Transient Energy Analysis of a Spatially Interconnected Model for 3D Poiseuille Flow

Saulat S. Chughtai and Herbert Werner

Abstract—In this paper a new model for 3D Poiseuille flow is presented. The model is based on applying a combined spectral-finite difference approach on the velocity-vorticity formulation of the Navier-Stokes equations. In 3D the dominating feature of the problem is non-normality of the eigenvectors. One measure to assess the non-normality is the transient energy response. The model is validated by comparing the maximum transient energy for two different cases; for both cases the Reynolds number is fixed at 5000, in the first case the span-wise spatial frequency is set to zero and the stream-wise spatial frequency is varied from 0.1 to 2, while for the second case the stream-wise frequency is set to zero and the span-wise frequency is varied from 0.1 to 3. It is known that for the first case the system is stable, while for the second case is highly non-normal with maximum transient energy reaching 4500. It is observed that the model predicts both of these characteristics.

Next, as a first step towards controller synthesis, a stabilizing state feedback controller is designed to minimize the transient energy response, by minimizing the induced $L_2$-norm. The closed loop response shows that the energy is reduced by the factor of 30.

I. INTRODUCTION

Control of flow patterns to improve efficiency or performance is of immense technological importance. The potential benefits of flow control include improved fuel efficiency and environmental compliance. In such schemes the control action is applied to delay/advance the transition, to suppress/enhance turbulence or to prevent/provoke separation. These phenomenon results in reduction of drag, enhancement of lift and reduction in flow-induced noise [1].

Initially most of the work in this field was concentrated on either passive control which resulted in design modification, or open-loop control. A nice survey on earlier open-loop control approaches is presented in [2], while some later results on this approach are presented in [3]. Later modern approaches for designing active controllers were used.

The dynamical behavior of flow is governed by the Navier-Stokes equations (NSE), which are nonlinear coupled partial differential equations. Due to the nonlinear nature of the phenomenon a lot of research has been done on nonlinear control approaches, a survey about the application of neural networks and chaos theory can be found in [1], while the application of model predictive control for flow control is presented in [4], and on application of adaptive control using back stepping is presented in [5]. Some other advances in the analysis of different flows and its implication on control are compiled in [6] and [7].

In order to be able to use linear controller synthesis techniques, NSE are linearized around the laminar flow. Converting linearized NSE into state space form will result in a singular system. In order to avoid this singularity, three possible formulations can be used [8]: a velocity-vorticity formulation, a stream wise function formulation and a velocity-pressure formulation. These linearized equations have infinite dimension in time and space, which can be reduced to finite dimension either by converting these into spectral domain or by using a finite difference approach. In [9] a finite dimensional model of plane Poiseuille in spectral domain is presented using a velocity-vorticity formulation, where Chebyshev basis functions are used in wall normal direction and Fourier series are used in stream-wise direction. By the analysis of this model it was found that the transition from laminar to turbulence at large Reynods number ($Re$) is due to the crossing of the imaginary axis by a pole pair at $Re > 5772$. Thus, transition control in plane Poiseuille flow can be viewed as a stabilization problem. Based on this model many researchers have proposed linear controllers. A detailed exposition of the application of linear control approaches can be found in [10] and references there in. In [11] the authors have designed simple $PI$ type controllers to achieve stability.

In the 3D case, experimental studies has shown that the phenomenon of transition may occur at $Re < 5000$. In [12] the authors proposed that this discrepancy between theoreti-
The model presented here is based on a velocity-vorticity formulation, which has numerical advantages as discussed in [20], [21].

2. The introduction of spatial shift operators has resulted in a model which represents an infinitely long channel. In [19], on the other hand, the authors have found a multi-input-multi-output (MIMO) model whose order increases with the number of finite elements used, thus making it difficult to use for controller synthesis of long channels with large number of inputs and outputs. In contrast, the model proposed here can be used to synthesize controllers using approaches recently proposed in [22], where there is no restriction on channel length or number of inputs and outputs.

In this paper, we have extended the approach presented in [18] to the 3D case. As pointed out earlier there are two main differences in 2D and 3D cases: one is that in the 3D case non-normality dominates the instability of the eigen modes, secondly in 2D we only need to solve for wall normal velocity, while in 3D vorticity equation must be solved together with wall normal velocity. The newly developed model is first validated by checking its transient energy response. It is shown that the model does show large transient energy growth for span-wise perturbations compared to its response for stream-wise perturbations, as has been reported else where in the literature e.g [12]. Next a state feedback controller is synthesized using the approach of [22] to minimize the induced $L_2$-norm of the closed loop system. It is shown that the closed loop system has considerably lower maximum transient energy.

The paper is organized as follows: Section 2 is devoted to the development of the spatially interconnected model. In section 3 the model is validated using the concept of transient energy. Section 4 deals with the state feedback controller synthesize problem. Finally conclusions are drawn and future directions are presented in section 5.

II. 3D Spatially interconnected model

Consider a three-dimensional steady state plane channel flow with centerline velocity $U_0$ and channel half-width $\delta$ as shown in Fig. 2.

The standard NSE can be linearized by first normalizing all velocities about the centreline velocity $U_0$ and half height $\delta$. Then, assuming laminar flow, the NSE can be linearized around the mean velocity profile in the streamwise direction ($x$). For laminar flow the mean velocity profile can be written as $U(y) = 1 - y^2$ in the domain $y \in [-1, 1]$. The equations governing small, incompressible, three-dimensional perturbations $\{u, v, w, p\}$ are then given by the linearized NSE and the continuity equations. By eliminating $p$ and defining a new variable $\omega$ (vorticity) we arrive at the following equivalent formulation of NSE, which is also known as
velocity-vorticity formulation,
\[
\Delta \dot{v} = \left\{-U \frac{\partial}{\partial x} \Delta + \frac{dU}{dy} \frac{\partial}{\partial x} + \Delta^2 \frac{\partial}{\partial x} \frac{\partial}{\partial x} + \frac{\Delta^2}{R_e} \right\} v
\] (1)
\[
\dot{\omega} = \left\{ -\frac{dU}{dy} \frac{\partial}{\partial z} \right\} v + \left\{-U \frac{\partial}{\partial x} + \Delta \frac{\partial}{\partial x} \frac{\partial}{\partial x} \frac{\partial}{\partial x} \right\} \omega
\] (2)

where vorticity is defined as \(\omega = \nabla \times \mathbf{u} \), with homogeneous boundary conditions,
\[
v(x, y = \pm 1, z, t) = 0 \quad \omega(x, y = \pm 1, z, t) = 0
\] (3)
\[
\frac{\partial v}{\partial y}(x, y = \pm 1, z, t) = 0 \quad \frac{\partial \omega}{\partial y}(x, y = \pm 1, z, t) = 0
\] (4)

Note that the above two equations have one directional coupling, i.e. the first equation does not depend on \(\omega\). The control inputs are applied at the boundary \((y = \pm 1)\) which then changes the homogenous boundary conditions into a non-homogeneous ones. However, the problem can be converted into non-homogeneous PDE with homogeneous boundary conditions by a change of variables. Let,
\[
v(x, y, z, t) = \phi(x, y, z, t) + q(t)g(x, z)f(y)
\] (5)
where, \(f(y)\) is any function satisfying the following condition,
\[
f(-1) = 1, \quad f(1) = 0, \quad \frac{df(\pm 1)}{dy} = 0
\] (6)
and \(g(x, z)\) defines the distribution of control force in the \(xz\)-plane. Using this new variable \(\phi\) we can recover the homogeneity in the boundary conditions. Thus the modified boundary value PDEs are given as:
\[
\Delta \phi + \dot{q}(gf) = -qU \frac{\partial}{\partial x} \Delta (gf) + q \frac{dU}{dy} \frac{\partial g}{\partial x}
\]
\[
+ \frac{\Delta^2}{R_e} (gf) - U \frac{\partial}{\partial x} \phi + \frac{dU}{dy} \frac{\partial \phi}{\partial x}
\]
\[
+ \frac{\Delta^2}{R_e} \phi
\]
\[
\dot{\omega} = \left\{ -qf \frac{dU}{dy} \frac{\partial g}{\partial z} - \frac{dU}{dy} \frac{\partial \phi}{\partial y} \right\}
\]
\[
+ \left\{-U \frac{\partial}{\partial x} + \Delta \frac{1}{R_e} \frac{\partial}{\partial x} \frac{\partial}{\partial x} \frac{\partial}{\partial x} \right\} \omega
\] (7)
\[
\phi(x, y = \pm 1, z, t) = 0 \quad \frac{\partial \phi}{\partial y}(x, y = \pm 1, z, t) = 0
\] (8)

To obtain finite dimensional approximation of the above PDEs, a Chebyshev expansion can be used while a finite difference scheme can be used in streamwise and spanwise directions.

A. Span-wise and stream-wise discretization

The \(x\)-direction is discretized into regularly spaced samples, using the following symmetric finite difference approx-
imation for the partial derivatives \(\frac{\partial^k}{\partial x^k}\)
\[
\left\{ \frac{\partial}{\partial x} \right\} = \frac{S_1 - S_1^{-1}}{2}
\]
\[
\left\{ \frac{\partial^2}{\partial x^2} \right\} = S_1 + S_1^{-1} - 2
\]
\[
\left\{ \frac{\partial^3}{\partial x^3} \right\} = \frac{S_1^2 - 2S_1 + 2S_1^{-1} - S_1^{-2}}{2}
\]
\[
\left\{ \frac{\partial^4}{\partial x^4} \right\} = \frac{S_1^2 - 4S_1 - 4S_1^{-1} + S_1^{-2} + 6}{2}
\] (10)

Here, \(S_1\) is the spatial shift operator along \(x\)-direction. Similarly for \(z\)-direction we have \(S_2\) as spatial shift operator. Applying these discretizations on (7), the variables \(\phi\) and \(\omega\) at \(i, j^{th}\) grid point in \(xz\)-plane will be functions of \(y\) and time \(t\). Then \(g(x, z)\) will represent control action distribution within one element. For simplicity, here, we have considered \(g(x, z) = 1\) this is equivalent to having uniform control action with in the element.

B. Wall normal discretization

Discretization in wall normal direction is done using Chebyshev polynomials with finite basis functions of degree \(N\). Thus, at any grid point \(ij\) in the \(xz\)-plane, \(\phi^{i,j}\) is given as,
\[
\phi^{ij}(y, t) = \sum_{m=0}^{N} a_m(t) \Gamma_m(y)
\] (11)
\[
\omega^{ij}(y, t) = \sum_{m=0}^{N} b_m(t) \Gamma_m(y)
\] (12)

where \(\Gamma_m(y)\) is the \(m^{th}\) Chebyshev basis function. For the present work we have used the basis functions proposed by McKernan [23], which are given as,
\[
\Gamma_1^M = \Gamma_1
\]
\[
\Gamma_2^M = \Gamma_2
\]
\[
\Gamma_3^M = \Gamma_3 - \Gamma_1
\]
\[
\Gamma_4^M = \Gamma_4 - \Gamma_2
\]
\[
\Gamma_{m<4, odd}^M = (\Gamma_m - \Gamma_1) - \frac{(m - 1)^2(\Gamma_{m-2} - \Gamma_1)}{(m - 3)^2}
\]
\[
\Gamma_{m<4, even}^M = (\Gamma_m - \Gamma_2) - \frac{((m - 1)^2 - 1)(\Gamma_{m-2} - \Gamma_2)}{(m - 3)^2 - 1}
\] (13)

The first 4 basis functions do not fulfill the Dirichlet and Neumann boundary conditions. This suggests that we should eliminate columns corresponding to these in (11), while for vorticity we only require Dirichlet boundary conditions which result in elimination of the first two basis functions. These basis functions can be decretized along the \(y\)-direction by making a grid of \(N+1\) Chebyshev-Gauss-Lobatto points \(y_k\), where \(y_k = \cos(\pi k/N)\), \(\forall k = 1, ..., N\). In order to make the resulting matrices square we can eliminate the two rows close to each of the upper and lower walls as these will be
basis functions and whose rows correspond to each of

Using (5), (10), (11) and (12), the measurement equations

Using the continuity equation and the definition of

Γ
corresponds to the first row of Γ′ and Γ′′ while y = −1 corresponds to the last row of Γ′ and Γ′′.

Remark 1: The assumption of estimating the Laplacian of skin friction can be relaxed by applying spatial integration using a suitable low pass spatial filter on (17). 

III. TRANSIENT ENERGY OF FINITE DIFFERENCE MODEL

In this section we will estimate the maximum transient energy, of the model proposed in the previous section, to validate the model. The transient energy is defined as [12],

where mass density of the fluid is assumed to be one. Since \( V = 2 \) and both \( v \) and \( \frac{dy}{dx} \) are assumed to remain constant in \( x \) and \( z \)-direction in a single element, we obtain using the continuity equation and the definition of \( \omega \)

Let us approximate the integral by a weighted sum as [24].

Using (5), (10), (11) and (12), the measurement equations can be written as,

where, \( y = 1 \) corresponds to the first row of \( \Gamma' \) and \( \Gamma'' \) while \( y = −1 \) corresponds to the last row of \( \Gamma' \) and \( \Gamma'' \).

\[ E(t) = 1 \int^{1}_{−1} \int^{\frac{1}{2}}_{−\frac{1}{2}} u^2 + v^2 + w^2 \, dz \, dx \, dy \]

\[ E(t) = \frac{1}{4} \int^{1}_{−1} v^2 + (\frac{\partial v}{\partial y})^2 + \omega^2 dy \]

\[ \forall n = 1, \, N - 1 \]

\[ w_n = \frac{2}{N} \sqrt{1 - y_n^2} \sum^{N-1}_{m=1} \frac{1}{m} \sin(m \pi n/N)(1 - \cos(m \pi)) \]

and \( w_0 = w_N = 0 \). Substituting (20) in (19) and arranging the terms in matrices we obtain

\[ E(t) = \frac{\Delta^T \Psi \Lambda + \Delta_g^T \Psi \Lambda_g + \Delta_\omega^T \Psi \Lambda_\omega}{4} \]
where
\[
\Psi = \begin{bmatrix}
w_0 & 0 \\
0 & \ddots & 0 \\
0 & & w_N
\end{bmatrix}
\quad \Lambda = \begin{bmatrix}
v_0 \\
v_1 \\
\vdots \\
v_N
\end{bmatrix},
\Lambda y = \begin{bmatrix}
\frac{\partial v}{\partial t} \\
\frac{\partial y}{\partial t} \\
\vdots \\
\frac{\partial v}{\partial t}
\end{bmatrix}, \Lambda_\omega = \begin{bmatrix}
\omega_0 \\
\omega_1 \\
\vdots \\
\omega_N
\end{bmatrix}
\]
(23)

In terms of the temporal state vector \(x^T = [a^T, b^T]\) of the linearized model (14), (22) can be written as
\[
E(t) = x^T \left( \frac{T^T \Psi T + T^T \Psi y}{4h^2} \right) x^t
\]
where \(T = \text{diag}(\Gamma, \Gamma)\) and \(T_y = \text{diag}(\Gamma', 0)\) together with the deleted rows corresponding to \(n = [0, 1, N - 1, N]\), respectively. If the system is diagonalizable then
\[
x^t = X_0 e^{\Phi t}
\]
where, \(\Phi\) is a diagonal matrix consisting of all the eigenvalues at its principal diagonal. Putting this in (24) will result in a matrix whose maximum singular value at any time \(t\) will be the maximum transient energy \((E_m(t))\) starting from any initial state.

A. Results and Discussion

First \(E_m(t)\) for \(Re = 5000, f_x = [0.1, \ldots, 2.5]\) and \(f_z = 0\) are calculated, where \(f_x\) is spatial frequency in \(x\)-direction and \(f_z\) is spatial frequency in \(z\)-direction, as shown in Fig. 3. It turns out that the system is stable, since the transient energy decreased after the initial peak. However, for a spatial frequency of nearly 1 the decay in the energy is less than for other frequencies. This is because of a pole pair which is close to the imaginary axis and for \(Re > 6000\) becomes unstable. The same behavior is reported elsewhere, e.g. [9].

Next, \(E_m(t)\) for \(Re = 5000, f_x = 0\) and \(f_z = [0.1, \ldots, 3]\) are calculated. These are plotted in Fig. 4. It shows a large peak of \(E_m(t)\) at \(f_z \approx 2\) at \(t \approx 380\). This shows that the system is highly non-normal at this frequency. The values obtained here are in agreement with the ones found by other researchers e.g. [12], [15], who found the peak of \(E_m(t)\) at \(t = 379\) at the same spatial frequency. The results presented above show that the model does capture the dominating features of the problem.

IV. CONTROLLER SYNTHESIS

Next, we demonstrate the use of the proposed model for designing a controller in physical domain. It is known [25], [10] that non-normality of an operator increases its induced \(L_2\)-norm by stretching its pseudo spectra. Hence, to improve non-normality one measure could be to minimize the closed loop \(L_2\)-norm. Here we have designed a state feedback controller by minimizing the closed loop induced \(L_2\)-norm based on the approach of [22], which is next summarized.

![Fig. 3. \(E_m(t)\) for \(Re = 5000, f_x = [0.1, \ldots, 2.5]\) and \(f_z = 0\)]

![Fig. 4. \(E_m(t)\) for \(Re = 5000, f_x = 0\) and \(f_z = [0.1, \ldots, 3]\)]

The model presented here can be written as,
\[
\begin{bmatrix}
ex^t \\
y
\end{bmatrix} = \begin{bmatrix}
\bar{A}^{t} & \bar{A}^{ts} & \bar{B}_0^s & \bar{B}_0^t \\
\bar{A}^{st} & \bar{A}^{st} & \bar{B}_0^s & \bar{B}_0^t \\
\bar{C}^{t} & \bar{C}^{s} & \bar{D}_{00}
\end{bmatrix} \begin{bmatrix}
ex^t \\
y
\end{bmatrix}
\]
(25)

where \(ex^t = [q^T, b^T]\), \(u = [\dot{q}^T, \dot{q}^T]^T\) and \(x^s\) contains the signals which are entering the \(i^{th}\) system from left and right. Correspondingly \(Sx^s\) are the signals exiting the right and left neighbors, where \(S = \text{diag}(S_1I, S_1^{-1}I, S_2I, S_2^{-1}I)\)

For brevity let us define \(x^T = [x^{tT}, x^{sT}]\). Then, for a system having state space representation (25), the following result holds [22]:

Theorem IV.1 A spatially interconnected system with state space representation (25), is well-posed, stable and has
induced $L_2$-norm $< \gamma$ if there exist $X_t \in X_t$ and $X_s \in X_s$ such that the following inequality is satisfied:

$$\begin{bmatrix} A^T X + X A & X B_3 & C^T_f \\ B_1^T X & -\gamma^2 I & D_{11}^T \\ C_1 & D_{11} & -I \end{bmatrix} < 0$$

(26)

where

$$X := \text{diag}(X^t, X^s)$$

$$X^t := \{ X^t \in R^{n_x \times n_s} : X^t = X'^T > 0 \}$$

$$X^s := \{ X^s = \text{diag}(X^{s_1}, \ldots, X^{s_r}) : X_{s_i} = X_{s_i}^T \in R^{n_x \times n_s} \}$$

Using the above result a state feedback controller is synthesized. The closed loop $E_{m}(t)$ is as shown in Fig. 5. It shows that the controller has reduced the peak transient energy from 4500 to 130.

![Graph showing the closed loop $E_{m}(t)$ for $Re = 5000$, $f_s = 0$ and $f_s = [0.1, \ldots, 3].$](image)

**Fig. 5.** Closed loop $E_{m}(t)$ for $Re = 5000$, $f_s = 0$ and $f_s = [0.1, \ldots, 3]$.  

**V. CONCLUSIONS**

In this paper a new model for 3D Poiseuille flow is presented. The model is based on a combined spectral-finite difference approach. The velocity-vorticity formulation of the NSE equations is used. By comparing the maximum transient energy for two different cases it has been demonstrated that the model predicts the dominating features of the Poiseuille flow. This suggests that it can therefore be used for synthesizing a controller to control the transition from laminar to turbulent flow. As a first step towards controlling synthesis, a stabilizing state feedback controller is designed to minimize the transient energy response. The closed loop response shows that the energy can be reduced considerably.

As a next step we will validate the model against nonlinear response and use the proposed model for the synthesis of low-order and fixed-structure controllers, based on the approach presented in [26].

**REFERENCES**