A Full Block S-Procedure Application to Distributed Control

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Abstract—This paper concerns the problem of distributed controller synthesis for a class of systems which we call “decomposable systems”. These systems have a special property that makes it possible to decompose them into a set of modal systems and approach the distributed control problem in an easy and effective way. In a recent paper of ours, we have shown a suboptimal $\mathcal{H}_\infty$ controller synthesis method based on the use of the multi-objective optimization technique making use of Linear Matrix Inequalities (LMIs); in this article we show that another suboptimal approach is possible, which exploits robust control techniques, namely the full block S-procedure. In the end we also briefly show a method based on small gain $\mathcal{H}_\infty$ robust performance synthesis, and the results of the three different methods are compared on a couple of test cases.

I. INTRODUCTION

In recent times, the control theory community has put much effort into the development of distributed control methods for large scale systems. This interest is due to the variety of applications that have been made possible by modern technological advances, for example, satellite formation flying [11], car platoons [8], unmanned aerial vehicles [1], [5] and large telescopes [7].

In this paper we focus on the class of “decomposable systems”, introduced in [10], where a distributed control method was presented. Decomposable systems can be imagined as the result of the interconnection of a set of identical subsystems or agents, which interact with each other following a pattern, as in Fig. 1.

In [10] we developed an LMI based method which can be used to design a controller that has the same interconnection structure as the plant. This method exploits the possibility of decomposing the system into a set of independent “modal” subsystems all depending on a parameter. The distributed structure of the controller is then achieved thanks to an application of the LMI based multi-objective optimization, at the cost of suboptimality with respect to a global controller.

In this paper, we show a different approach to the same class of systems, which exploits a result from robust controller synthesis (namely, the full block S-procedure) to yield a different (still suboptimal) solution for the $\mathcal{H}_\infty$ state feedback problem. The idea is basically to pull the parameter out of the model of the system in its decomposed form and deal with it as if it were an uncertainty. A similar idea has been explored recently in [12], where the small gain theorem and $\mu$ synthesis are used to guarantee the stability with respect to variations in the parameter. The developed method applies only to state feedback and is less general than other methods in the literature, like [9], but it has the advantage of having a computational complexity that does not depend on the size of the system, i.e. the synthesis LMI does not grow with the number of agents.

The paper is organized as follows. In Section II we introduce the notation and the basic definitions, and we briefly summarize the results of [10], adapted for continuous time systems. Section III shows the tools that we employ to develop the novel approach. Section IV contains the new decentralized controller synthesis method, and Section V shows how this result can be extended to distributed controller synthesis. Section VI reports another result that can be obtained by the small gain theorem. Finally, Section VII contains some numerical results and the comparison of the performance of the three different methods, and the conclusions are drawn in Section VIII.

II. PRELIMINARIES

We denote the field of real numbers by $\mathbb{R}$, the field of complex numbers by $\mathbb{C}$ and the set of real (complex) $n \times m$ matrices by $\mathbb{R}^{n \times m}$ ($\mathbb{C}^{n \times m}$). Let $\otimes$ indicate the Kronecker product and $I_n$ the identity matrix of size $n$. The notation $A \succ 0$ ($A \prec 0$) indicates that all the eigenvalues of the Hermitian matrix $A$ are strictly positive (negative) and the bullet • denotes a symbol that is either not relevant or clear from the context.

We call the systems that are discussed in this paper “decomposable systems” (the reason of the name will be clear shortly). Before presenting their formal definition, we...
can say that these systems can be interpreted as the result of the interconnection of $N$ identical subsystems, each of order $l$. The interconnection follows a pattern that is described by a “pattern matrix” $P$. Some examples of systems that are comprised in this category can be found in vehicle formation control problems [4], paper machines [15] and others.

**Definition 1 (Decomposable systems):** Let us consider the $Nl$-th order linear dynamical system described by:

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) + B_ww(t) + B_uu(t) \\
y(t) &= C_x(x(t) + D_{zw}w(t) + D_{zu}u(t) (1)
\end{align*}
\]

Where $x \in \mathbb{R}^{Nl}$ is the state, $u \in \mathbb{R}^{N_{us}}$ is the control input, $w \in \mathbb{R}^{N_{ws}}$ is the disturbance, $y \in \mathbb{R}^{N_{rv}}$ is the measured output and $z \in \mathbb{R}^{N_{rz}}$ is the performance output. We call such systems “decomposable systems” iff there exists a state-space realization that can be parameterized as:

\[
\begin{align*}
A &= I_N \otimes A_a + P \otimes A_b \\
B_s &= I_N \otimes B_{a,s} + P \otimes B_{a,b} \\
C_s &= I_N \otimes C_{a,s} + P \otimes C_{a,b} \\
D_s &= I_N \otimes D_{a,s} + P \otimes D_{a,b} (2)
\end{align*}
\]

where $P \in \mathbb{R}^{N \times N}$ is a diagonalizable matrix ($P = Z^{-1} \Lambda Z$, where $\Lambda$ is the diagonal matrix containing the eigenvalues), which we call the “pattern matrix”. If the pattern matrix $P$ is symmetric, then we call the system a “symmetric decomposable system” (and the eigenvalues of $P$ are all real).

The diagonal part of the matrices (those with the subscript “a”) can be considered to represent the internal dynamics of the subsystems, while the part depending on the pattern matrix (with the subscript “b”) accounts for the interactions between subsystems. A sparse pattern matrix would indicate that each subsystem interacts only with a limited set of the others, e.g. its neighbors.

These systems have an interesting property that allows them to be decomposed into a set of $N$ independent “modal” subsystems. This property and its consequences for control design have been already explained in detail in [10], and we invite the interested reader to look into this reference for more insight. However, we summarize a few results that are used in the continuation of this paper.

**Definition 2:** Let us define $\mathcal{M}^{P,p,q}$ as the set of all matrices $\mathcal{M} \in \mathbb{R}^{n \times q}$ for which there exist two matrices $M_a, M_b \in \mathbb{R}^{p \times q}$ such that:

\[
\mathcal{M} = I_N \otimes M_a + P \otimes M_b (3)
\]

**Lemma 3:** Let $Z \in \mathbb{C}^{n \times n}$ be a non-singular matrix such that $\Lambda = Z^{-1} P Z$ is diagonal. If $M \in \mathcal{M}^{P,p,q}$ then:

\[
M = (Z \otimes I_p)^{-1} \mathcal{M}(Z \otimes I_q) (4)
\]

is block diagonal, with blocks of size $p \times q$.

**Proof:** From Definition 2, we can write:

\[
M = (Z \otimes I_p)^{-1} (I_N \otimes M_a + P \otimes M_b)(Z \otimes I_q) (5)
\]

then from the properties of the Kronecker product [2] we have:

\[
M = (Z^{-1} I_N Z \otimes I_p M_a I_q) + (Z^{-1} P Z \otimes I_p M_b I_q) (6)
\]

which is equivalent to

\[
M = I_N \otimes M_a + \Lambda \otimes M_b (7)
\]

Since $I_N$ and $\Lambda$ are diagonal, then $M$ is block diagonal.

**Remark 4:** We will use the bold font to denote matrices that are parameterized as in (7). A matrix in bold with subscript $i$ will indicate the $i^{th}$ block in the diagonal: $M_i = M_a + \lambda_i M_b$, where $\lambda_i$ is the $i^{th}$ eigenvalue of $\Lambda$ (or the $i^{th}$ element in the diagonal of $\Lambda$). Moreover, notice that the inverse of (4) turns a block diagonal matrix into a matrix in $\mathcal{M}^{P,p,q}$ if and only if the matrix is “bold”:

\[
(Z \otimes I_q)M(Z \otimes I_p)^{-1} \in \mathcal{M}^{P,p,q} \iff M = I_N \otimes M_a + \Lambda \otimes M_b (8)
\]

We can now report the theorem that we are going to use in order to develop the synthesis method.

**Theorem 5 (Decomposition Theorem):** A decomposable system of order $Nl$ as described in Definition 1 is equivalent to $N$ independent subsystems of order $l$. Each of these subsystems has only $m_u$ inputs, $m_d$ disturbances, $r_z$ performance outputs and $r_y$ control outputs.

**Proof:** According to Lemma 3, every matrix $M$ appearing in the state-space description of the system can be rewritten as:

\[
M = (Z \otimes I_p)M(Z \otimes I_q)^{-1}
\]

with $M$ block diagonal parameterized as in Remark 4. Then, with the following (invertible) change of variables:

\[
x = (Z \otimes I_l)\hat{x} \quad y = (Z \otimes I_{r_y})\hat{y} \\
u = (Z \otimes I_{m_u})\tilde{u} \quad z = (Z \otimes I_{r_z})\tilde{z} \\
w = (Z \otimes I_{m_w})\tilde{w}
\]

the system finally becomes:

\[
\begin{align*}
\hat{x}(k+1) &= A \hat{x}(k) + B_w \hat{w}(k) + B_u \hat{u}(k) \\
\hat{z}(k) &= C_x \hat{x}(k) + D_{zw} \hat{w}(k) + D_{zu} \hat{u}(k) \\
\hat{y}(k) &= C_y \hat{x}(k) + D_{yw} \hat{w}(k)
\end{align*}
\]

where the system matrices $A$, $B_w$, $B_u$, $C_x$, etc. are all block diagonal. This is equivalent to the following set of $n$ independent $i^{th}$ order systems:

\[
\begin{align*}
\hat{x}_i(k+1) &= A_i \hat{x}_i(k) + B_{w,i} \hat{w}_i(k) + B_{u,i} \hat{u}_i(k) \\
\hat{z}_i(k) &= C_z \hat{x}_i(k) + D_{zw,i} \hat{w}_i(k) + D_{zu,i} \hat{u}_i(k) \\
\hat{y}_i(k) &= C_y \hat{x}_i(k) + D_{yw,i} \hat{w}_i(k)
\end{align*}
\]

for $i = 1, \ldots, n$ (10)

where $\hat{x}_i$ is the $i^{th}$ block of size $l \times 1$ of $\hat{x}$, and $\hat{w}_i$, $\hat{u}_i$, $\hat{z}_i$ and $\hat{y}_i$ are similarly defined. We stress that these subsystems are different from the physical subsystems that may compose the global plant (i.e., the diagonal part of $A$); for this reason, we will sometimes call them “modal subsystems” to emphasize this fact.

In [10] we have shown a synthesis method for a class of distributed systems (“symmetric decomposable systems”), that can be basically summarized in the following steps:
1) Decompose the decomposable system into a set of \( N \) smaller systems; all these systems have state-space matrices parameterized by the eigenvalues \( \lambda \) of a “pattern matrix” \( P \).
2) Write the \( N \) synthesis LMIs for each of these subsystems.
3) Introduce a constraint that forces the state-space matrices of the controller to have the same parameterization according to \( \lambda \) (for state feedback in the continuous time case, this boils down to fixing a common Lyapunov matrix in all the \( N \) synthesis LMIs).
4) It turns out that all the synthesis LMIs are now affine in \( \lambda \), so \( \lambda \) being real they can be reduced from \( N \) to only 2 LMIs by the convexity.

This approach is based on a technique for multi-objective optimization called “Lyapunov shaping” [13], that naturally introduces conservatism to the result due to the necessity of forcing the Lyapunov matrices of a set of parameter dependent LMIs to be equal. In the discrete time case, the conservatism can be reduced by employing the extended LMI parameterization [3].

We summarize in a theorem the result for the state feedback \( \mathcal{H}_\infty \) synthesis for continuous time system ([10] contains only the discrete time case explicitly).

**Theorem 6:** Consider a continuous time symmetric decomposable system (Definition 1), with \( B_{u,b} = 0 \), \( D_{z,u,b} = 0 \). A sufficient condition for the existence of a static state feedback controller of the kind:

\[
u(t) = (I_N \otimes K_a + \mathcal{P} \otimes K_b)x(t)
\]

that yields a closed-loop \( \mathcal{H}_\infty \) norm smaller than \( \gamma \) is that the following set of LMIs has a feasible solution:

\[
\begin{bmatrix}
A_tX + B_{u,t}L_{t,*} + XA_t^T + L_{t,*}^T B_{u,t}^T & X C_{t,*} + L_{t,*}^T D_{z,u}^T \\
* & -I_{m,w} & D_{z,u}^T \\
* & * & -I_{r_x,2}^T
\end{bmatrix} < 0
\]

for the two values of \( \gamma \) corresponding to maximum and minimum eigenvalue \( \lambda_1 \), where \( X, \mathbf{L}_i = L_a + \lambda_i L_b \) are optimization variables, \( K_a = L_aX^{-1}, K_b = L_bX^{-1} \).

**Proof:** Start from the LMI for \( \mathcal{H}_\infty \) performance in [6] and use the procedures shown in [10].

**III. FULL BLOCK S-PROCEDURE**

Another possibility is to employ techniques from robust control, by assuming that \( \lambda \) is an uncertainty of the system in (10). There is a convex solution to the state feedback synthesis problem that is obtained by means of the so-called “full block S-procedure” [14]. We report here the result that we are going to use.

**Theorem 7 (continuous time):** (adapted from [14]) Let us consider an uncertain continuous time system described by the equations:

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + B_1p(t) + B_2w(t) + Bu(t) \\
q(t) &= C_1x(t) + D_1p(t) + D_{12}w(t) + E_1u(t) \\
z(t) &= C_2x(t) + D_{21}p(t) + D_2w(t) + E_2u(t)
\end{align*}
\]

where \( x(t) \) is the state, \( w(t) \) the disturbance, \( u(t) \) the control input, \( z(t) \) the performance output, \( q(t) \) and \( p(t) \) signals for which it holds:

\[
p(t) = \Delta q(t)
\]

where \( \Delta \) is a (time varying) uncertainty which assumes values in the convex hull generated by a set \( \{ \Delta_j \} \) for \( j = 1, ..., J \), which includes the zero vector.

The state feedback controller yielding an \( \mathcal{H}_\infty \) norm from \( u \) to \( z \) smaller than \( \gamma \) for all the valid uncertainties can be found via the LMI set in (15), (16) and (17) (at the top of the next page), where \( Y = Y^T, M, \bar{R} = \bar{R}^T, \tilde{S}, \tilde{Q} = \tilde{Q}^T \) are the decision variables, and the stars * replace the symbols that would make the left hand side of the inequalities symmetric. The controller gain \( K \) is given by \( K = MY^{-1} \).

Notice that this theorem offers only a sufficient condition, so its use implies conservatism. Also notice that the theorem involves a time varying uncertainty, which means that some more conservatism will be introduced when applying it to the decentralized or distributed control methods, as in that case the uncertainties (the eigenvalues of the pattern matrix) are time invariant.

**IV. FULL BLOCK S-PROCEDURE AND DECENTRALIZED CONTROL**

We consider a subclass of continuous time decomposable systems, where only the state matrix is built with the pattern and the other state-space matrices are diagonal. We assume a symmetric pattern matrix \( \mathcal{P} \), with real eigenvalues. The method in [10] would find a suboptimal \( \mathcal{H}_\infty \) controller as in (11) with a diagonal part and a part with the sparsity described by the pattern matrix. We call such controllers “distributed controllers”, meaning that the control action on each agent is determined on the basis of the local output and the output of the neighboring agents. It is possible to constrain the controller to have \( K_b = 0 \), which would yield a decentralized controller. By “decentralized controller” we mean a controller made of a set of local controller that interact only with each single agent, and not with their neighbors; for this reason, such controllers have a block diagonal gain matrix. The same kind of decentralized controller can be obtained with the new methodology. The result is summarized in the following theorem.

**Theorem 8:** Let us consider a continuous time symmetric decomposable systems described by the equations:

\[
\begin{align*}
\dot{x}(t) &= (I_N \otimes A_u + \mathcal{P} \otimes A_b)x(t) + I_N \otimes B_au(t) + I_N \otimes B_uw(t) \\
z(t) &= I_N \otimes Cx(t) + I_N \otimes Du(t)
\end{align*}
\]

where \( x(t) \) is the state, \( w(t) \) the disturbance, \( u(t) \) the control input and \( z(t) \) the performance output. There exists a suboptimal decentralized controller of the kind:

\[
u(t) = (I_N \otimes K)x(t)
\]

with \( \mathcal{H}_\infty \) norm from \( w \) to \( z \) smaller than \( \gamma \), if the LMIs in (20), (21) and (22) (at the top of next page) are feasible, where \( \mathcal{X} \) and \( \Delta \) are the highest and lower eigenvalue of \( \mathcal{P} \).
and $Y = Y^T$, $M$, $\hat{R} = \hat{R}^T$, $\hat{S}$, $\hat{Q} = \hat{Q}^T$ are the decision variables. The controller gain $K$ is given by $K = MY^{-1}$.

The discrete time version can be easily obtained as well.

V. FULL BLOCK S-PROCEDURE AND DISTRIBUTED CONTROL

With the new method it is not possible to introduce directly a $K_b$ term as in (19) that would produce a distributed controller instead of a decentralized one. There is a “trick” though that allows using the decentralized controller synthesis method of Theorem 8 to generate a distributed controller. It is just sufficient to extend (18) by adding another input channel that is influenced by the pattern matrix. In fact, we can prove that:

$$\begin{align*}
\dot{x}(t) &= (I_N \otimes A_a + P \otimes A_b)x(t) + (I_N \otimes B_a)u(t) + \\
&\quad + (I_N \otimes B_w)w(t) \\
z(x) &= (I_N \otimes C_{z,a} + P \otimes C_{z,b})x(t) + (I_N \otimes D_{z,a})u(t)
\end{align*}$$

in closed loop with:

$$u(t) = (I_N \otimes K_a + P \otimes K_b)x(t)$$

(23)

is equivalent to:

$$\begin{align*}
\dot{x}(t) &= (I_N \otimes A_a + P \otimes A_b)x(t) + (I_N \otimes [B_a 0] + \\
&\quad + P \otimes [0 B_a])\hat{u}(t) + (I_N \otimes B_w)\hat{w}(t) \\
z(x) &= (I_N \otimes C_{z,a} + P \otimes C_{z,b})x(t) + (I_N \otimes [D_{z,a} 0] + \\
&\quad + P \otimes [0 D_{z,a}])\hat{u}(t)
\end{align*}$$

(25)

in closed loop with:

$$\hat{u}(t) = I_N \otimes \begin{bmatrix} K_a \\ K_b \end{bmatrix} x(t)$$

(26)

They both yield:

$$\begin{align*}
\dot{x}(t) &= (I_N \otimes (A_a + B_uK_a) + P \otimes (A_b + B_uK_b))x(t) + \\
&\quad + (I_N \otimes B_w)\hat{w}(t) \\
z(x) &= (I_N \otimes (C_{z,a} + D_{z,u}K_a) + P \otimes (C_{z,b} + D_{z,u}K_b))x(t)
\end{align*}$$

(27)

The system in (25) in closed loop with the decentralized controller (26) is equivalent to the controller in (23) in closed loop with the distributed controller in (24). So one can use (25) and (26) as a reference for computing the distributed controller of the system in (23). Again we summarize in a theorem.

**Theorem 9:** Let us consider a continuous time symmetric decomposable systems described by the equations:

$$\begin{align*}
\dot{x}(t) &= (I_N \otimes A_a + P \otimes A_b)x(t) + (I_N \otimes B_a)u(t) + \\
&\quad + (I_N \otimes B_w)w(t) \\
z(x) &= (I_N \otimes C_{z,a} + P \otimes C_{z,b})x(t) + (I_N \otimes D_{z,a})u(t)
\end{align*}$$

(28)

with $H_\infty$ norm from $w$ to $z$ smaller than $\gamma$, if the LMIs in (30) (31), and (32) (at the top of next page) are feasible, where:

$$\begin{bmatrix}
\mathcal{A} & \mathcal{B}_1 & \mathcal{B}_2 & \mathcal{B}_3 \\
\mathcal{C}_1 & \mathcal{D}_1 & \mathcal{D}_2 & \mathcal{D}_3
\end{bmatrix} =
\begin{bmatrix}
A_a & A_b & B_a & B_u & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & I \\
C_a & C_b & D & 0 & 0
\end{bmatrix}
$$

(33)

and $\lambda$, $\Delta$ are the maximum and minimum eigenvalue of $\mathcal{P}$, and $Y = Y^T$, $M$, $\hat{R} = \hat{R}^T$, $\hat{S}$, $\hat{Q} = \hat{Q}^T$ are the decision variables. The controller gains $K_a$, $K_b$, are given by the relation:

$$K = \begin{bmatrix} K_a \\ K_b \end{bmatrix} = MY^{-1}$$

(34)

**Proof:** Consider the equivalent system (25) in its
decomposed form (obtained as in [10]):
\[
\begin{align*}
\dot{x}_i(t) &= (A_u + \lambda_i \otimes A_b)x_i(t) + ([B_u 0] + \lambda_i [0 B_u])\hat{u}_i(t) + B_ww_i(t) \\
z_i(x) &= (C_{z,a} + \lambda_i C_{z,b})x_i(t) + ([D_{zu} 0] + \lambda_i [0 D_{zu}])\hat{u}_i(t)
\end{align*}
\]
with \( \hat{u}_i = [\hat{u}_{a_i}^T \hat{u}_{b_i}^T]^T \). This set of systems (depending on \( \lambda_i \)) can be interpreted as a system depending on the (uncertain) parameter \( \lambda_i \). Define the output signal \( q_i(t) \) as:
\[
q_i(t) = [x_i^T \hat{u}_{b_i}^T]^T
\]
and the input signal \( p_i(t) \) as:
\[
p_i(t) = \lambda_i q_i(t)
\]
Then the system can be rewritten as:
\[
\begin{align*}
\dot{x}_i(t) &= A_u x_i(t) + [A_b B_u] p_i(t) + [B_u 0] \hat{u}_i(t) + B_ww_i(t) \\
q_i(t) &= \left[ \begin{array}{c} I \\
0 \end{array} \right] x_i(t) + \left[ \begin{array}{c} 0 \\
0 \\
0 \\
0 \end{array} \right] \hat{u}_i(t) \\
z_i(x) &= (C_{z,a} + \lambda_i C_{z,b})x_i(t) + ([D_{zu} 0] + \lambda_i [0 D_{zu}])p_i(t) + [D_{zu} 0] \hat{u}_i(t)
\end{align*}
\]
from which the matrices in (33) can be derived by inspection. Then apply Theorem 7.

A discrete time version of Theorem 9 can be as easily derived.

**Remark 10:** Note that the cases to which Theorem 9 applies are fewer with respect to the method in [10], because there is no possibility of having a part in the input matrix or in the feedthrough matrix depending on the parameter \( \lambda \). Theorem 9 applies only if \( B_{u,b} = 0, D_{z,u,b} = 0 \).

**Remark 11:** The technical condition that \( \underline{\lambda} \leq \lambda \leq \bar{\lambda} \) does not reduce the applicability of the theorem, as the parameterization in (7) is not unique. For example, in the case that \( 0 \leq \lambda \leq \bar{\lambda} \), then pick an \( a \) such that \( \underline{\lambda} \leq a \leq \bar{\lambda} \); choose a new pattern matrix \( \mathcal{P}' = \mathcal{P} - aI_N \), and replace the state-space matrices \( M_a \) with \( M_a' = M_a + aM_0 \). The global state-space matrices are unchanged and the eigenvalues of \( \mathcal{P}' \) satisfy the condition.

### VI. SMALL GAIN THEOREM FOR DISTRIBUTED CONTROL

It is possible to use the small gain theorem [16] as an alternative to the full block S-procedure in order to guarantee the robust \( \mathcal{H}_\infty \) performance for all the possible values of the parameter \( \lambda \), similarly to what is done in [12]. This would easily lead to the following theorem.

**Theorem 12:** Let us consider a continuous time symmetric decomposable systems described by (28), and put it in the form as in (13) via the relations in (33) considering \( \lambda \) as the perturbation. There exists a suboptimal distributed controller as in (29) that yields a closed loop \( \mathcal{H}_\infty \) norm smaller or equal to \( \gamma \) if the \( \mathcal{H}_\infty \) norm of the closed loop system from the joint inputs \( [w^T p^T]^T \) to the joint outputs \( [z^T q^T]^T \) is \( \leq \gamma \) and \( \max_i(|\lambda_i|) \) is too far from \( \frac{1}{\gamma} \).

The synthesis can be executed by iterating an \( \mathcal{H}_\infty \) synthesis LMI on the value of \( \gamma \), rescaling the disturbance signal \( p \) if \( \max_i(|\lambda_i|) \) is too far from \( \frac{1}{\gamma} \).

### VII. NUMERICAL RESULTS

It is natural to compare the new method of Theorem 9 with the old method of [10] (Theorem 6) and method in Theorem 12. The three methods have been tried on randomly generated systems and compared. It appears that most of the times (more than 95\% of the cases, in our tests) the method of [10] yields better performance. Yet there are some cases where the opposite happens, for example in the system shown in (39) at the top of the next page, with pattern matrix shown in Fig. 2. The eigenvalues of the pattern matrix are in the interval \([-2.2361, 2.2361]\) and the local order is \( l = 3 \).

![Pattern matrix for the example, and the graph connections that it represents.](image-url)
\[
\begin{align*}
\dot{x}(t) &= \left( I_N \otimes \begin{bmatrix} 0.1 & -0.2 & 0.7 \\ -0.9 & -0.6 & -0.4 \\ -0.9 & 0.6 & -0.5 \end{bmatrix} + \mathcal{P} \otimes \begin{bmatrix} 0.1 & -0.3 & -0.1 \\ -0.3 & -0.1 & -0.1 \\ -0.2 & -0.1 & 0 \end{bmatrix} \right) x(t) + \\
&\quad + I_N \otimes \begin{bmatrix} 0.7 \\ 0 \\ 0.5 \end{bmatrix} u(t) + I_N \otimes \begin{bmatrix} 0.1 \\ 0.1 \\ 0.1 \end{bmatrix} w(t) \\

z(x) &= \left( I_N \otimes \begin{bmatrix} 0.9 & 0.2 & 0.3 \\ 0 & 0 & 0 \end{bmatrix} \right) \left( I_N \otimes \begin{bmatrix} 0.5 & 0.3 & 0.5 \\ 0 & 0 & 0 \end{bmatrix} \right) x(t) + I_N \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)
\end{align*}
\] (39)

\begin{table}[h]
\begin{tabular}{|c|c|c|}
\hline
Method & \mathcal{H}_\infty norm & \gamma \\
\hline
Open loop & 1.9637 & \text{--} \\
Multi-objective optimization (Theorem 6) & 0.2170 & 0.2952 \\
Full block S-procedure (Theorem 9) & 0.1833 & 0.2779 \\
Small gain (Theorem 12) & 0.2141 & 1.4953 \\
\hline
\end{tabular}
\caption{Optimization results for case 1}
\end{table}

\begin{table}[h]
\begin{tabular}{|c|c|c|}
\hline
Method & \mathcal{H}_\infty norm & \gamma \\
\hline
Open loop & 0.2481 & \text{--} \\
Multi-objective optimization (Theorem 6) & 0.0513 & 0.0521 \\
Full block S-procedure (Theorem 9) & 0.1716 & 0.2254 \\
Small gain (Theorem 12) & 0.2181 & 1.4998 \\
\hline
\end{tabular}
\caption{Optimization results for case 2}
\end{table}

VIII. CONCLUSION

A new method for distributed state feedback control of decomposable systems [10] has been developed. The method is based on full block S-procedure [14] and it involves some conservatism due to the nature of the tools used. This conservatism though is of a “different kind” than the conservatism due to the need for a common Lyapunov matrix, which is the key of the old method. The method has also been compared to a similar approach using the small gain theorem. The numerical results show that in some cases, the S-procedure based method gives better performance than the other methods.

The most interesting result of the paper is, in our opinion, that the three different methods are all conservative, but in a different way; so it appears that they might be considered as complementary to one another.

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REFERENCES