Cooperative Control With Improvable Network Connectivity

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Abstract—In this paper, an efficient approach to improve the connectivity of any given network is provided by introducing an additional node into the network. More specifically, the additional node acts as a neighbor to certain pair of nodes in the network with only changing one link among the existing nodes. The location of the additional node is designed according to the current topology as well as the states, and it has been proved that, controlling the most-connected node will in general improve the convergence rate of the network. Moreover, the proposed method overcomes the dependence of the convergence rate on the Fiedler value. In particular, the convergence problem is converted to a standard stability problem by using the cooperative control Lyapunov function, and a state-feedback controller with switching logic in finding the observed node is designed to guarantee the fast convergence. Simulation results demonstrate the effectiveness of the proposed strategy in speeding up consensus in a network.

I. INTRODUCTION

Distributed cooperative control problem has received tremendous amount of interests in the past decades. Numerous solid results have been proposed in application areas ranging from military battle systems to mobile robot/sensor network to civilian commercial highway and air transportation systems, leading to significant theoretical developments [1] in the areas of formation control [2], attitude synchronization [3], flocking [4].

Despite its popularity in both theoretical developments and practical implementations, cooperative control poses significant challenges as applied in consensus seeking problems, among which the limitation on connectivity is the most critical and challenging one. In essence, the connectivity is often justified by the second smallest eigenvalue of the graph laplacian [5], or as is well known the Fiedler value or algebraic connectivity.

Moreover, since the numerical formulation of the Fiedler value has been thoroughly explored [6][7], it is thus straightforward to apply nonlinear optimization technique to maximize the magnitude of the Fiedler value so as to improve the connectivity as well as the robustness. In [8], an iterative decentralized supergradient algorithm is proposed to increase the connectivity, in which the computation of the eigenvector with respect to graph Laplacian is optimized to yield the maximum Fiedler value. Xiao and Boyd proposed a convex optimization approach to the design of fast-converging algorithms in the fixed network [9]. In [10], the Fiedler value is maximized using an iterative greedy-type optimization algorithm, although the convergence of this algorithm is proved to be local, the global maximum value could be attained under certain conditions.

In addition to the optimization based studies, improving network connectivity with adding additional node(s) into the network has also received certain interests. In [11], the connectivity of a static ad-hoc network is improved by adding additional nodes, whose locations have been optimized by greedy tessellation algorithm. In [12], a UAV is treated as the additional node, and the movement of UAV is optimized to improve the connectivity of ground-based wireless ad hoc networks, the nonlinear programming technique and adaptive schemes are utilized in the optimization process, this work was extended in [13]. In [14], it is proved the union of star graphs is the optimal topology for connectivity. Moreover, it is well-established in [15] that pinning of the most highly connected nodes is shown to require a significantly smaller number of local controllers as compared to the randomly pinning schemes, which means controlling the most connected nodes could efficiently improve the connectivity. In addition, extensive simulations in [14] demonstrate that establishing a link between the nodes with maximum out-degree and minimum in-degree can in general yield a fast convergence.

However, the quantitative relation between the convergence rate and the states has not been fully explored. Inspired by the virtual leader approach [16][1], this paper provides an efficient approach to improve the convergence rate of any given network with the introduction of a supervisory node, which acts as a neighbor to certain pair of nodes through observing one node and controlling another node. The proposed scheme only needs to change one link among the existing communication tunnels, it can be treated as a methodical algorithm for selecting the additional rewiring link [17]. In particular, the Lyapunov method is utilized in this paper to convert the convergence problem into standard stability problem, and the location of the additional node is selected according to the current topology and states, and a state-feedback control law is proposed to ensure the convergence rate can be improved definitely.

II. PRELIMINARY RESULTS ON COOPERATIVE CONTROL

Consider a group of $n$ agents with linear dynamics

$$\dot{x}_i = u_i$$

(1)

The interactions between the group members are characterized using the following binary communication matrix and its corresponding time sequence $\{t_k : k \in \mathbb{N}\}$ as
$S(t) \in \{0, 1\}^{n \times n} = S(k) = S(tk) = S(tk), \forall t \in [tk, tk+1)$ \[1\],
where $\mathbb{N} = \{0, 1, \ldots, \infty\}$

$$S(t) = \begin{bmatrix}
1 & s_{12}(t) & \cdots & s_{1n}(t) \\
s_{21}(t) & 1 & \cdots & s_{2n}(t) \\
\vdots & \vdots & \ddots & \vdots \\
s_{nl}(t) & s_{l2}(t) & \cdots & 1
\end{bmatrix}$$ \[2\]

where $s_{ij}(t) = 1$ if information of the $j$th node is available to the $i$th node, and $s_{ij}(t) = 0$ if otherwise. Time sequence \{tk : k \in \mathbb{N}\} and the corresponding changes in the row $S_i(t)$ of $S(t)$ are detectable instantaneously by and locally at the $i$th agent, but they are not predictable or prescribed or known apriori or modeled in any way \[1\][18].

The cooperative control input for system (1) is \[1\]

$$u_i = -x_i + \sum_{j=1}^{n} s_{ij}(t) \omega_{ij} x_j$$ \[3\]

If the node state is a scalar, substituting (3) into (1), the close-loop dynamics is

$$\dot{x} = [-I_n + D(t)]x = -L(t)x$$ \[4\]

with $d_{ij}(t) = \frac{s_{ij}(t) \omega_{ij}}{\sum_{l=1}^{n} s_{il}(t) \omega_{il}}$.

where $x = [x_1 \ldots x_n]^T$ is the overall state, $\omega = [\omega_{ij}] \in \mathbb{R}^{n \times n}$ is a row-stochastic gain matrix, $I_n$ is the $n$-dimensional identity matrix. If the state is a vector, then the Kronecker product should be used, as is well known. In this paper, in order not to obscure our development, we assume that $x_i \in \mathbb{R}$.

In (4), $D(t) = [d_{ij}] \in \mathbb{R}^{n \times n}$ is a nonnegative, piecewise-constant and row-stochastic matrix. $L(t) = -D(t) + I_n$ is the closed-loop system matrix, has the same zero-row-sum property as graph Laplacian. As such, $L(t)$ has a simple eigenvalue: $\lambda_0(L) = 0$ associated with a right eigenvector as $\omega_0 = [1 \ 1 \ldots 1]^T \in \mathbb{R}^n$, and a unity left eigenvector as $\gamma = [\gamma_1 \ \gamma_2 \ \ldots \ \gamma_n]^T \in \mathbb{R}_{\geq 0}^n$ with $\sum_{i=1}^{n} \gamma_i = 1$ and $D^T \gamma = \gamma$ \[1\]. Thus,

$$\frac{d}{dt}(\gamma^T x) = \gamma^T \dot{x} = \gamma^T Dx - \gamma^T x = 0$$ \[5\]

Hence, $\gamma^T x_0$ is the centroid of all the states. In fact, $\gamma^T x_0$ is exactly the final consensus value provided $S(t)$ is time-invariant. Moreover, the in-index, $q_i^{\text{in}}$, and out-index, $q_i^{\text{out}}$, of node $i$ in $D(t)$ is defined, respectively, as follows

$$q_i^{\text{in}} = \sum_{j=1}^{n} d_{ij}(t) - d_{ii} \quad q_i^{\text{out}} = \sum_{j=1}^{n} d_{ji}(t)$$ \[6\]

Apparently, the out-index has the similar properties as the out-degree in the graph theory \[1\][5], denoting the standing \[19\] of one node to the network, and according to definition of $D(t)$, the in-index $q_i^{\text{in}} \geq 0$ is similar to in degree in the graph theory \[1\], it characterizes the influences come from its neighbors.

Before proceeding further, we propose the following Lemmas regarding the existing results on the cooperative control theory:

**Lemma 1**: \[1\] For linear system (1), under input (3), the resulted closed-loop system (4) is both Lyapunov stable and asymptotically cooperative stable \[1\] if and only if the communication matrix (2) is uniformly sequentially complete, or from graph point of view, from any $t_k$ on, the union of future graphs has at least one globally reachable node.

Also, in control theory, it is convenient to convert a convergence problem into a standard stability problem. In order to accomplish this, the error state $e_i$ for node $i$ is defined as

$$e_i = [(x_1 - x_i) \ldots (x_{i-1} - x_i) \ (x_{i+1} - x_i) \ldots (x_n - x_i)]^T$$ \[7\]

Therefore, system (4) is asymptotically cooperative stable if and only if $\lim_{t \to \infty} \sum_{i=1}^{n} e_i(t) = 0$. The following Lemma provides the existing results on the cooperative control Lyapunov function of reducible and irreducible networks.

**Lemma 2**: \[1\] For any nonnegative, row-stochastic, reducible network matrix $D(t)$, system (4) has a cooperative control Lyapunov function if and only if its lower triangular form $F_\gamma$ is lower triangular complete \[1\], and the cooperative control Lyapunov function is of the form

$$V_c = \sum_{i=1}^{n} \gamma_i e_i^T G_i^T \gamma G_i e_i$$ \[8\]

where $G_i \in \mathbb{R}^{n \times (n-1)}$ is the identity matrix $I_n$ eliminating the $i$th column, $\gamma = \text{diag}\{\gamma_1, \ \gamma_2, \ \ldots, \ \gamma_n\}$, $\gamma_i > 0$, $\forall i > 0$. The time derivative of (8) is

$$\dot{V}_c = -2 \sum_{i=1}^{n} \gamma_i e_i^T Q_i e_i$$ \[9\]

with $Q_i = G_i^T [\gamma (I_n - D(t)) + (I_n - D(t))^T \gamma] G_i$.

Furthermore, for any nonnegative, row-stochastic, irreducible network matrix, it has the same cooperative control Lyapunov function (8), except that in this case $\gamma$ is the unity left eigenvector of $D(t)$ associated with the eigenvalue $\lambda_0(D) = 1$.

**III. MOTIVATION AND MAIN THEOREM**

For close-loop system (4), its Fiedler value is

$$\lambda_1(L) = \min_{x \in \mathbb{R}^n, x \neq 0, x \perp 1} \frac{x^T L x}{x^T x}$$

Thus, a straightforward way to improve the connectivity is to maximize the Fiedler value by employing sophisticated optimization techniques \[10\], or alternatively but more efficiently, introducing additional physical or virtual node(s) into the network, the connectivity will be improved with allocating the additional node(s) properly \[11][12\]. More specifically, introducing addition node is equivalent to establish or modify the current communication tunnels, and the consensus is that connecting the node with maximum out-degree and the node with maximum in-degree will result a better connectivity \[14\].

However, it is well known that the convergence rate of one particular state depends mainly on which eigenspace it belongs to. In particular, given linear system (4), for any
TABLE I
COMPARISON OF THE CONSENSUS TIME (SEC)

<table>
<thead>
<tr>
<th></th>
<th>$D_1$</th>
<th>$D_2$</th>
<th>$D_3$</th>
<th>switching</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>10.97</td>
<td>8.11</td>
<td>3.96</td>
<td>3.62</td>
</tr>
<tr>
<td>$x_2$</td>
<td>6.09</td>
<td>6.17</td>
<td>6.09</td>
<td></td>
</tr>
<tr>
<td>$x_3$</td>
<td>2.41</td>
<td>2.01</td>
<td>2.14</td>
<td>1.57</td>
</tr>
</tbody>
</table>

initial state $v_i$ with network $D_i$, it can be decomposed into the following form

$$v_i = \chi_1 v_i^T e_0 + \chi_2 \xi_1 + \ldots + \chi_n \xi_{n-1}$$

where $\chi_i \in \mathbb{R}$ are gains, $v_i^T$ is the centroid of all the states, $\xi_i$ are the right eigenvectors, indexed by the order of eigenvalues, and as such $\xi_1$ is the eigenvector associated with the Fiedler value.

It follows that, if $\chi_2 \neq 0$, the convergence rate of $v_i$ will be determined by the Fiedler value. In fact, it will converge exponentially since system (4) is linear. As such, the Fiedler value actually represents the lower bound of the convergence rate (the upper bound is a function of Laplacian, initial state, and Fiedler value, as defined in [20]). Therefore, the Fiedler value can not be used to justify the convergence rate for all the states. In this regard, it can be concluded that the topology-based approaches can not always ensure a definitely improved convergence rate. This statement can be verified in the following example:

**Example 1:** Taking the following network matrices and initial states $x_1 = [0 \ 12 \ 3]^T$, $x_2 = [0 \ 0 \ 2]^T$, $x_3 = [12 \ 14 \ 10]^T$ for instance,

$$D_1^x = \begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 0.5 & 0 \\ 0.8 & 0.1 & 0.1 \end{bmatrix}, \quad D_2^x = \begin{bmatrix} 0.1 & 0 & 0.9 \\ 0.5 & 0.5 & 0 \\ 0.8 & 0.1 & 0.1 \end{bmatrix}, \quad D_3^x = \begin{bmatrix} 0.1 & 0.9 & 0 \\ 0.5 & 0.5 & 0 \\ 0.8 & 0.1 & 0.1 \end{bmatrix}$$

Note that all the network matrices are lower triangularly complete, satisfying Lemma 1. Therefore, substituting (10) into (4), the consensus will be reached in a finite time. In what follows, the consensus time is defined as the time needed to reach vicinity (95%) of the final consensus value. Table I provides the needed consensus time for network $D_i^x$.

In addition, the Fiedler values are

$$\lambda_1(L_1^c) = 0.5, \quad \lambda_1(L_2^c) = 0.5734, \quad \lambda_1(L_3^c) = 0.9$$

where $L_i^c = I_3 - D_i^c$.

Intuitively that, $D_2^c$ should converges faster than $D_1^c$ since $\lambda_1(L_2^c) > \lambda_1(L_1^c)$. However, as indicated in Table I, the numerical results shows otherwise with states $x_1$ and $x_3$. Also, it should be noted that $D_2^c$ is the modified version of $D_1^c$ when connecting the maximum out-degree node (i.e., node 1) and maximum in-degree node (i.e., node 3), as suggested in [14]. Obviously that simply connecting any pair of nodes is not good enough to ensure a definitely improved convergence rate. Or in other words, only considering topology in attempting to improve the convergence rate is not sufficient enough, the states of all the nodes should be taken into account during the control development.

Motivated by the above analysis as well as the virtual leader approach initiated in [1][16], we introduce an additional supervisory node into the network with a switching-type communication tunnel, and the location of the supervisory node is determined by both the topology and states. In particular, we have the following assumption regarding the supervisory node:

**Assumption 1:** The supervisory node has global knowledge of the network at every $t_k$, it can only control one node and observe another node at every $t_k$. In what follows, we termed these two nodes as controlled node and observed node, respectively. We further assume the supervisory node has the same state with the observed node.

Taking $3 \times 3$ network for instance, suppose the supervisory node observes node 3 and controls node 1, then the network matrix $D(t)$ is augmented to

$$\hat{D} = \begin{bmatrix} d_{11} - f_1 & d_{12} & d_{13} + f_1 \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{bmatrix}$$

where $0 \leq f_1 \leq d_{11}$ is the gain associated with the supervisory node’s behavior.

Furthermore, defining the cooperative control Lyapunov function (8) for linear system with $D(t)$, and invoking Lemma 1, it follows that system (1) is both Lyapunov stable and cooperative Lyapunov stable provided $D(t)$ is uniformly sequentially complete and lower triangularly complete. That is,

$$\dot{V}_c = \sum_{i=1}^{3} \gamma_i e_i^T Q_i e_i < 0$$

Consequently, using the same cooperative control Lyapunov function for the augmented network (12), (13) becomes

$$\dot{\hat{V}}_c = \dot{V}_c + 3 \sum_{i=1}^{3} \gamma_i e_i^T \Delta Q_i e_i = \dot{V}_c + \Delta \dot{V}_{31}$$

where $\Delta Q_i = G_i^T [\gamma \Delta \hat{D} + \Delta D^T \gamma] G_i$ [1], and

$$\Delta \hat{D} = \begin{bmatrix} -f_1 & 0 & 0 \\ 0 & 0 & f_1 \\ 0 & 0 & 0 \end{bmatrix}$$

Then, substituting (15) into (14), we obtain

$$\Delta \dot{V}_{31} = 3 \sum_{i=1}^{3} \gamma_i e_i^T \Delta Q_i e_i = -4 \gamma_1 \gamma_3 f_1 (x_3 - x_1)^2$$

$$-4 \gamma_1 \gamma_2 f_1 (x_2 - x_1)^2 + 4 \gamma_1 \gamma_2 f_1 (x_1 - x_2) (x_3 - x_2)$$

Noted that (16) has two negative definite terms and one indefinite term. Clearly the convergence rate will be improved definitely if $\Delta \dot{V}_{31} < 0$, which leads to $\dot{V}_c < \dot{V}_c$. 


Furthermore, for any \( n \times n \) network, if the supervisory node controls node \( i \) and observes node \( j \), \( \Delta \hat{V}_{ji} \) can be formulated analogously. That is
\[
\Delta \hat{V}_{ji} = -4\gamma_i f_i \sum_{k=1}^{n} \gamma_k (x_i - x_k)^2 + 4\gamma_i f_i \sum_{k=1}^{n} \gamma_k (x_i - x_k)(x_j - x_k)
\]
(17)

However, because of the time-varying states, there is possibility that \( \Delta \hat{V}_{ji} > 0 \), and it occurs if and only if
\[
\sum_{k=1}^{n} \gamma_k x_k - \gamma_i x_i > x_i > x_j
\]
or
\[
\sum_{k=1}^{n} \gamma_k x_k - \gamma_i x_i < x_i < x_j
\]
(18)

Note that (17) and (18) applies in any order/type network except in the irreducible network associated with simple eigenvalue \( \lambda(D(t)) = 1 \). As such, in this case (18) becomes
\[
\gamma^T x > x_i > x_j \quad \gamma^T x < x_i < x_j
\]
(19)

Therefore, the convergence rate of \( \hat{D}(t) \), regardless of the location of the supervisory node, is not only a function of the topology, but the states.

Then, the input generated by the supervisor node imposing on the controlled node \( i \) while observing node \( j \) is
\[
u_{sm}^* = -f_i(x_{j^*} - x_j)
\]
(20)

where \( i^* \) and \( j^* \) are the indexes of the controlled node and observed node, respectively.

Note that (20) is a classical P-type state feedback controller [21], proposed to eliminate the discrepancy between the states of controlled node and observed node. Hence, if the switching logic is designed properly, the discrepancy among all the states will be eliminated promptly, thus leading to a fast convergence. The main results in this paper is provided in the following theorem:

**Theorem 1:** Given any network \( D(k) \) at \( t_k \), the convergence rate will be improved definitely provided it is augmented by a supervisory node with control action (20).

In addition, for the reducible network, the controlled node is the node with minimal in-index while for the irreducible network, the controlled node is the node with maximum out-index. Also, in the case of the switching topology, the controlled node should be refreshed at every interval using the same logic and the current topology \( D(k) \).

Furthermore, in either case the state of the observed node \( x_j \) is determined by the following switch logic
\[
\begin{cases}
x_{j^*} = \min\{x_k, \forall k \neq i^*\} & \text{if} \quad x_i > \Omega \\
x_{j^*} = \max\{x_k, \forall k \neq i^*\} & \text{if} \quad x_i < \Omega
\end{cases}
\]
(21)

where for the reducible network, \( \Omega = \gamma^T x \) with \( \gamma \) as the unity left eigenvector associated with eigenvalue \( \lambda(D(k)) = 1 \), while for the reducible network \( \Omega = \sum_{k=1}^{n} \gamma_k x_k - \gamma_i x_i \) with \( \gamma_i \in \mathbb{R} \) are nonnegative constants.

### IV. Proof of the Main Theorem

As established in the previous section simply changing the location of the supervisory node can not ensure a definitely improved connectivity because of the time-varying states, and vice versa. In this section, Theorem 1 will be proved using the cooperative stability results (17). To accomplish this, the controlled node and the observed node will be identified, respectively.

Before proceeding to the proof of Theorem 1, the following Lemma is introduced regarding the relation between the unity left eigenvector and the topology:

**Lemma 3:** For any nonnegative, row-stochastic matrix \( D \), if node \( i \) has the maximum out-index as defined in (6), then its corresponding component \( \gamma_i \) of the unity left eigenvector \( \gamma \) associated with the eigenvalue \( \lambda(D) = 1 \) has the maximum value.

**Proof:** The proof of Lemma 3 can be found in the Appendix.

#### A. Selection of the controlled node

It follows from (17) that a better convergence rate will be achieved if \( \Delta \hat{V}_{ji} \) reaches its minimal value. To determine the controlled node, suppose the observed node is chosen perfectly such that
\[
-\sum_{k=1}^{n} \gamma_k(x_i - x_k)^2 + \sum_{k=1}^{n} \gamma_k(x_i - x_k)(x_j - x_k) < 0
\]
(22)

According to (17), \( \Delta \hat{V}_{ji} \) are linear functions of the gains \( \gamma_i f_i \). Therefore, a better convergence rate will be ensured if \( \gamma_i f_i = \max\{\gamma_j f_j, \forall j\} \) [22], or simply choosing \( f_i = d_{ii}(t) \) and \( \gamma_i = \max\{\gamma_j, \forall j\} \). Consequently, in order to determine the maximum \( f_i \) according to topology, we need to find the node with maximum \( d_{ii} \). Suppose node \( i \) in network \( D(t) \) has the minimal in-index as defined in (6), that is \( \phi_i^m = \min\{\phi_i^m, \forall j\} \). Then, since \( d_{ii} = 1 - \phi_i^m \), it follows node \( i \) has the maximum \( d_{ii} \). Therefore, choosing node \( i \) as controlled node, and letting \( f_i = d_{ii} \) will yield a smaller \( \Delta \hat{V}_{ji} \). In other words, controlling the node with minimal in-index will yield a faster convergence.

In addition, for an irreducible network, since \( \gamma_i \) are components of the unity left eigenvector. Then recalling Lemma 3, it follows that if node \( i \) has the maximum out-index, then \( \gamma_i = \max\{\gamma_k, \forall k\} \).

#### B. Selection of the observed node

As established in the last section, controlled node can be determined instantaneously by the topology at the supervisory node at every \( t_k \). However, there are still \( n-1 \) possible ways in allocating supervisory in a \( n \times n \) network under this development, identifying which node should be the observed node is thus in order such that the supervisory node can be distributed in a way to improve the connectivity definitely.

According to (17), a straightforward way to find the observed node is comparing the resulted \( \Delta \hat{V}_{ji} \) provided node \( i \) is the controlled node, the observed node should be the node with minimal \( \Delta \hat{V}_{ji} \). In addition, \( \gamma_i f_i \) of \( \Delta \hat{V}_{ji} \) associated with
all the examined cases is the same since they all control the same node. Therefore, the observed node will be determined only by the states.

For any \( n \times n \) network \( D \), the controlled node can be identified intuitively using topology, and in what follows, we assume node \( i^* \) is the controlled node at \( t_k \), that is \( \gamma_i^* f_{i^*} = \max \{ \gamma_j f_j, \forall j \} \). It follows from (17) that if \( \Delta \hat{V}_{j^*} < \Delta \hat{V}_i \), we obtain

\[
x_j < x_i, \text{ and } x_i > \sum_{m=1}^{n} \gamma_m x_m - \gamma_i x_i \sum_{k=1}^{n} \gamma_k - \gamma_i
\]

or

\[
x_j > x_i, \text{ and } x_i < \sum_{m=1}^{n} \gamma_m x_m - \gamma_i x_i \sum_{k=1}^{n} \gamma_k - \gamma_i
\]

Therefore, if \( x_i > \sum_{m=1}^{n} \gamma_m x_m - \gamma_i x_i \sum_{k=1}^{n} \gamma_k - \gamma_i \), the observed node should be the node with the minimal state. Otherwise, we need to observed the node with the maximum state, so as to yield a smaller \( \Delta \hat{V}_{j^*} \).

Then, the following input can be designed for the supervisory node

\[
u^{i^*}_{sn} = -f_{i^*}(x_{i^*} - x_{j^*}) \tag{23}
\]

with the switching logic to select the observed node

\[
\begin{cases}
x_{j^*} = \min \{x_k, \forall k \neq i^*\} & \text{if } x_{i^*} > \Omega \\
x_{j^*} = \max \{x_k, \forall k \neq i^*\} & \text{if } x_{i^*} < \Omega
\end{cases} \tag{24}
\]

where \( \Omega = \gamma^T x \) for the irreducible network, and \( \Omega = \sum_{k=1}^{n} \gamma_k x_k - \gamma_i x_i \sum_{m=1}^{n} \gamma_m x_m - \gamma_i x_i \) for the reducible network, this concludes the proof of Theorem 1. Also, in order to eliminate too often switching or oscillation around the threshold, in the implementation of switching logic, if \( \|x_{i^*} - \Omega\| < \epsilon \), where \( \epsilon \) is a positive sufficiently small constant, no switch is needed.

Remarks 1: The inequality sets determined by different cases for \( \Delta \hat{V}_{j^*} < 0 \) are exclusive with each other (except when \( x_i = \max \{x_j, \forall j\} \), in which case all \( \Delta \hat{V}_{j^*} \) are negative and the switching logic (24) is still valid in finding the minimal \( \Delta \hat{V}_{j^*} \)), this validates that the proposed switching logic covers all the space without overlapping.

Remarks 2: It follows from (24) that the state of controlled node will be used as a reference to determine whether observing the node with minimal state or the node with maximum state.

V. Numerical Results

Taking 3x3 irreducible case for instance, the reducible case is provided in Table I. Assuming the initial states: \( x_1 = [0 - 5 - 10]^T \), \( x_2 = [6 0 10]^T \), and \( x_3 = [10 1 10]^T \), the network matrices used are

\[
\begin{align*}
D_1 &= \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.4 & 0.2 & 0.4 \\ 0.4 & 0.2 & 0.4 \end{bmatrix} \\
D_2 &= \begin{bmatrix} 0.5 & 0.2 & 0.3 \\ 0.4 & 0.2 & 0.4 \\ 0.6 & 0.2 & 0.2 \end{bmatrix} \tag{25}
\end{align*}
\]

Apparently, complying with Theorem 1, node 1 should be the controlled node in either case. In what follows, system (1) with the network matrices (25) has been rigorously examined, the results are provided in Table II, where the case \( j \rightarrow i \) means the supervisory node controls node \( i \) and observes node \( j \). Also, only the results of controlling node 1 will be provided for the sake of brevity.

It follows from Table II, the proposed switching scheme can definitely improve the convergence rate. Also, as shown in case \( D_2 - x_2 \), the convergence is degraded even controlling the most-connected node, while the proposed control scheme can still ensure a faster consensus. Moreover, the frequency of the switching is determined by the states. In particular, the network switches to \( 3 \rightarrow 1 \) instantly at the beginning in \( D_2 - x_1 \) and \( D_1 - x_1 \), whereas, as shown in Fig. 1, the network keeps switching among \( 3 \rightarrow 1 \) and \( 2 \rightarrow 1 \) in case \( D_2 - x_2 \), this due to the fact that the state of the controlled node is closer to the consensus value determined by \( \gamma^T x \).

Also, the reason why the proposed scheme exhibits superior performance in terms of connectivity is because the resulted \( \Delta \hat{V}_{j^*} \) remains negative definite throughout the engagement, while, in other cases, \( \Delta \hat{V}_{j^*} > 0 \) occurs occasionally as states evolved, which will degrade the convergence rate eventually, as shown in Fig. 2.

VI. Conclusion

This paper proves that the convergence rate of any network attributes to both the topology and the states, controlling either of them is not theoretically enough to ensure a better convergence rate, while controlling the node with maximum out-index or minimal in-index will in general provide a better convergence results.

However, the proposed scheme can only ensure a definitely improved convergence, its relation to convergence time has not been explored. Or in other words, finite-time convergence problem with the proposed strategy should be addressed in the future work. Moreover, the supervisory node is assumed
to have the global knowledge of the network, this assumption may pose challenges in certain applications and should be released in the future work.

**APPENDIX**

**PROOF OF LEMMA 3**

Without loss of any generality, we take \( 3 \times 3 \) network for instance, the \( n \times n \) case can be proved analogously. Thus, according to property of the left eigenvector, we have \( \gamma_1 + \gamma_2 + \gamma_3 = 1 \), and

\[
\begin{cases}
d_{11}\gamma_1 + d_{21}\gamma_2 + d_{31}\gamma_3 = \gamma_1 \\
d_{12}\gamma_1 + d_{22}\gamma_2 + d_{32}\gamma_3 = \gamma_2 \\
d_{13}\gamma_1 + d_{23}\gamma_2 + d_{33}\gamma_3 = \gamma_3
\end{cases}
\]  
\tag{A.26}

We assume that node 1 has the maximum out-index, that is \( \sum_{i=1}^{3} d_{1i} \) has the maximum value versus other rows, and since \( \sum_{i=1}^{3} \sum_{j=1}^{3} d_{ij} = 3 \). Therefore, we have

\[
1 < \sum_{i=1}^{3} d_{1i} < 3 
\tag{A.27}
\]

We start our proof by contradiction. Firstly, we assume that \( \gamma_1 < \gamma_2 < \gamma_3 \), thus according to the first equation of (A.26), we have

\[
d_{11}\gamma_1 + d_{21}\gamma_2 + d_{31}\gamma_3 < \gamma_1
\]

That is \( \gamma_1 (\sum_{i=1}^{3} d_{1i}) < \gamma_1 \). Obviously, this contradicts to (A.27). Therefore, assumption of \( \gamma_1 < \gamma_2 < \gamma_3 \) is not valid.

Again, if we assume that \( \gamma_2 < \gamma_1 < \gamma_3 \) or \( \gamma_3 < \gamma_1 < \gamma_2 \), then, according to the first equation of (A.26), we have

\[
\gamma_1 (\sum_{i=1}^{3} d_{1i}) < \gamma_1 + (\gamma_3 - \gamma_1) d_{31}
\]

\[
\gamma_1 (\sum_{i=1}^{3} d_{1i}) < \gamma_1 + (\gamma_1 - \gamma_2) d_{21}
\]

which obviously contradicts condition (A.27). Therefore, \( \gamma_1 \) should be the maximum value among \( \gamma_i \). For a \( n \times n \) network, the same proof procedure applies, and the same conclusion stands, that is \( \gamma_i = \max\{\gamma_j, \forall j\} \) provided \( \sum_{j=1}^{n} d_{ji} = \max\{\sum_{k=1}^{n} d_{ki}, \forall i\} \), which completes the proof of Lemma 5. Note that similar approach was employed in [23] to prove that the determinants of any \( n \times n \) matrix \( D \) are non-zero provided \( |d_{ii}| > \sigma_{in} \).

Furthermore, the exact average consensus can be ensured (i.e., \( \gamma_1 = \gamma_j = 1/n \)) if and only if all the nodes have the same out-index [1].

**REFERENCES**


