Teaching Linear-Quadratic Optimal Control to Undergraduate Students

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Abstract—In this paper we show that the solution to a dynamic optimization problem: optimize an integral quadratic performance criterion along trajectories of a linear dynamic system over an infinite time period (steady-state linear-quadratic optimal controller) can be solved using elementary mathematics as a static optimization problem. The derivations require only a basic (undergraduate) usage of linear algebra and state space linear system analysis. Having full understanding of derivations of the linear-quadratic optimal controller, students and engineers will feel confident to use these controllers in numerous engineering and scientific applications.

I. INTRODUCTION

Optimal control theory is considered as an advanced graduate level topic and taught as an advanced graduate level engineering course. Optimal control is hardly even mentioned to undergraduate engineering students. The main reason is certainly the use of advanced mathematics needed to formulate the optimality concept and derive the corresponding optimal results. Either the calculus of variations or dynamic programming has to be used to solve the general problem of optimal control theory: minimize a performance criterion that carries information about the system state variables and system inputs along trajectories of a system whose dynamics is represented by either differential (concentrated parameter systems) or partial differential (distributed parameter systems) equations. The dynamic programming technique was solely developed by Belman in the middle of the 19950s, [1]. Belman formulated his famous principle of optimality. The calculus of variations, a mathematical discipline that was rapidly developing in the past century, was used by Russian mathematicians headed by Pontryagin to derive the well-known principle, the minimum principle, [2], also known as the Pontryagin minimum principle. Both approaches, dynamic programming and calculus of variations are essential for the development of optimal control theory. These approaches are out of reach of undergraduate engineering students. They are commonly taught in advanced graduate courses on optimal control theory and its applications. It should be pointed out that there are several standard optimal control theory textbooks used to teach engineering graduate courses in optimal control, [3]-[14].

In the following, we will present a particular class of optimal controllers that can be fully understood and derived by undergraduate students. This class of controllers, linear-quadratic (LQ) optimal controllers, has the greatest potential for actual applications among all optimal controllers. In this paper, we will provide the problem formulation of the LQ optimal controller at steady state, and analytically justify all steps in the process of deriving the optimal LQ-controller gain. To that end, we will use only mathematical knowledge of elementary linear algebra and basic state space results about linear dynamic systems. These tools are in general known to most of undergraduate engineering students.

In general undergraduate students have excellent knowledge and understanding of linear dynamic systems and corresponding linear system control problems. Formulating a quadratic performance measure that has to be optimized along trajectories of a dynamic system, should not be a strange concept to undergrads. The corresponding optimal control problem is known in the literature as the linear-quadratic (LQ) optimal control problem. It is interesting to observe that at steady state the linear-quadratic optimal controller is a proportional (P) controller. All undergraduate students are very well familiar with the proportional-integral-derivative (PID) controller, and its variants proportional-integral (PI), proportional-derivative (PD), and proportional (P) controllers - the LQ optimal controller might be considered as a special class of the P-controllers.

II. PROBLEM FORMULATION

Consider a time invariant linear system in the state space form

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t), \quad x(0) = x_0 \quad (1)$$

where $x(t)$ is the state space vector of dimension $n$, and $u(t)$ is the system input vector of dimension $m$. Matrices $A$ and $B$ are constant and of appropriate dimensions. An integral quadratic performance criterion to be optimized with respect to the system input is associated with (1)

$$J = \frac{1}{2}\int_0^\infty \left[ x^T(t)R_1x(t) + u^T(t)R_2u(t) \right] dt \quad (2)$$

$$R_1 = R_1^T \geq 0, \quad R_2 = R_2^T > 0$$

In the performance criterion, the matrices $R_1$ and $R_2$ are symmetric penalty matrices, respectively, positive.
semidefinite (all the eigenvalues of \( R_1 \) are in the closed right half of the complex plane (including the imaginary axis)) and positive definite (all the eigenvalues of \( R_2 \) are strictly in the right half complex plane (excluding the imaginary axis)). It follows from (2) that the optimization is done over an infinite time interval, and the corresponding controller is also known as the steady-state optimal linear-quadratic controller.

The choice of a quadratic performance criterion in (2) is pretty reasonable. Namely, the goal is that (2) be minimized at every time instant, meaning that the square of the state variables and the “square” of the control input signals should be minimized. Hence, the absolute goal is to regulate these variables to zero. This is very realistic when the considered state space model represents system deviations from the nominal trajectories, in which case regulating state space variables to zero means bringing the system back to the nominal trajectories. Such state space models are usually obtained by performing linearization of nonlinear systems with respect to their nominal trajectories and nominal control inputs. Minimizing the square of the system input we simply want to avoid large signals to enter the system. In the case when the state space model represents dynamics of a pure linear system, the regulation of the system state space trajectories to zero might not be the desired goal. Fortunately, there is a technique that allows regulation of state space trajectories of a linear system to non-zero values, set-point optimal controllers. Such optimal controllers basically have the same problem formulation as (1) and (2) with a minor modification in the implementation of the system closed-loop in order to achieve the desired non-zero set point. This modification will be presented in Section III.2.

**A. Choice of the Penalty Matrices**

The penalty matrices are chosen by control engineers using their experience dealing with particular control problems, and very often they are diagonal matrices. In the case when they are diagonal matrices, the engineering experience indicates that the more weight we put on a given state space variable or a given input variable the more important that variable is. For example, for a system of order \( n = 2 \), the choice \( R_1 = \text{diag}[10,1] \) indicates that for this particular optimization problem, it is much more important that the state variable \( x_1(t) \) is as close as possible to zero than the state variable \( x_2(t) \). If there is no need to regulate a particular state variable to zero, or even to keep it close to zero, the corresponding weight should be equal to zero. For example, the choice \( R_1 = \text{diag}[10,0] \) indicates that the state variable \( x_2(t) \) can take arbitrary values in the optimization process. The choice of penalty matrices gives a lot of freedom to control engineers to achieve their goal of optimally controlling a system. By increasing diagonal elements in the matrix \( R_2 \), we will get the optimal solution and the corresponding controller that has smaller input signals. For a mathematical reason, in order to have a unique solution to the optimization problem defined in (1) and (2), the matrix \( R_2 \) must be nonsingular. This will become obvious from the expression that gives the solution for the considered optimization problem to be derived in Section III of this paper.

We assume that full-state feedback is available and define feedback control as

\[
u(x(t)) = -Fx(t) \quad (3)
\]

This optimization problem, defined by (1)-(3), was solved in 1960 by R. Kalman in his famous paper [15]. The corresponding linear-quadratic optimal controller is often called in the control literature the Kalman regulator (controller).

In the next section, we will show how the solution to the optimization problem defined in (1)-(3) can be obtained using the undergraduate knowledge of linear algebra and state space analysis, so that the corresponding controller can be introduced to undergraduate students. As a matter of fact, the state feedback controller given in (3) is a simple constant feedback controller (assuming that all state variables are available for feedback, full-state feedback). It is much simpler for implementation than dynamic controllers that require either signal differentiation or signal integration.

Having full understanding of derivations of the linear-quadratic optimal controller, students and engineers will feel confident to use this controller in numerous engineering and scientific applications.

**III. Solution of the Linear-Quadratic Optimal Control Problem**

The solution to the optimization problem defined in (1)-(3) will be obtained via constrained static optimization. To that end, we will need four fundamental mathematical results that will be derived AND explained in detail using simple linear algebra knowledge.

Under state feedback control (3), formulas (1) and (2) become

\[
\frac{dx(t)}{dt} = (A - BF)x(t), \quad x(0) = x_0
\]

and

\[
J = \frac{1}{2} \int_{0}^{\infty} x^T(t) \left(R_1 + F^T R_2 F\right) x(t) dt \quad (5)
\]

The state trajectory under feedback control can be found from (4) in terms of the matrix exponential [7] using the well-known state space result [7], [14]

\[
x(t) = e^{(A - BF)t} x(0)
\]

Substituting (6) into (5) gives

\[
J = \frac{1}{2} x^T(0) \left\{ \int_{0}^{\infty} e^{(A - BF)t} \left(R_1 + F^T R_2 F\right) e^{(A - BF)t} dt \right\} x(0)
\]

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This integral can be evaluated via the solution of the following algebraic Lyapunov equation [16]

\[
(A-BF)^T P + P(A-BF) + R_1 + F^T R_2 F = 0
\]  

(8)

This fact can be easily established as demonstrated in Result 1. 

**Result 1:** It can be shown using simple algebra that the solution of the algebraic Lyapunov equation defined by

\[ M^T P + PM + N = 0, \quad N = N^T \]

is equal to

\[ P = \int_0^\infty e^{M^T t} Ne^M dt \]

This can be proved by observing that

\[
\frac{d}{dt} \left\{ e^{M^T t} Ne^M t \right\} = M^T e^{M^T t} Ne^M t + e^{M^T t} Ne^M M
\]

and integrating both sides of this expression from 0 to \( \infty \) leading to

\[
e^{M^T t} Ne^M t \bigg|_{t=0}^{t=\infty} = e^{M^T \infty} Ne^{M \infty} - e^{M^T 0} Ne^{M 0} = -N
\]

for the left-hand side and

\[
M^T \int_0^\infty e^{M^T t} Ne^M dt + \int_0^\infty e^{M^T t} Ne^M dt M = M^T P + PM
\]

for the right-hand side, that is

\[-N = M^T P + PM\]

which verifies the stated result. Note that in the above derivations we have used one of the main properties of the state transition matrix (matrix exponential), the derivative property given by

\[
\frac{d}{dt} e^{M^T t} = M e^{M^T t} = e^{M^T} M
\]

\[ e^{M0} = I, \quad e^{M \infty} = 0 \] for \( M \) asymptotically stable.

The performance criterion can be now evaluated as

\[
J = \frac{1}{2} x^T(0) P x(0) + \frac{1}{2} \text{trace} \left[ P x(0) x^T(0) \right]
\]  

(9)

Where the trace operator represents the sum of all diagonals elements of a square matrix. The last step can be easily justified as demonstrated in Result 2.

**Result 2:** We can use the property of the trace operator that states \( \text{trace}(MN) = \text{trace}(NM) \), assuming that the matrices \( M \) and \( N \) are of appropriate dimensions so that both products \( MN \) and \( NM \) are defined (square matrices are obtained in both cases). Using this property we have the equality \( \text{trace} \left[ x^T(0) P x(0) \right] = \text{trace} \left[ P x(0) x^T(0) \right] \). Also, we have used the fact that since \( x^T(0) P x(0) \) is a scalar, then \( x^T(0) P x(0) = \text{trace} \left[ x^T(0) P x(0) \right] \). Students can demonstrate this property easily starting with a simple two by two example and generalizing the result obtained to an arbitrary order.

To avoid the dependence of the performance criterion on the system initial conditions (problem dependent), it is common to assume that in general the system initial conditions are distributed on the unit sphere, so that the performance criterion is given by

\[
J = \frac{1}{2} \text{trace} \left[ P \right]
\]  

(10)

With the above derivations, the dynamic optimization problem: minimize (2) subject to (1) and (3), is converted into the static optimization problem: minimize (10) subject to (8). In this optimization problem, the unknown constant matrix gain \( F \) has to be determined.

**IV. DERIVATION OF THE OPTIMAL SOLUTION VIA STATIC OPTIMIZATION**

The solution to the defined static optimization problem in (8) and (10) can be obtained using the basic calculus result which requires that the Lagrangian be formed with \( L \) playing the role of a Lagrange multiplier

\[
\mathcal{L} = \text{trace} \left[ P + \left[ (A-BF)^T P + P(A-BF) + R_1 + F^T R_2 F \right] \right]
\]

(11)

and the necessary conditions for a minimum (in general an extremum) be applied

\[
\frac{\partial \mathcal{L}}{\partial F} = 0, \quad \frac{\partial \mathcal{L}}{\partial L} = 0, \quad \frac{\partial \mathcal{L}}{\partial P} = 0
\]

(12)

At this point we need a definition of the matrix operator stated in the previous formulas. Note that \( \mathcal{L} \) is a scalar function of matrix variables and that \( F, L, P \) are matrices. The above partial derivatives are termed as gradient matrices [17] and defined by a matrix whose \( i, j \) elements are given by (for example in the case of \( \frac{\partial \mathcal{L}}{\partial P} \))

\[
\frac{\partial \mathcal{L}}{\partial P} = \left[ \frac{\partial \mathcal{L}}{\partial P_{ij}} \right], \quad i, j = 1, 2, \ldots, n, \quad p_{ij} = [P]_{ij}
\]  

(13)

The known formula for the gradient matrix derivatives are given by

\[
\frac{\partial}{\partial X} \text{trace} \left[ MX \right] = M^T
\]

\[
\frac{\partial}{\partial X} \text{trace} \left[ MX^T \right] = M
\]  

(14.a)

\[
\frac{\partial}{\partial X} \text{trace} \left[ MXNX^T \right] = M^T X N^T + N^T X^T M^T
\]

(14.b)

\[
\frac{\partial}{\partial X} \text{trace} \left[ MX^N \right] = M^T X N + M X N
\]

Using the facts that \( \text{trace}(MN) = \text{trace}(NM) \) and \( \text{trace}(M) = \text{trace}(M^T) \), we have also variants of these formulas such as
Interestingly enough, it can be used to determine the system asymptotic stability [16].

**Comment:** Note that if we had kept the initial condition in the expression for the performance criterion as given in (9), the last equation would have been given by

\[
(A - BF)^T L + L(A - BF) + x_0 x_0^T = 0
\]

Since (15) and (16) are not affected by the system initial conditions, the expression for the optimal feedback gain is obviously not affected by the choice of the system initial conditions.

At this point we summarize the formulas obtained for the optimal controller, optimal feedback system, and optimal performance criterion

\[
u_{opt}(t) = -R_2^{-1} B^T P x_{opt}(t) = -F_{opt} x_{opt}(t)
\]

\[
A^T P + PA + R_1 - PBR_2^{-1} B^T P = 0
\]

\[
\frac{dx_{opt}(t)}{dt} = (A - BF_{opt})x_{opt}(t), \quad x_{opt}(0) = x_0
\]

\[
J_{opt} = \frac{1}{2} x_0^T P x_0 = \frac{1}{2} \text{trace}\{P x_0 x_0^T\}
\]

Note that the optimal feedback gain must be such that the closed-loop system is asymptotically stable, otherwise the performance criterion will become infinity. Since the algebraic Riccati equation is nonlinear and in the matrix form, it has many solutions. It was shown in the original work of Kalman, [15] that for controllable and observable systems the algebraic matrix Riccati equation has a unique solution that stabilizes the closed-loop systems, that is, the matrix \(A - BF_{opt}\) is asymptotically stable. Even more, the unique stabilizing solution is positive definite, \(P > 0\). These facts are formally stated as Result 4.

**Result 4:** Under the assumption that the system is controllable and observable, the unique positive definite solution \(P > 0\) of the algebraic Riccati equation exists such that the closed-loop system matrix \((A - BF_{opt})\) is asymptotically stable, [15]. The controllability is apparently with respect to the pair \((A, B)\). The observability is required with respect to the pair \((A, C)\) where \(R_1 = C^T C\) \((C\) is the Cholesky factor that can be obtained via the Cholesky decomposition under which any positive semidefinite matrix \(M \succeq 0\) can be written as \(M = N^T N\). One way of interpreting and understanding from a practical point of view the observability condition imposed on the pair \((A, C)\) is to assume that in the quadratic performance criterion (2) instead of \(x^T(t) R_1 x(t)\) we minimize the “square” of the system output, that is, \(y^T(t) y(t)\) with \(y(t) = C x(t)\), which leads to \(y^T(t) y(t) = x^T(t) C^T C x(t)\). In that case, the requirement \((A, C)\) is observable seems to be very natural. Hence, before solving the algebraic Riccati equation (16), we should first test for controllability and observability, which can be done using MATLAB to find the
that controllability and observability conditions imposed respectively on \((A,B)\) and \((A,C)\) can be relaxed to
the stabilizability condition of \((A,B)\) and detectability condition of \((A,C)\). Note that stabilizability means
controllability of only unstable system modes (state space variables) and detectability means observability of only
unstable system modes, [18].

In summary, the optimal feedback gain that minimizes (2) subject to (1) is given by (15), where the matrix \(P\) satisfies
the algebraic Riccati equation (16). The optimal (minimal) value for the performance criterion is given by (10)
with \(P\) obtained from (16). The solution of the algebraic Riccati equation and optimal feedback gain can be obtained
using the MATLAB function lqr as \([\text{Fopt},P]=\text{lqr}(A,B,R1,R2)\).

Note that the optimal controller can be easily implemented since we have only constant state feedback (proportional
controller). This block diagram of this controller is represented in Figure 1.

\[ u_{\text{opt}}(x(t)) \quad \text{Linear System} \quad \begin{bmatrix} x(t) \\ -F_{\text{opt}} \end{bmatrix} \]

**Figure 1**: Block diagram for the linear-quadratic optimal controller

**V. OPTIMAL DETERMINISTIC NONZERO SET-POINT CONTROLLER**

In many applications our goal is to regulate all, some, or
only one state variable(s) to constant value(s). Denote
the \(n_1\)-dimensional vector \((n_1 < n)\) of controlled (regulated)
state variables by \(z(t)\), and relate it with the state variables by
\[
z(t) = Dx(t) \quad (19)
\]

Our goal is that \(z(t)\) is as close as possible to a certain constant
vector, say \(z(t) \approx r\) (\(r\) is chosen by control
engineers to meet their requirements), at least at steady
state, that is
\[
z_{ss} = Dx_{ss} = r = \text{const} \quad (20)
\]

From the original system equation (1), the steady state
values of the state space variables satisfy
\[
0 = Ax_{ss} + Bu_{ss} \quad (21)
\]

From the last two equations, we can find the steady state
values for the system state variables and the system input, by
solving the following system of linear algebraic equations

\[
\begin{bmatrix} A & B \\ D & 0 \end{bmatrix} \begin{bmatrix} x_{ss} \\ u_{ss} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (22)
\]

This system produces solutions for \(x_{ss}\) and \(u_{ss}\) if the
following holds
\[
\text{rank} \begin{bmatrix} A & B \\ D & 0 \end{bmatrix} = n + n_1 \quad (23)
\]

The unique solutions for \(x_{ss}\) and \(u_{ss}\) will be obtained
if \(m = n_1\), which is the case when the dimension of the system
input (number of system inputs) is equal to the dimension of
the controlled (regulated) variables (number of the
controlled variables).

Having obtained the values for \(x_{ss}\) and \(u_{ss}\), we can introduce
the change of variables as
\[
\overline{x}(t) = x(t) - x_{ss}, \quad \overline{u}(t) = u(t) - u_{ss} \quad (24)
\]

and derive from (1) and (24) a new linear system, which in
the translated coordinates has the form
\[
\frac{d\overline{x}(t)}{dt} = Ax(t) + Bu(t) + Ax_{ss} + Bu_{ss} = 0 \quad (25)
\]

For the linear system (25) we can use the performance
criterion (2) and choose the penalty matrices according to
our needs. The optimal gain (for the zero set-point
controllers) is
\[
\overline{u}_{\text{opt}}(t) = -F_{\text{opt}}\overline{x}_{\text{opt}}(t) \quad (26)
\]

Going back to the original coordinates, we have
\[
\begin{align*}
\overline{u}_{\text{opt}}(t) &= \overline{u}_{\text{opt}}(t) + u_{ss} = -F_{\text{opt}}x_{\text{opt}}(t) + u_{ss} \\
&= -F_{\text{opt}}(x_{\text{opt}}(t) - x_{ss}) + u_{ss} \quad (27)
\end{align*}
\]

The corresponding block diagram is presented in Figure 2.

\[ x_{\text{opt}}(t) \quad \text{Linear System} \quad \begin{bmatrix} x(t) \\ F_{\text{opt}} \end{bmatrix} \quad \text{D} \quad z(t) = r \quad (28)
\]

**Figure 2**: Optimal linear nonzero set-point controller

**VI. CONCLUSION**

We have shown how elementary knowledge of linear
algebra and state space linear system analysis can be used
to completely derive and understand standard optimal control
problem (linear-quadratic controller), so it can be taught to
undergraduate students.
REFERENCES


