A Unified Approach for Robust Stability Design of PID Controllers

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Abstract—In this paper a graphical technique is introduced for finding all continuous-time or discrete-time proportional integral derivative (PID) controllers that satisfy a robust stability constraint for an arbitrary order transfer function with time delay. These problems can be solved by finding all achievable PID controllers that simultaneously stabilize the closed-loop characteristic polynomial and satisfy constraints defined by a set of related complex polynomials. The key advantage of this procedure is that this method depends only on the frequency response of the system. The ability to include the time delay in the nominal model of the system will often allow for designs with reduced conservativeness in plant uncertainty and an increase in size of the set of all PID controllers that robustly stabilize the system. The delta operator is used to describe the controllers in a discrete-time model, because it not only possesses numerical properties superior to the discrete-time shift operator, but also converges to the continuous-time controller as the sampling period approaches zero. A unified approach allows us to use the same procedure for discrete-time and continuous-time robust stability design of PID controllers.

I. INTRODUCTION

Due to the extensive use of proportional integral derivative (PID) controllers in industrial and bioengineering applications, there has been a significant effort to determine the set of all PID controllers that meet specific design goals. As the target of this research is to develop design methods that can be applied in industry, these methods should have several key elements. First, they should be applicable to a broad set of plants. In order for the methods to be applicable in the process control industry, it is particularly important that they handle time-delays. Ideally, the design methods should be simple to understand and easy to implement. Methods that depend only on the frequency response of the system eliminate the need for a plant model, which may not be available in some applications. In biological systems, for example, rational transfer functions models often do not exist. Finally, designs done directly in the digital domain allow for easy computer implementation.

Not surprisingly, most of the early work in this area sought to find all continuous-time PID controllers that stabilized the nominal plant model. Much of early work in this area was done by Bhattacharyya and colleagues and assumed knowledge of the system transfer function model [1], [2], [3], [4], and [5]. The method introduced by Tan in [6] broke the numerator and denominator of the plant transfer function into even and odd parts. Shafiei and Shenton found all PID controllers that placed the closed-loop poles in certain D-partitions [7]. In [8] and [9], a new method, which did not involve complex mathematical derivations, was used to solve the problem of stabilizing an arbitrary order transfer function, when only the frequency response of the plant transfer function was known. Beyond stability, investigators have also looked at performance and robustness. The authors in [6] and [8] found regions where the controllers were guaranteed to meet certain gain and phase margin requirements. PID controllers that also satisfy gain crossover, phase crossover, and bandwidth requirements for double integrator systems with delay were found in [10].

As these controllers must be implemented on real systems, design methods that deal with robustness are of particular importance. In [11], [12], and [13], Saeki and colleagues looked at different methods for $H_\infty$ controller design of PID controllers. Bhattacharyya and Keel looked at $H_\infty$ design of first-order controllers in [14]. They looked at weighted sensitivity and weighted complementary sensitivity design of PID controllers for plants with no poles or zeros on the $j\omega$ axis in [15]. Unfortunately, none of these methods that dealt with robustness worked directly with systems with time-delay, which are prevalent in the process control industry. In [16], Keel and Bhattacharyya did allow for time-delays in the nominal model when they investigated the stability problem for plants with no poles or zeros on the $j\omega$ axis and with known time delay. All of the methods in [1]-[16] are based in the continuous-time.

As more and more controllers are implemented as digital compensators, design methods that work directly in the digital domain become more important. Unfortunately, most of the work in this area has concentrated on design of continuous-time PID controllers. In [17], the delta operator was used to obtain a unified approach for finding stability boundaries of PID controllers for arbitrary order transfer functions with time delay in the frequency domain. The delta operator was used to describe controllers in the discrete-time, because it provides not only numerical properties superior to the discrete-time shift operator, but also converges to the continuous-time as the sampling period approaches zero [18] and [19]. In [20], Suchomski used the delta operator to design robustly stable PID controllers for low order known system transfer functions. In [21], a discrete PID controller was designed in the Z-domain for a reduced second order model by using pole-zero cancellation.
in order to meet the desired performance specification. In [22], Vu found the optimal gains of discrete PID controllers in the Z-domain by minimizing the variance of the output variable.

In [23], [24], [25], [26], and [27] and the authors of these papers developed techniques for finding all achievable PID controllers that simultaneously stabilize the closed-loop continuous-time system and satisfy an $H_{\infty}$ sensitivity, $H_{\infty}$ complementary sensitivity, weighted sensitivity, robust stability, or robust performance constraint, respectively. In [28], [29], and [30], this method was extended to a unified approach for continuous-time and discrete-time $H_{\infty}$ sensitivity, $H_{\infty}$ complementary sensitivity, or weighted sensitivity design of PID controllers.

In this paper the goal is to define a unified approach for continuous-time and discrete-time robust stability design of all achievable PID controllers. This method is applicable for single-input-single-output (SISO) linear time invariant (LTI) proper transfer functions of any order with time delay. A unified approach using the delta operator allows us to apply the same procedure for discrete-time and continuous-time robust stability design. As this work builds upon the straightforward development in [17] and [28], it does not require the plant transfer function model, but only the frequency response of the system. If the plant transfer function is known, we can apply the same procedures by first computing the frequency response.

The remainder of this paper is organized as follows. In Section II, the design methodology is introduced. In Section III, this method is applied to a numerical example that demonstrates the application of this methodology. Finally the conclusion is presented in Section V.

II. DESIGN METHODOLOGY

A SISO LTI continuous-time plant transfer function is defined as:

$$G_p(s) = G_0(s)e^{-\tau s},$$  \hspace{1cm} (1)

where $G_0(s)$ is an arbitrary-order transfer function and $\tau$ is the time delay. The ability to include the time delay in the nominal model allows the designer to find much tighter uncertainty bounds in the system with known delays than would be possible otherwise [31]. The equivalent model of (1) in the delta domain, when the output of plant is sampled and a zero-order hold is placed at the input, can be found from [18] as:

$$G_p(\gamma) = \frac{\gamma}{1+\gamma T_0} \left[ L^{-1} \left\{ \frac{1}{s} G_p(s) \right\} \right],$$  \hspace{1cm} (2)

where $T_0$ is the sampling period, $\mathcal{T}$ is the generalized transform, $L$ is the Laplace transform, and $\gamma$ is defined in [19] as:

$$\gamma = \begin{cases} s, & T_0 = 0 \\ e^{\omega T_0} - 1, & T_0 \neq 0. \end{cases}$$  \hspace{1cm} (3)

Consider the SISO system shown in Figure 1, where $G_p(\gamma)$ is the nominal plant, $G_c(\gamma)$ is the PID controller, $W_T(\gamma)$ is the multiplicative weight, and $|\Delta_T(\gamma)| \leq 1$ is the uncertain perturbation. The input signal and the output signal of the controlled plant are $R(\gamma)$ and $Y(\gamma)$, respectively. The PID controller is defined as:

$$G_c(\gamma) = K_p + K_i + K_d \frac{\gamma}{1+T_0\gamma},$$  \hspace{1cm} (4)

where $K_p$, $K_i$, and $K_d$ are the proportional, integral, and derivative gains, respectively.

![Fig. 1. Block diagram of the system with multiplicative uncertainty.](image)

The transfer functions in Figure 1 can be expressed in the frequency domain. The plant transfer function and the multiplicative weighted function can be written in terms of their real and imaginary parts as:

$$G_p(\beta) = R_c(\beta) + j I_m(\beta),$$  \hspace{1cm} (5)

and

$$W_T(\beta) = A_T(\beta) + jB_T(\beta),$$  \hspace{1cm} (6)

where $\beta = \begin{cases} j\omega & T_0 = 0 \\ e^{j\omega T_0} - 1 & T_0 \neq 0. \end{cases}$ The PID controller is defined in the frequency domain as:

$$G_c(\beta) = K_p + K_i + K_d \frac{\beta}{1+T_0\beta}. \hspace{1cm} (7)$$

The deterministic values of $K_p$, $K_i$, and $K_d$ for which the closed-loop characteristic polynomial is Hurwitz stable have been found in [17] with a small difference in the parameterization of the PID controllers. In this paper, the problem is to find all PID controllers that stabilize the system and satisfy the robust stability constraint as:
$\|W_T(\beta)T(\beta)\|_\infty \leq \gamma_0$, \hspace{1cm} (8)

where $T(\beta) = \frac{G_p(\beta)G_c(\beta)}{1 + G_p(\beta)G_c(\beta)}$ is the complementary sensitivity function and $\gamma_0$ is a positive real scalar. The complex function in (8) for a SISO system for each value of $\omega$ can be written in terms of its magnitude and phase angles as:

$\|W_T(\beta)T(\beta)e^{i\theta T}\| \leq \gamma_0$, \hspace{1cm} $\forall \omega$. \hspace{1cm} (9)

If (9) holds, then for each value of $\omega$

$W_T(\beta)T(\beta)e^{i\theta T} \leq \gamma_0$, \hspace{1cm} (10)

must be true for some $\theta_T \in [0, 2\pi)$, where $\theta_T = -\angle W_T(\beta)T(\beta)$. Consequently, all PID controllers that satisfy (8) must lie at the intersection of all controllers that satisfy (10) for all $\theta_T \in [0, 2\pi)$ [26].

To accomplish this, for each value of $\theta_T \in [0, 2\pi)$ we will find all PID controllers on the boundary of (10). It is easy to show from (10), that all the PID controllers on the boundary must satisfy:

$P(\omega, \theta_T, \gamma_0, T_0) = 0$. \hspace{1cm} (11)

where $P(\omega, \theta_T, \gamma_0, T_0) = 1 + G_p(\beta)G_c(\beta) - \frac{1}{\gamma_0} G_p(\beta)G_c(\beta)W_T(\beta)e^{i\theta T}$. Note that (11) reduces to the frequency response of the standard closed-loop characteristic polynomial as $\gamma_0 \to \infty$.

Substituting, (5), (6), (7), and $e^{i\theta T} = \cos \theta_T + j \sin \theta_T$ into (11), and solving for the real and imaginary parts yields:

$X_{Rp} K_p + X_{Ri} K_i + X_{Rd} K_d = -\omega$, \hspace{1cm} (12)

$X_{Ip} K_p + X_{Ii} K_i + X_{Id} K_d = 0$, \hspace{1cm} (13)

where $X_{Rp} = \frac{\omega}{\gamma_0} \left( R_c(\beta)(\Phi) + I_m(\beta)(\Psi) \right)$,

$X_{Ri} = \frac{1}{\gamma_0} \left( -R_c(\beta) \left( \frac{2\omega T_0^2}{\gamma_0}(\Phi) \cos(\omega T_0) + 1 \right) \right)$

$X_{Rd} = \frac{\omega^2}{\gamma_0} \left( -R_c(\beta) \left( \frac{2\omega T_0^2}{\gamma_0}(\Phi) \cos(\omega T_0) + 1 \right) \right)$

$X_{Ip} = \frac{\omega}{\gamma_0} \left( -R_c(\beta)(\Psi) + I_m(\beta)(\Phi) \right)$

$X_{Ii} = \frac{1}{\gamma_0} \left( I_m(\beta) \left( \frac{2\omega T_0^2}{\gamma_0}(\Psi) \cos(\omega T_0) + 1 \right) \right)$

$X_{Id} = \frac{\omega^2}{\gamma_0} \left( I_m(\beta) \left( \frac{2\omega T_0^2}{\gamma_0}(\Psi) \cos(\omega T_0) + 1 \right) \right)$

$\Phi = -A_T(\beta) \cos \theta_T + B_T(\beta) \sin \theta_T + \gamma_0$, \hspace{1cm} $\Psi = A_T(\beta) \sin \theta_T + B_T(\beta) \cos \theta_T$.

This is a three-dimensional system in terms of the controller parameters $K_p$, $K_i$, and $K_d$. The boundary of (11) can be found in the $(K_p, K_i)$ plane for a fixed value of $K_d$. After setting $K_d$ to the fixed value $\tilde{K}_d$, (12) and (13) can be rewritten as:

$\begin{bmatrix} X_{Rp} & X_{Ri} \end{bmatrix} \begin{bmatrix} K_p \cr K_i \end{bmatrix} = -\omega \begin{bmatrix} -X_{Rd} \tilde{K}_d \cr -X_{Id} \tilde{K}_d \end{bmatrix}$. \hspace{1cm} (14)

Solving (14), for all $\omega \neq 0$ and $\theta_T \in [0, 2\pi)$, gives the following equations:

$K_p(\omega, \theta_T, \gamma_0, T_0) = \frac{-R_c(\beta)\left( (\Phi)(\cos(\omega T_0) + 1) - (\Psi)(\sin(\omega T_0)) \right) - I_m(\beta)(\Phi)(\sin(\omega T_0) + (\Psi)(\cos(\omega T_0) + 1))}{\gamma_0 D(\beta)(\cos(\omega T_0) + 1)}$, \hspace{1cm} (15)

$K_i(\omega, \theta_T, \gamma_0, T_0) = \omega^2 \tilde{K}_d \frac{2\sin^2(\omega T_0)}{\cos(\omega T_0) + 1}$

$\frac{2\omega \sin(\omega T_0)(R_c(\beta)(\Psi) - I_m(\beta)(\Phi))}{\gamma_0 D(\beta)(\cos(\omega T_0) + 1)}$, \hspace{1cm} (16)
\[
D(\beta) = \left| G_p(\beta) \right|^2 \left( 1 - \frac{2}{\gamma_0} (A_T \cos \theta_T - B_T \sin \theta_T) + \frac{1}{\gamma_0^2} W_T(\beta) \right)^2,
\]

\[
\left| G_p(\beta) \right|^2 = R_c^2(\beta) + I_m^2(\beta),
\]

\[
\left| W_T(\beta) \right|^2 = A_T^2(\beta) + B_T^2(\beta).
\]

Setting \( \omega = 0 \) in (14), we obtain \( K_p(0, \theta_T, \gamma_0, T_0) \) is arbitrary and \( K_i(0, \theta_T, \gamma_0, T_0) = 0 \), unless \( I_m(0) = R_c(0) = 0 \), which holds only when \( G_p(\gamma) \) has a zero at the origin. By letting \( T_0 \to 0 \) in (15) and (16), the continuous-time robust stability boundaries, which are equivalent to those in [26], are found.

The procedure can be repeated in the \((K_p, K_d)\) plane. After setting \( K_i \) to a fixed value \( \tilde{K}_i \), (12) and (13) can be rewritten as:

\[
\begin{bmatrix}
X_{Ri} & X_{Rd} \\
X_{Li} & X_{Id}
\end{bmatrix}
\begin{bmatrix}
K_p \\
K_i
\end{bmatrix}
= \begin{bmatrix}
-\omega - X_{Ri} \tilde{K}_i \\
- X_{Li} \tilde{K}_i
\end{bmatrix}.
\]

Solving (17) for all \( \omega \neq 0 \), \( \theta_T \in [0, 2\pi) \) gives the same expression as (15) for \( K_p(\omega, \theta_T, \gamma_0, T_0) \), and the following equation for \( K_d(\omega, \theta_T, \gamma_0, T_0) \):

\[
K_d(\omega, \theta_T, \gamma_0, T_0) = \tilde{K}_i \frac{\cos(\omega T_0) + 1}{2\omega^2 \sin^2(\omega T_0)} + \frac{(-R_c(\beta)(\Psi) + I_m(\beta)(\Phi))}{\omega \gamma_0 D(\beta) \sin(\omega T_0)}.
\]

At \( \omega = 0 \), \( \tilde{K}_i \) must be equal to zero for a solution to exist. Furthermore, as \( I_m(0) = 0 \) for all real plants, in these special cases, \( K_d(0, \theta_T, \gamma_0, T_0) \) is arbitrary and

\[
K_p(0, \theta_T, \gamma_0, T_0) = \frac{-\gamma_0}{R_c(0) \cos \theta_T + B_T(0) \sin \theta_T + \gamma_0}.
\]

Letting \( T_0 \to 0 \) in (15) and (18), the continuous-time robust stability boundaries, which are equivalent to those in [26], are found.

Lastly, the solution is found in the \((K_i, K_d)\) plane. After setting \( K_p \) to a fixed value of \( \tilde{K}_p \), (12) and (13) are rewritten as:

\[
\begin{bmatrix}
X_{Ri} & X_{Rd} \\
X_{Li} & X_{Id}
\end{bmatrix}
\begin{bmatrix}
K_i \\
K_d
\end{bmatrix}
= \begin{bmatrix}
-\omega - X_{Ri} \tilde{K}_p \\
- X_{Li} \tilde{K}_p
\end{bmatrix}.
\]

Although the coefficient matrix is singular, a solution will exist in two cases. First, at \( \omega = 0 \), \( K_d(0, \theta_T, \gamma_0, T_0) \) is arbitrary and \( K_i(0, \theta_T, \gamma_0, T_0) = 0 \), unless \( I_m(0) = R_c(0) = 0 \), which holds only when the plant has a zero at the origin. In such case, a PID compensator should be avoided as the PID pole cancels the zero at the origin and the system becomes internally unstable. A second set of solutions occurs at any frequency \( \omega_1 \) where \( K_p(\omega_1, \theta_T, \gamma_0, T_0) \) from (15) is equal to \( \tilde{K}_p \). At these frequencies, \( K_d(\omega_1, \theta_T, \gamma_0, T_0) \) and \( K_i(\omega_1, \theta_T, \gamma_0, T_0) \) must satisfy the following straight line equation:

\[
K_d(\omega_1, \theta_T, \gamma_0, T_0) = K_i(\omega_1, \theta_T, \gamma_0, T_0) \frac{\cos(\omega T_0) + 1}{2\omega^2 \sin^2(\omega T_0)} + \frac{(-R_c(\beta)(\Psi) + I_m(\beta)(\Phi))}{\omega_1 \gamma_0 D(\beta) \sin(\omega_1 T_0)}.
\]

Letting \( T_0 \to 0 \) in (21), the continuous-time robust stability boundaries, which are equivalent to those in [26], are found.

### III. Example

In this section, a PID controller is designed to regulate the shaft position of a DC motor. The feedback loop has an unknown communication delay between 0.05 and 0.15 seconds. The discrete-time PID controller should robustly stabilize the system with the uncertain communication delay. The sampling period is \( T_0 = 0.1 \) seconds. The nominal model of the DC motor is given by

\[
G_p(s) = \frac{65.5}{s(s + 34.6)} e^{-\tau},
\]

where \( \tau \) has been selected to be the mean value of the uncertain communication delay, 0.1 seconds. The multiplicative weight is chosen to bound the multiplicative errors for different communication delays [31], [32] as

\[
W_T(s) = \frac{s}{0.357 s + 20}.
\]

Note, by including the time delay in the nominal model we are able to reduce the conservativeness in our plant uncertainty and increase the size of the set of discrete-time PID controllers that robustly stabilize the system.

The procedure for finding the discrete-time stability boundary of the nominal system and the discrete-time robust
stability region in the \((K_p, K_i)\) plane for a fixed value of \(\hat{K}_d = 0.2\) is given as:

1) Using (2), the discrete-time delta-domain equivalent of the system in (22) is given by:

\[
G_p(\gamma) = \frac{1.3631\gamma + 18.3357}{\gamma(\gamma + 9.6857)}(1 + T_0/\gamma)^{-0.1}.
\]  
(24)

2) The discrete-time delta-domain bilinear approximation of (23) is given by:

\[
W_T(\gamma) = \frac{0.7368\gamma}{\gamma + 14.7368}.
\]  
(25)

3) Equations (5) and (6) are used to find the real and imaginary parts of (24) and (25) in the frequency domain, respectively.

4) The discrete-time PID stability boundary of the nominal system can be found by setting \(\gamma_0 = \infty\) in (15) and (16).

5) All PID controllers that satisfy the robust stability constraint in (8) are found by setting \(\theta_T = 1\) in (15) and (16) for \(\theta_T \in [0, 2\pi] \) and finding the intersection of all regions.

The discrete-time stability boundary and the region that satisfies the discrete-time robust stability constraint are shown in Figure 2. The intersection of all regions inside the stability boundary of the \((K_p, K_i)\) plane is the discrete-time robust stability region.

To verify the results, an arbitrary controller from this region is chosen, giving us the discrete-time PID controller:

\[
G_c(\gamma) = 1.7 + \frac{3.42}{\gamma} + \frac{0.2\gamma}{1 + 0.1\gamma}.
\]  
(26)

The substitution of (24), (25) and (26), into (8) gives \(\|W_T(\beta)T(\beta)\|_{\infty} = 0.578\). As \(\|W_T(\beta)T(\beta)\|_{\infty} \leq 1\), the system is robustly stable.

The second method uses (15) and (18) in the \((K_p, K_d)\) plane for a fixed value of \(\hat{K}_p\). Plots of \(K_p(\omega_i, \theta_T, \infty, T_0)\) and \(K_p(\omega_i, \theta_T, \gamma_0, T_0)\), from (15), for various values of \(\theta_T \in [0, 2\pi]\) are shown in Figure 5. From Figure 5 it is clear that \(\hat{K}_p\) must be chosen between [0, 3.2]. Here \(\hat{K}_p\) is chosen to be 0.5. For each curve, the \(\omega_i\)'s are the frequencies at which \(K_p(\omega_i, \theta_T, \gamma_0, T_0) = \hat{K}_p = 0.5\). Each \(\omega_i\) is substituted into (21) to find the required boundaries. In addition, we have the boundary at \(K_i(0, \theta_T, \gamma_0, T_0) = 0\).

The region that satisfies the discrete-time robust stability constraint and the stability boundary is shown in Figure 6. The intersection of all regions inside the discrete-time stability boundary of the \((K_i, K_d)\) plane is the robust stability region.
To verify the results, an arbitrary controller from this region is chosen, giving us the discrete-time PID controller

\[ G_c(\gamma) = 0.5 + \frac{0.56}{\gamma} + \frac{0.06\gamma}{1+0.1\gamma}. \]  

(27)

The substitution of (24), (25), and (27) into (8) gives \( \|P_T(\beta)T(\beta)\|_{\infty} = 0.181 \), which is less than \( \gamma_0 = 1 \). Consequently, the design goal is met.

IV. CONCLUSIONS

A graphical technique was introduced for finding all achievable continuous-time or discrete-time PID controllers that satisfy a robust stability constraint of a given SISO transfer function with time delay. This method is simple to understand and requires only the frequency response of the plant. A numerical example of a DC motor with an unknown communication delay in the feedback loop was presented to demonstrate the application of this method. It was shown that the continuous-time and discrete-time designs can be understood under a common framework through the delta operator.

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