Optimal Output Trajectory Design and Tracking in Preview-based Nonperiodic Tracking-Transition Switching for Nonminimum-Phase Linear Systems

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Abstract—In this article, the problem of nonperiodic tracking-transition switching with preview is considered, where, multiple switching between tracking and transition occur, and the output needs to track desired trajectory during the tracking sections, then rapidly transits to another point during the transition sections with no post-transition oscillations. Due to the coupling between the control of the tracking sections and that of the transition ones, these control objectives become challenging, particularly for nonminimum-phase systems. In the proposed approach, (1) the optimal desired output trajectory for the transition sections is designed through direct minimization of the output energy, and (2) the needed control input that maintains the smoothness of the system state across all tracking-transition switching instances is obtained by using a preview-based stable-inversion approach, and the needed preview time is quantified.

I. INTRODUCTION

We present an inversion-based optimal control approach to solve the problem of nonperiodic tracking-transition switching with preview for nonminimum-phase linear systems. Such a control problem arises in many applications [1], [2], where, multiple switching between output tracking and output transition occur, and the control objectives are two folds: (1) During each tracking section, the desired trajectory shall be tracked accurately to meet the specific needs of the application; And (2) during the transition section immediately after, the output shall be rapidly transited, with no induced post oscillations, to the desired position. The tracking-transition switching are, in general, nonperiodic, and can be previewed for a finite amount of time (i.e., a finite preview). In this article, we propose a stable-inversion approach to achieve the above two control objectives. The proposed approach combines the stable-inversion approach with optimal control technique to maintain both the smoothness of the system state across the tracking-transition switching intervals, and the output precision throughout the entire tracking-transition switching course.

Challenges exist in the problem of nonperiodic tracking-transition with preview for nonminimum-phase systems. Note that point-to-point output transition with minimal oscillations is needed in many applications. The desired output trajectory obtained from the optimal state transition (OST) or the optimal output transition (OOT) [3] techniques, however, can be highly oscillatory when the system dynamics is lightly-damped, due to the minimization of input energy [4]. Although such large output oscillations can be mitigated by modifying the system dynamics with a pre-filter [4], it is desirable to directly minimize the output energy rather than the input energy in the output transition. More importantly, smooth tracking of such a desired trajectory across switching instants may not be achieved, because smoothness of the system state—not just the output—across the switching instances needs to be maintained which cannot be achieved by using existing approaches such the OST, the OOT, or the input-shaping techniques [3]. Moreover, for nonminimum-phase systems, as pre-actuation is needed to track a given desired output trajectory [5], the control of tracking sections is coupled with that of transition sections [5], and vice versa, which becomes challenging in general, as usually the tracking-transition switching are nonperiodic, and the system state at the switching instants are unknown. Therefore, nonperiodic tracking-transition switching with preview for nonminimum-phase systems cannot be addressed with existing approaches.

The main contribution of the article is the development of an inversion-based approach to achieve optimal tracking-transition switching with preview for nonminimum-phase linear systems. In the proposed approach, the tracking-transition switching problem is transformed to design the desired output trajectory for the transition sections first, and then obtaining the corresponding control input for the trajectory consisting of both tracking and transition sections. The desired output trajectory for the transition sections are designed to not only minimize the output energy but also guarantee the smoothness of the desired output across the transition-tracking switching instants. The required control input is obtained through the preview-based approach. It is shown that at tracking-transition switching instants, smoothness of the entire system state is maintained, and the tracking error finite preview caused finite preview can be rendered arbitrarily small by having a large enough preview time. Therefore, the proposed approach extends the stable-inversion theory to the nonperiodic output tracking-transition switching with preview for nonminimum-phase systems.

II. PROBLEM FORMULATION: OPTIMAL OUTPUT TRACKING-TRANSITION WITH PREVIEW

Consider the following square LTI system

\[ \dot{x} = Ax + Bu, \quad y = Cx, \]  

(1)
with the same number of inputs and outputs, \( u(\cdot), y(\cdot) \in \mathbb{R}^p \), and \( x(\cdot) \in \mathbb{R}^n \). We assume that

**Assumption 1:** System (1) is controllable, observable, and hyperbolic (i.e., has no zeros on the imaginary axis), with a well-defined vector relative degree \( r \in [r_1, r_2, \ldots, r_p] \) [6]. Without loss of generality, we assign all transition sections \( T_k \) for \( k \in \mathbb{N} \) (the set of natural numbers), to be closed,

\[
T_k = [t_{k,i}, t_{k,f}], \quad k \in \mathbb{N},
\]

where \( t_{k,i} \) and \( t_{k,f} \) are defined as the initial instant and the final instant for the \( k^{th} \) transition section, respectively. Correspondingly, we assign all tracking sections \( I_q \) for \( q \in \mathbb{N} \) to be open, i.e.,

\[
I_q = (t_{q,f}, t_{q+1,i}), \quad q \in \mathbb{N}.
\]

Thus, the entire trajectory is a well-defined function of time, and is partitioned accordingly into the user-defined desired trajectory for the tracking intervals, \( y_{dtr}(\cdot) \), and the to-be-designed output trajectory for the transition intervals, \( y_{dtn}(\cdot) \), i.e.,

\[
y_d(\cdot) = (\cup_q y_{dtr,q}(\cdot)) \cup (\cup_k y_{dtn,k}(\cdot)),
\]

where \( y_{dtr,q}(\cdot) \) and \( y_{dtn,k}(\cdot) \) are defined for the \( q^{th} \) tracking interval \( I_q \) and the \( k^{th} \) transition interval \( T_k \), respectively, and \( q, k = 1, 2, \ldots \). Correspondingly, the control input \( u_{ff}(\cdot) \) for the tracking intervals and the transition intervals are denoted as \( u_{tr}(\cdot) \) and \( u_{tn}(\cdot) \), respectively,

\[
u_{ff}(\cdot) = (\cup q u_{tr,q}(\cdot)) \cup (\cup_k u_{tn,k}(\cdot)),
\]

Moreover, as discontinuity of the output, the input, or the system state, may occur at the boundary points, we further denote the left-hand limit as \( f(t^-) \), and the right-hand limit of a signal \( f(t) \) at time instant \( t \) as \( f(t^+) \), as given below

\[
f(t^-) = \lim_{\Delta t \to 0^-} f(t - \Delta t), \quad f(t^+) = \lim_{\Delta t \to 0^+} f(t + \Delta t),
\]

The left-hand limit and the right-hand limit of the \( r^{th} \) derivative of the signal \( f(t) \), \( f^{(r)}(t^-) \) and \( f^{(r)}(t^+) \), are defined similarly.

**Assumption 2:** The desired output trajectory is piecewisely-smooth enough, i.e., during any given tracking interval, \( I_q \) for \( q \in \mathbb{N} \), the desired output trajectory for the \( m^{th} \) output channel, \( y_{dtr,m}(\cdot) \), is differentiable up to the \( r_{m}^{th} \) order at all the interior points of the interval \( I_q \). The number of output transition in any finite time interval is finite.

**Assumption 3:** At any given time instant \( t_c \), there exists a finite preview time \( T_p \) of future desired tracking-transition switching consisting of \( N \) number of transitions \( (N = 0, 1, 2, \ldots, \) and \( N < \infty \), such that the desired output trajectory after the \( N^{th} \) transition is known for any given time duration \( \varepsilon > 0 \).

**Remark 1:** The above assumption essentially is to require that within the preview time window, the boundary conditions for all tracking-transition switching are well defined.

**Remark 2:** Assumption 3 implies that there exists an instant increase of preview time (in length) when the final instant of the preview window, \( t_c + T_p \), approaches to the initial instant of a transition interval.

**Problem Formulation** Let Assumptions 1–3 be satisfied, then the preview-based optimal output tracking and transition (POOTT) problem is to (1) design the desired output trajectory for each transition interval, \( y_{dtn,k}(t) \) for \( t \in T_k \) and each \( k = 1, 2, \ldots, N \), and (2) obtain the corresponding feedforward control input \( u_{ff}(\cdot) \), such that

\[
\begin{align*}
\mathcal{O}_1 & \text{ The required output transition is achieved, i.e., } \\
y_{dtr}(t_{k,i}) = y_{dtn}(t_{k,i}), y_{dtn}(t_{k,f}) = y_{dtr}(t_{k,f}), \\
& \text{ for } k = 1, 2, \ldots, N,
\end{align*}
\]

\[
\mathcal{O}_2 & \text{ During each transition, the energy of the output along with its derivatives up to the } r^{th} \text{ order are minimized, i.e., } \\
\min_{y_{dtn}(\cdot)} \int_{t_k}^{t_{k+1}} Q(H_{Y_{dtr}}(t)) dt,
\]

where \( Q \in \mathbb{R}^{p \times p} \) is a semidefinite positive matrix, \( H_{Y_{dtr}}(\cdot) \) is the vector of the desired output trajectory and its derivatives up to the \( r^{th} \) order (during the transition intervals),

\[
y_{dtr}(t) = \left[ \xi_{1,dtr}(t)^T, \xi_{2,dtr}(t)^T, \ldots, \xi_{p,dtr}(t)^T \right]^T, \quad \text{with}
\]

\[
\xi_{k,dtr}(t) = \left[ y_{k,dtr}(t), y_{k,dtr}(t), \ldots, \frac{d_{k-1}}{dt^n} y_{k,dtr}(t) \right]^T
\]

and \( H_Y \in \mathbb{R}^{p \times m} \) is a block diagonal matrix with \( m = p + \sum_{k=1}^{r_k} r_k \),

\[
H_{Y} = \begin{bmatrix}
H_{Y_1} & 0 & \cdots & 0 \\
0 & H_{Y_2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & H_{Y_p}
\end{bmatrix}
\]

with \( H_{Y_k} \in \mathbb{R}^{1 \times (r_k + 1)} \) for \( k = 1, 2, \ldots, p \) as

\[
H_{Y_k} = [H_{k}]_1, \quad \text{where}
\]

\[
[H_{k}]_1 = [-h_{k,1}, -h_{k,2}, \ldots, -h_{k,r_k}, 1]
\]

is the coefficient of a stable polynomial, i.e., the following polymer \( P(s) \)

\[
P(s) = s^{k+1} + h_{k,1}s^{k+1} + \ldots + h_{k,r_k}s + h_{k,1}
\]

has all its roots on the left open half complex plane.

\[
\mathcal{O}_3 \text{ When there exists an enough (but finite) preview, precision output tracking is maintained throughout the entire output tracking and transition course, i.e., at any time instant } t, \text{ the error relative to the exact tracking input is within any chosen positive number } \varepsilon_i > 0,
\]

\[
\|e_{p,in}(t)\|_2 \leq \|u_{tr}(t) - u_{tr}(t)\|_2 \leq \varepsilon_i,
\]

where \( \|a\|_2 \) is the standard vector 2-norm for vector \( a \in \mathbb{R}^n \), \( u_{tr}(\cdot) \) denotes the input that achieves exact tracking of the desired output trajectory, and \( u_{tr}(\cdot) \) denotes the input solution to the POOTT problem.

**III. STABLE-INVERSION-BASED SOLUTION TO THE POOTT PROBLEM**

We start with transforming system (1) to the output-tracking form.

**A. Output-Tracking Form**

Under Assumption 1, there exist (i) a state transformation, \( M : \mathbb{R}^n \rightarrow \mathbb{R}^n \) and (ii) an input law to transform system (1) into the output-tracking form. The needed state transformation \( M \) is given by

\[
\begin{bmatrix}
\xi(t) \\
\eta(t) \\
\eta(t)
\end{bmatrix} = Mx(t) = \begin{bmatrix}
M_{\xi} \\
M_{\eta,s} \\
M_{\eta,u}
\end{bmatrix} x(t)
\]

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where $\xi(t) = M_\xi x(t)$ are the output and its derivatives as in
\[
\xi(t) = [\xi_1^T(t), \xi_2^T(t), \cdots, \xi_p^T(t)]^T
\]  \hspace{1cm} (16)
with $\xi_k(t)$ being defined in (9), and $\eta_s$ and $\eta_u$ are the stable and the unstable subspaces of the internal dynamics, respectively. In (15), the matrices $M_\xi$, $M_{\eta,s}$, and $M_{\eta,u}$ are partitioned according to the rows of the output vector $\xi$, the stable internal dynamics $\eta_s$, and the unstable internal dynamics $\eta_u$, respectively. The input needed for the transformation in (15), in general, can be represented as
\[
\begin{align*}
\mathbf{u}_{m}(t) &= M_\xi \mathbf{z}(t) + M_\gamma \mathbf{r}(t) + M_\eta \eta_s(t) + M_u \eta_u(t) \\
&= M_\gamma \mathbf{Y}(t) + M_\eta \eta_s(t) + M_u \eta_u(t)
\end{align*}
\]  \hspace{1cm} (17)
where $\mathbf{Y}(\cdot)$ is as defined in (9). The above input is called the inverse input. The readers are referred to [7] for the expressions of matrices $M_\gamma$, $M_\eta$, $M_u$ in (17). By using the stable-inversion theory, it can be shown [5] that for square systems under Assumptions 1, 2, the bounded solution to the unstable internal dynamics $\eta_u(\cdot)$ is unique, so is the inverse input in (17). Using (15) and (17), system (1) is transformed to the output tracking form
\[
\begin{align*}
\mathbf{\dot{\xi}}(t) &= \mathbf{\dot{u}}_{m}(t) + \mathbf{B}_\gamma \mathbf{r}(t) \\
\mathbf{\eta}(t) &= A_i \eta(t) + B_i \mathbf{Y}(t)
\end{align*}
\]  \hspace{1cm} (18)
where
\[
\begin{align*}
\mathbf{\dot{u}}_{m}(t) &= \mathbf{\dot{u}}_{m,1}(t) + \mathbf{B}_\gamma \mathbf{r}(t) \\
\mathbf{\eta}(t) &= A_i \eta(t) + B_i \mathbf{Y}(t)
\end{align*}
\] (19)
with $E_k$ denoting the identity matrix of dimension $k \times k$, and the eigenvectors of $A_s$ and $A_u$ spanning the stable and the unstable subspaces of the internal dynamics, respectively.

### B. Design of the Optimal Output Transition Trajectory

The output tracking form (18) reveals that the $r^{th}$ derivative of the output acts as the input to the transformed system dynamics with the output and its (lower than $r^{th}$ order) derivatives constitute one part of the system state. Thus, the optimal output design problem (8) can be rendered as an optimal input design problem.

The output subdynamics (18) is stabilized by using a static state-feedback control, i.e., by choosing,
\[
\gamma^{(r)}(t) = H_\xi \mathbf{z}(t) + \gamma(t),
\]  \hspace{1cm} (20)
with the following output subdynamics becomes exponentially stable,
\[
\begin{align*}
\mathbf{\dot{\xi}}(t) &= (\mathbf{I}_{up} + \mathbf{B}_\xi H_\xi) \mathbf{\dot{\xi}}(t) + \mathbf{B}_\xi \gamma(t) \\
&= \mathbf{A}_\xi \mathbf{\dot{\xi}}(t) + \mathbf{B}_\xi \gamma(t)
\end{align*}
\]  \hspace{1cm} (21)
By (19–21), it can be verified that, the state feedback gain $H_\xi$ should be designed so that all the poles of the output subdynamics $\{\mathbf{A}_\xi, \mathbf{B}_\xi\}$ are real.

With the output subdynamics stabilized, the optimal desired transition output trajectory that satisfies $\mathcal{O}_1$ and $\mathcal{O}_2$ can be obtained as follows: for any given $k^{th}$ output transition within the preview time window, the optimal desired transition output trajectory can be obtained by solving the following optimal state transition problem,
\[
\min \mathbf{J}(T_k, \gamma) = \min \int_{T_{k-1}}^{T_k} \gamma^T(t, k) \mathbf{R}_\gamma \gamma(t, k) d\tau,
\]  \hspace{1cm} (22)
for given boundary desired output and its derivatives for the $k^{th}$ transition, i.e., $\xi_{dr}(t_k)$ and $\xi_{dr}(t_{k+})$, respectively.

**Remark 3:** Combining (20, 22) with (8–12), it is now clear that the cost function in (22) is transformed to that in the POOT problem formulation is (8–9) by setting $\mathbf{R}_\gamma = Q$ (8), and $H_{\xi_k} = H_k$ in (11). Thus, the solution to (22) satisfies $\mathcal{O}_1$ and $\mathcal{O}_2$.

The solution to the above minimization problem (22) can be readily obtain (See, e.g., [8]),
\[
\begin{align*}
\mathbf{y}^{(r)}(t, k) &= R_\gamma^{-1} \mathbf{B}_\xi^T e^{\mathbf{A}_\xi (T_{k-1}-t)} g^{-1}(T_k) \\
&\quad \times \left[ \xi_{dr}(t_{k+}) - e^{\mathbf{A}_\xi (T_{k-1}-t_k)} \xi_{dr}(t_k) \right],
\end{align*}
\]  \hspace{1cm} (23)
where $\mathcal{G}(T_{k+})$ is the controllability Gramian,
\[
\mathcal{G}(T_{k+}) = \int_{T_k}^{T_{k+}} e^{\mathbf{A}_\xi (T_{k+}-\tau)} \mathbf{B}_\xi^T e^{\mathbf{A}_\xi (\tau-T_k)} d\tau.
\]  \hspace{1cm} (24)
For the $k^{th}$ output transition, the desired optimal output trajectory and its derivatives, $\bar{\xi}_{dd}(t)$ ($t \in [t_k, t_{k+}]$), is obtained as
\[
\xi_{dr}^{(r)}(t) = H_\xi \bar{\xi}_{dd}(t) + \mathbf{y}^{(r)}(t, k), \quad \text{for } t \in [t_k, t_{k+}].
\]  \hspace{1cm} (25)

**Lemma 1:** Let Assumptions (1) to (3) be satisfied, then,
1) The desired optimal output trajectory and its derivatives up to the $(r-1)^{th}$, $\mathbf{\hat{\xi}}_{dd}(t)$, that satisfy Objectives $\mathcal{O}_1$ and $\mathcal{O}_2$ are given by
\[
\mathbf{\hat{\xi}}_{dd}(t) = \left\{ \begin{array}{ll}
\bar{\xi}_{dr}(t), & t \in I_k \\
\bar{\xi}_{dd}(t), & t \in T_k
\end{array} \right.
\]  \hspace{1cm} (26)
2) At the transition instants, the desired output trajectory is smooth up to the $(r-1)^{th}$ derivative, i.e., for each $k = 1, 2, \cdots, N$, where $\xi_{dd}(t)$ is the user-specified desired output trajectory for the tracking intervals, and $\bar{\xi}_{dd}(t)$ is obtained by solving the output subdynamics (21) with the optimal external input $\gamma^{(r)}(t, k)$,
\[
\mathbf{\dot{\xi}}_{dd}(t_{k-}) = \bar{\xi}_{dd}(t_{k+}), \quad \bar{\xi}_{dr}(t_{k+}) = \bar{\xi}_{dr}(t_{k-}),
\]  \hspace{1cm} (27)
where the function $\Gamma(\cdots, \cdots)$ is as defined in (23);

**Remark 4:** Note that discontinuity can occur at the transition instants in the $r^{th}$ derivative of the desired output,
\[
\mathbf{y}^{(r)}(t_k) \neq \mathbf{y}^{(r)}(t_{k+}), \quad \mathbf{y}_{dr}^{(r)}(t_{k+}) \neq \mathbf{y}_{dr}^{(r)}(t_{k-})
\]  \hspace{1cm} (29)

### C. Preview-based Input for the Tracking-Transition Period

We start with considering the case of infinite preview.

\[
\xi(t) = M_\xi x(t)
\]  \hspace{1cm} (30)
1) Inverse Input For Exact Tracking: Infinite Preview
Case: The above optimal desired output trajectory exists for nonminimum-phase systems with infinite preview of the desired trajectory, i.e., [2], [9].
\[ u_{\text{opt}}(t) = M_q \mathbf{V}_d(t) + M_q \eta_v(t) + M_q \eta_v^*(t), \] (31)
where the stable and the unstable internal dynamics, \( \eta_v(t) \) and \( \eta_v^*(t) \), respectively, are obtained by,
\[ \eta_v(t) = e^{A_v(t-\tau)} \eta_v(\tau) + \int_0^t e^{A_v(t-\tau)} B_u \mathbf{V}_d(\tau) d\tau \] (32)
\[ \eta_v^*(t) = -\int_t^\infty e^{-A_v(\tau-\tau)} B_u \mathbf{V}_d(\tau) d\tau. \] (33)

2) Optimal-Preview-based Inverse Input: Finite Preview Case: Obtaining the preview-based inverse control amounts to solving the unstable part of the internal dynamics (18) in the presence of tracking-transition switching as,
\[ \eta_{u,p}(t) = e^{-A_u(t-t_c)} \eta_u(t_c + T_p) - \int_{t_c}^{t_c+T_p} e^{-A_u(\tau-t_c)} B_u \mathbf{V}_d(\tau) d\tau, \] (34)
where \( \hat{\eta}_u(t_c + T_p) \) is the estimated future boundary value of the unstable internal dynamics at the end of the preview time window \((t_c + T_p)\) [7], unknown future boundary value is set to zero,
\[ \hat{\eta}_u(t_c + T_p) = 0. \] (35)
However, an optimal estimation of the boundary condition [7] can be obtained and incorporated into the proposed approach (see Sec. III-D later).

With the preview-based solution to the unstable internal dynamics (33), the preview-based inverse input to the POOT problem is obtained as
\[ u_{\text{pre}}(t) = M_q \mathbf{V}_d(t) + M_q \eta_v(t) + M_q \eta_{u,p}(t). \] (36)

Next, we show that the above preview-based solution achieves the objective O3.

**Lemma 2:** Let Assumptions 1 to 3 be satisfied, and the preview-based optimal output-input trajectory be given by (32) to (35). Then:
1) The stable part of the internal dynamics is continuous at both the initial and the final instants of each transition interval, i.e.,
\[ \eta_v(t^-_{k,i}) = \eta_v(t^+_{k,i}), \text{ and } \eta_v(t^-_{k,f}) = \eta_v(t^+_{k,f}), \text{ for } k = 1, 2, \ldots, N \] (37)
2) When the estimated boundary condition of the unstable internal dynamics, \( \hat{\eta}_u(t_c + T_p) \), is continuous, the preview-based solution to the unstable internal dynamics (33) is continuous at both the initial and the final instants of each transition interval, i.e.,
\[ \eta_v(t^-_{k,i}) = \hat{\eta}_u(t^+_{k,i}), \text{ and } \eta_v(t^-_{k,f}) = \hat{\eta}_u(t^+_{k,f}), \text{ for } k = 1, 2, \ldots, N. \] (38)
3) Smooth state transition is achieved at both the initial and the final transition instants,
\[ x_{tr}(t^-_{k,i}) = x_n(t^-_{k,i}), \text{ and } x_{tr}(t^-_{k,f}) = x_n(t^-_{k,f}). \] (39)

**Lemma 3:** Let Assumptions in Theorem 2 be satisfied, and the future boundary value of the unstable internal dynamics be set to zero, i.e., \( \hat{\eta}_u(t_c + T_p) = 0 \), then for any given precision in the preview-based inverse input \( \varepsilon_i \), there exists a finite preview time \( T_p < \infty \) such that the required input precision (14) can be achieved. Particularly, the required preview time is given by
\[ T_p \geq \frac{1}{\alpha} \ln \left( \frac{K}{\alpha \varepsilon_i} \right), \] (39)
where \( \alpha > 0 \) is the bound of the exponentially-decaying rate of the unstable internal dynamics,
\[ \|e^{-A_u}\| \leq K e^{-\alpha}, \quad \forall t \geq 0; \] (40)
and the positive constant \( K \) is defined as
\[ K \equiv \sum_{k=1}^N \|B_u\| \|\mathbf{V}_d(\tau)\|, \] (41)
with \( \|A\| \triangleq \sup_{\tau \in \mathbb{R}} \|A(\tau)\| \).

D. Optimal Estimation of the Unknown Boundary Value of the Unstable Internal Dynamics

In applications such as robot manipulation and nanomanufacturing [9], reducing the amount of needed preview time is crucial due to the stringent amount of preview time. The needed preview time can be reduced through the minimization of the following cost function with respect to the boundary value,
\[ \min_{\hat{\eta}_u(t_c + T_p)} \int_{t_c}^{t_c+T_p} \|\xi(\tau) - \xi_d(\tau)\|^2 + \|u_{\text{pre}}(\tau)\|^2 d\tau \] (41)
where, \( \|A\| \triangleq \hat{\eta}_u(t_c + T_p), \) and \( \|M\| \triangleq M^T W M \).

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where, \( \|A\| \triangleq \hat{\eta}_u(t_c + T_p), \) and \( \|M\| \triangleq M^T W M \).
B. Output Tracking Form and Inverse Input

The piezo dynamics model in (43) has a relative degree of two. The minimal state-space representation of the piezoelectric dynamics model (43) was obtained by using a balance realization, and the inversion based input was obtained as,

\[ u_{inv}(t) = [20.3906 \quad -0.2032 \quad 0.0342] \dot{y}_d(t) + [-0.5609 \quad 4.5764] \eta_1(t) + [10.9054 \quad -20.0614] \eta_2(t), \]

where

\[ \dot{y}_d(t) = [\dot{y}_d(t) \quad \ddot{y}_d(t) \quad \dddot{y}_d(t)]^T \]

The above inverse control input and the corresponding internal dynamics were used in output tracking during both the tracking and the transition sections. The key was to (1) obtain the optimal output trajectory for the transition sections (see Sec. III-B), and (2) determine the preview-based internal dynamic states \( \eta_1 \) and \( \eta_2 \) in the inverse input (44) across the switching instants (see Sec. III-B).

C. Implementation of the POOTT Technique

Desired Transition Trajectory Design For the simplicity of implementation (with no loss of similarity to experimental scenario), we designed, in the simulation, the optimal output transition trajectory for all transition sections at once, but only used the desired trajectory within the given preview time window for control. Six tracking sections of three different types of trajectories (ramp, sinusoidal, and exponential signals) were incorporated into the tracking sections of the desired trajectory (see Fig. 1 (a)). Then the desired trajectory for the transition sections was designed according to Eqs. (21, 23, 24). The obtained optimal output transition trajectory is shown in Fig. 1 (a) for \( Q = \text{diag}([1.001]) \), and \( R = [1] \), and Fig. 1 (b) for a different set of \( Q \) and \( R \) at \( Q = \text{diag}([.001 \quad .001]) \) and \( R = [1] \).

The simulation results clearly demonstrate that the proposed method guaranteed the smoothness of the desired output trajectory across tracking-transition switching instants (see the insert in Fig. 1 (a)). Comparing Fig. 1 (a) with Fig. 1 (b), it is evident that different stabilization of the output subdynamics results in different desired output trajectory for transition sections. The desired trajectory in Fig. 1 (a) was used in the simulation.

Optimal preview-based inverse input The stable part of the internal dynamics was solved first (see Sec. III-C.2), where the desired output \( \dot{y}_d(t) \) in (45) was specified according to Lemma 1 (see (26–29)), and the initial boundary value \( \eta_1^*(t_0) \) was set to zero (as initially both the system and the output were at zero). Then, the preview-based solution to the unstable internal dynamics was solved for a given finite-preview time (see (33)), where the unknown future boundary value of the unstable internal dynamics (at the end of the preview time window) was optimally estimated (see Sec. 3.3.2). To ensure the tracking precision, preview time of \( T_p = 0.8 \) ms (estimated by using the settling time of the unstable internal dynamics) was used.

For comparison, the input shaping technique was also applied to design and track the desired output trajectory during the transition sections. Readers are referred to [10] for the details of the input shaping control design. The obtained reference command signal and the modified command signal convolved with the input shaper is shown in Fig. 2 (a), and the desired trajectory of transition sections obtained by using the reference command signal and that by using the modified command signal are compared in Fig. 2 (b). A PI controller was designed for the tracking sections by first, using a notch filter to “cancel” the dominant resonant mode at 24.6 KHz and 31.3 KHz. In the simulation, the control input was switched between the input-shaping feedforward control (for the transition sections) and the PI-notch filter feedback control (for the tracking sections).

D. Simulation results and discussion

Output tracking at two different speeds were evaluated in the simulation. (The higher speed one was obtained by speeded up the desired trajectory shown in Fig. 1 (a).) The output tracking results for the low-speed tracking are compared in Fig. 3 for the proposed approach (with a preview time of 0.8 ms) in (a), (c), and the input shaping-PI control in (b), (d). The high speed tracking results are shown in Fig. 4 where tracking with the input-shaping PI-control was not conducted.

The results show that by using the proposed inversion-based POOTT technique, precision output tracking can be maintained throughout the entire tracking-transition course. At low speed, relatively good tracking can be achieved by using the input-shaping-PI control except at the switching instants. Such large oscillations at switching instants, caused by the mismatch of the system state, were dramatically reduced by using the proposed technique (The relative RMS tracking error \( E_2(\%) \) was over 70 times smaller than that of using the input shaping-PI control, see Fig. 3). Such a precision tracking across all tracking-transition switching was maintained even when the trajectory became much

\[
\begin{align*}
Q & = \text{diag}([10.001]), \ R = [1] \\
Q & = \text{diag}([0.01 \quad 0.01]), \ R = [1]
\end{align*}
\]
(a), whereas the output tracking obtained by using the zero-boundary POOTT technique was completed lost (see Fig. 5 (a)). Therefore, the proposed POOTT technique extends the preview-based stable-inversion approach to nonperiodic output tracking-transition switching.

Effect of Preview-Time on Output Tracking

Two different preview times ($T_p = 0.08\text{ms}$ and $0.8\text{ms}$) were used in the simulation. Additionally, we also compared the output tracking by using the optimal estimation of the boundary value in the proposed method (called the optimal boundary POOTT technique) with that by using zero boundary value (see (34)) in the proposed method (called the zero boundary POOTT technique). The tracking results obtained by using these two methods are compared in Fig. 5 for the preview time $T_p = 0.08\text{ms}$ and $0.8\text{ms}$, along with that obtained with zero preview time.

The simulation results show that having an enough amount of preview time was critical to the tracking of nonperiodic output tracking-transition switching with preview. For both the zero-boundary and the optimal boundary POOTT methods, the tracking error decreased as the preview time increased (see Fig. 5). Particularly, with preview time of $0.8\text{ ms}$, output tracking error as small as $< 0.4\%$ was achieved. Thus, with enough preview time, precision output tracking over nonperiodic tracking-transition switching can be achieved by using the proposed POOTT technique.

The simulation results also show that the amount of preview time for achieving precision output tracking can be substantially reduced by using the optimal boundary instead of the zero boundary POOTT technique. When the preview time was as small as $T_p = 0.08\text{ms}$, relatively good output tracking over the entire tracking-transition course can still be obtained by using the optimal-boundary POOTT method (see Fig. 5).

V. CONCLUSION

An inversion-based approach to achieve precision tracking in nonperiodic tracking-transition switching with preview was proposed. The optimal desired output trajectory for the transition sections was designed by directly minimizing the output energy, and the required control input was obtained by using a preview-based stable-inversion approach. The proposed approach maintained the smoothness of system state across the tracking-transition switching, and the required preview time was quantified in terms of the system internal dynamics. The preview time was further minimized by incorporating with the optimal preview-based inversion approach. A nanomanipulation simulation example was presented.

REFERENCES