamma_Q \in \mathbb{R}^{p \times p} \) are positive definite and  \( e(t) = x_l(t) - x_m(t) \), guarantees that the solution  \( (e(t), \Theta(t), Q(t)) \) of the closed-loop system given by (11), (44), (48), (49), and (50) is Lyapunov stable for all \( (e_0, \Theta_0, Q_0) \in \mathbb{R}^{(m+p) n \times \mathbb{R}^{p \times p}} \) and  \( t \geq 0 \), and  \( x_l(t) \rightharpoonup x_m(t) \) as  \( t \to \infty \). Furthermore,  \( x_p(t), t \geq 0 \), satisfying (1) is bounded for all  \( x_{p0} \in \mathbb{R}^n \).

**Proof.** Note that with  \( u(t), t \geq 0 \), given by (48), it follows from (11) that

\[
\dot{x}_l(t) = A_l x_l(t) + B_l \Theta(t) x_l(t) + B_l Q(t) r(t), \quad x_l(0) = x_{l0}, \quad t \geq 0, \quad (51)
\]

or, equivalently, using (46) and (47),

\[
\dot{x}_l(t) = A_l x_l(t) + B_l \Theta(t) (\Theta(t) - \Theta^*) x_l(t) + B_l [Q^* + Q(t) - Q^*] r(t)
\]
\[
= [A_l + B_l \Theta(t)] x_l(t) + B_l (\Theta(t) - \Theta^*) x_l(t)
\]
\[
+ B_l Q^* r(t) + B_l \Psi(t) r(t)
\]
\[
= A_m x_l(t) + B_m r(t) + B_l [\Theta(t) - \Theta^*]
\]
\[
+ B_l \Psi(t) r(t), \quad x_l(0) = x_{l0}, \quad t \geq 0, \quad (52)
\]

where  \( \Psi(t) \triangleq \Theta(t) - \Theta^* \),  \( t \geq 0 \), and  \( \Psi(t) \triangleq Q(t) - Q^* \),  \( t \geq 0 \). Now, it follows from (44) and (52) that

\[
\dot{e}(t) = A_m e(t) + B_l \Psi(t) r(t) + B_l \Phi(t) r(t), \quad e(0) = x_{0} - x_{m0}, \quad t \geq 0. \quad (53)
\]

To show Lyapunov stability of the closed-loop system (49), (50), and (53), consider the Lyapunov function candidate

\[
V(\epsilon, \Phi, \Psi) = \epsilon^T P_m e + tr \Gamma_Q^{-1} \Psi^* \epsilon^T + tr \Gamma_\Theta^{-1} \Phi^* \Phi^T, \quad (54)
\]
here $P_m > 0$ satisfies (45), and note that $V(0,0,0) = 0$. Since $P_m$, $Q$, $Q^*$, and $Q^*$ are positive definite, $V(e, \Phi, \Psi) > 0$ for all $(e, \Phi, \Psi) \neq (0,0,0)$. In addition, $V(e, \Phi, \Psi)$ is radially unbounded. Now, using (49) and (50), it follows that the derivative of (54) along the closed-loop system trajectories is given by

$$
\dot{V}(e, \Phi, \Psi) = e^T(t)[A_n^TP_m + P_mA_m]e(t) + 2e^T(t)P_mB_B\Phi^T(t)x_I(t) + 2e^T(t)P_mB_B\Psi^T(t)y(t) + 2\text{tr}(\Gamma_1\Phi(t)Q^*-1\Phi^T(t)) + 2\text{tr}(\Gamma_Q\Psi(t)Q^*-1\Psi^T(t)) \\
= e^T(t)[A_n^TP_m + P_mA_m]e(t) + 2e^T(t)P_mB_B\Phi^T(t)x_I(t) + 2e^T(t)P_mB_B\Psi^T(t)y(t) - 2\text{tr}(\Psi(t)Q^*-1B_mB_m^T)e(t)r(t) - 2\text{tr}(\Phi(t)Q^*-1B_mB_m^T)e(t)x_I(t) \\
\leq -\epsilon_m e^T(t)e(t), \quad t \geq 0. \quad (55)
$$

Hence, the closed-loop system given by (11), (44), (48), (49), and (50) is Lyapunov stable for all $(e_0, \Theta_0, Q_0) \in \mathbb{R}^{(m+n)p} \times \mathbb{R}^{p \times (m+n)p} \times \mathbb{R}^{p \times p}$ and $t \geq 0$. Now, by the LaSalle-Yoshizawa theorem [17], $\lim_{t \to \infty} e_m(t) = 0$, and hence, $x(p)(t) \to x_m(t)$ as $t \to \infty$.

Finally, to show that $x(p)(t) \to 0$, satisfying (1) is bounded, note that since $x_m(t) > 0$, is the state of a stable reference model given by (44), it follows that $x(p)(t)$, $t \geq 0$, is bounded. In addition, since the first $p$ components of $x(p)(t)$, $t \geq 0$, given by (13) correspond to the filtered output of the original system $y(t)$, $t \geq 0$, and the $(p+1)$ to $(n+m)$ components of $x(p)(t)$, $t \geq 0$, given by (13) correspond to the filtered input $u(t)$, $t \geq 0$, of the original system input $u(t)$, $t \geq 0$, it follows that $y(t)$, $t \geq 0$, and $u(t)$, $t \geq 0$, are bounded. Now, since the filter constant by (14) is asymptotically stable, it follows that $y(t)$, $t \geq 0$, and $u(t)$, $t \geq 0$, are bounded. Similarly, $y(t)$, $y_1(t)$, $y_2(t)$, $\ldots$, $y(n-1)(t)$, and $u(t)$, $u_1(t)$, $u_2(t)$, $\ldots$, $u(n-1)(t)$, $t \geq 0$, are bounded, and hence, uniformly continuous. Hence, it follows from the minimality of $(A_p, B_p, C_p)$ that $x(p)(t)$, $t \geq 0$, is bounded. □

A block diagram showing the adaptive control architecture given in Theorem 3.1 is shown in Fig. 1.

**Remark 3.1.** Theorem 3.1 shows that $x(p)(t) \to x_m(t)$ as $t \to \infty$. Since the first $p$ components of $x(p)(t)$, $t \geq 0$, given by (13) correspond to the filtered output of the original system $y(t)$, $t \geq 0$, we can always choose an appropriate reference model for (44) that captures a desired tracking behavior for $y(t)$, $t \geq 0$. Furthermore, since the filter constant $\lambda$ is a positive design parameter, it can be chosen large enough to minimize the transient response difference between $y(t)$ and $y(t)$ for all $t \geq 0$. Hence, Theorem 3.1 guarantees adaptive output stabilization and command following for the original uncertain dynamical system (1) and (2).

**Remark 3.2.** It is important to note that the adaptive laws (49) and (50) do not require explicit knowledge of $Q^*$ and $\Theta^*$. In addition, no specific structure on the uncertain matrices $(A_p, B_p, C_p)$ within $A_I$ in (11) are required as long as the expanded compatibility conditions (46) and (47) are satisfied. Furthermore, the expanded compatibility conditions (46) and (47) are less restrictive than standard matching conditions for model reference adaptive control appearing in the literature involving the actual system model dynamics $(A_p, B_p, C_p)$; see, for example, Chapter 5 in [18] or Assumption 4.1 in [7]. This is due to the fact that the states of the original system (1) and (2) are unknown, and hence, matching conditions of the form given by (46) and (47) cannot be used in the context of output feedback adaptive control.

**IV. ILLUSTRATIVE NUMERICAL EXAMPLES**

In this section, we present two numerical examples of increasing complexity to illustrate the efficacy of the proposed adaptive control architecture.

**Example IVA (Minimum phase, damped, asymptotically stable plant).** Consider the plant given by

$$
\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -20 & -1.5 \end{bmatrix} x(t) + \begin{bmatrix} -2 \\ 1 \end{bmatrix} u(t), \quad t \geq 0, \quad (56)
$$

$$
y(t) = \begin{bmatrix} -1 & 0 \end{bmatrix} x(t), \quad (57)
$$

with $x(0) = [2, -1]$, and poles $\{-1.25 \pm 4.29j\}$ and zero $\{ -1.00 \}$. Let $\lambda = 10$ and let the reference model matrices in (44) be given by

$$
A_m = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -5 & -12.5 & -12.5 & -5 \end{bmatrix}, \quad B_m = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} . \quad (58)
$$

Note that $A_m$ and $B_m$ given in (58) satisfy the expanded compatibility conditions in (46) and (47). Finally, let $\varepsilon_m = 50$ and $\Gamma_0 = 1000$. Our aim here is to achieve output stabilization. The closed-loop response along with the control signal and adaptive gain is shown in Fig. 2. △

**Example IVB (Nonminimum phase, unstable plant).** Consider the plant given by

$$
\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -20 & 0.5 \end{bmatrix} x(t) + \begin{bmatrix} -2 \\ 1 \end{bmatrix} u(t), \quad t \geq 0, \quad (59)
$$

$$
y(t) = \begin{bmatrix} -1 & 0 \end{bmatrix} x(t), \quad (60)
$$

with $x(0) = [2, -1]$, and poles $\{0.25 \pm 4.47j\}$ and nonminimum phase zero $\{1.00\}$. Here we use the same control design as in Example IVA. Fig. 3 shows the closed-loop system.
response along with the control signal and the adaptation gain.

Fig. 2. Closed-loop response for Example IV.A.

Fig. 3. Closed-loop response for Example IV.B.

V. CONCLUSION

In this paper we presented an output feedback direct adaptive control architecture for linear multivariable uncertain systems with nonminimum phase zeros. The proposed adaptive control algorithm is predicated on a nonminimal state space realization involving an expanded set of states with filtered versions of the system inputs and outputs. The controller does not require any information of the system poles and zeros. Future work will include extensions to discrete-time systems and nonlinear uncertain dynamical systems with unstable zero dynamics.

REFERENCES