Observer Design for Lipschitz Nonlinear Systems Using Riccati Equations

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Abstract—This paper presents a new observer design technique for Lipschitz nonlinear systems. Necessary and sufficient conditions for existence of a stable observer gain are developed using a S-Procedure Lemma. The developed condition is expressed in terms of the existence of a solution to an Algebraic Riccati Equation in one variable. Thus, the need to solve Linear Matrix Inequalities in multiple variables is eliminated. The advantage of the developed approach is that it is significantly less conservative than other previously published results for Lipschitz systems. It yields a stable observer for larger Lipschitz constants than other techniques previously published in literature.

I. INTRODUCTION

For Linear Time Invariant (LTI) systems, the observer design problem is dual of the control system design problem. However, this is not the case for nonlinear systems. While the introduction of geometric techniques has led to great success in the development of controllers for nonlinear systems (Isidori, 1998, Khalil, 2001), it has not been possible to obtain results of wide applicability for state estimation. Early papers by Krener and Respondek (1985), Krener and Isidori (1983) and Xiao-Hua and Gao (1985) attempted to find a coordinate transformation so that the state estimation error dynamics were linear in the new coordinates. Necessary and sufficient conditions for the existence of such a coordinate transformation have been established but in practice are extremely difficult to satisfy.

There has been work by authors to propose observers for more specialized classes of nonlinear systems. These include results on observers for Lipschitz nonlinear systems (Thau, 1973, Kou, et al., 1975, Raghavan and Hedrick, 1992, Rajamani, 1998), results on sliding mode observers (Misawa, et al., 1989), results for sector nonlinear systems (Arcak and Kokotovic, 2001), and results on observer based control for a fully linearizable nonlinear system (Esfandiari and Khalil (1992)). The extended Kalman Filter also continues to be used frequently as a deterministic observer for nonlinear systems (Reif and Unbehauen, 1996).

This paper focuses on some new observer design results for the class of Lipschitz nonlinear systems. A major limitation of the existing results for Lipschitz nonlinear systems is that they work only for adequately small values of the Lipschitz constant. When the Lipschitz constant is large or when the equivalent Lipschitz constant has to be chosen large due to the non-Lipschitz nature of the nonlinearity, most existing observer design results fail to provide a solution. This paper develops a solution methodology that works for significantly larger Lipschitz constants compared to exiting results. Furthermore, the methodology requires only an algebraic Riccati equation in one variable to be solved and does not require solving a multi-variable LMI problem. Thus the design procedure is easier to implement.

II. PROBLEM STATEMENT

This paper presents an efficient methodology for designing observers for the class of nonlinear systems described by

$$
\dot{x} = Ax + Bu + \Phi(x,u) \\
y = Cx
$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^p$ is the input vector, and $y \in \mathbb{R}^m$ is the output measurement vector. $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n \times p}$, and $C \in \mathbb{R}^{m \times n}$ are appropriate matrices. The functions $\Phi(x,u)$ is a Lipschitz nonlinearity with a Lipschitz constant $\gamma$ i.e.

$$
\|\Phi(x,u) - \Phi(\hat{x},u)\| \leq \gamma \|x - \hat{x}\|, \forall x, \hat{x}
$$

(2)

To begin with, note that any nonlinear system of the form

$$
\dot{x} = f(x,u)
$$

(3)

can be expressed in the form of equation (1), as long as $f(x,u)$ is differentiable with respect to $x$. Further, many nonlinearities can be assumed to be Lipschitz, at least locally. For instance, the sinusoidal terms usually encountered in many problems in robotics are all globally Lipschitz. Even terms like $x^2$ can be regarded as Lipschitz provided we know that the operating range of $x$ is bounded. Thus, the class of systems being considered in this paper is fairly general, with the linearity assumption being made only on the output vector $y$.

The observer will be assumed to be of the form

$$
\dot{\hat{x}} = A\hat{x} + Bu + \Phi(\hat{x},u) + L(y - C\hat{x})
$$

(4)

The estimation error dynamics are then seen to be given by

$$
\dot{\hat{x}} = (A - LC)\hat{x} + \Phi(x,u) - \Phi(\hat{x},u)
$$

(5)

where $\hat{x} = x - \hat{x}\).

The Lyapunov function candidate for observer design is defined as

$$
V = \hat{x}^T P\hat{x}
$$

(6)

where $P > 0$ and $P \in \mathbb{R}^{n \times n}$. Its derivative is
\[
\dot{V} = \dot{x}^T \left( (A - LC)^T P + P(A - LC) \right) \dot{x} + \dot{x}^T P \left[ \Phi(x) - \Phi(\hat{x}) \right] + \left[ \Phi(x) - \Phi(\hat{x}) \right]^T P \dot{x}.
\]

This can be rewritten in matrix form as
\[
\dot{V} = \dot{x}^T \left[ \Phi(x) - \Phi(\hat{x}) \right]^T \left[ (A - LC)^T P + P(A - LC) \right] \dot{x} + \dot{x}^T P \left[ \Phi(x) - \Phi(\hat{x}) \right] + \left[ \Phi(x) - \Phi(\hat{x}) \right]^T P \dot{x}.
\]

(7)

IV. NONLINEAR OBSERVER

4.1 Observer for Lipschitz Nonlinear Systems

Theorem 1: For the class of systems and observer forms described in equations (1)-(2) and (4), if an observer gain matrix \( L \) can be chosen such that
\[
\left[ (A - LC)^T P + P(A - LC) + \epsilon^2 I \right] / P < 0
\]

for some positive definite symmetric matrix \( P \), then this choice of \( L \) leads to asymptotically stable estimates by the observer (4) for the system (1). Likewise, if there exists any symmetric positive definite matrix \( P \) such that the derivative of the Lyapunov function \( V \) in equation (8) is negative definite, then there also exists a matrix \( L \) such that equation (13) is satisfied. In this sense, equation (13) is both a necessary and sufficient condition for observer stability.

Proof: Suppose there exist matrices \( L \) and \( P \) which satisfy equation (13). Let this choice of \( L \) be used in the observer (4) for state estimation of the system given by (1).

Consider the traditional Lipschitz nonlinear system which satisfies
\[
\| \Phi(x,u) - \Phi(\hat{x},\hat{u}) \| \leq \gamma \| x - \hat{x} \|, \forall x, \hat{x}.
\]

In this case, we find
\[
\left[ \Phi(x,u) - \Phi(\hat{x},\hat{u}) \right] ^T \left[ \Phi(x,u) - \Phi(\hat{x},\hat{u}) \right] \leq \gamma^2 \left[ x - \hat{x} \right] ^T \left[ x - \hat{x} \right]
\]

Hence
\[
\left[ \Phi(x,u) - \Phi(\hat{x},\hat{u}) \right] ^T \left[ \Phi(x,u) - \Phi(\hat{x},\hat{u}) \right] \leq \gamma^2 \left[ x - \hat{x} \right] ^T \left[ x - \hat{x} \right]
\]

(15)

Thus the nonlinear function \( \Phi(x,u) \) satisfies an inequality of the type
\[
\left[ (x - \hat{x})^T \left( \Phi(x,u) - \Phi(\hat{x},\hat{u}) \right) \right] ^T M \left[ (x - \hat{x}) - \Phi(x,u) - \Phi(\hat{x},\hat{u}) \right] \leq 0
\]

(17)

where \( M \in \mathbb{R}^{2n \times 2n} \) is a symmetric matrix and is given by
\[
M = \left[ -\gamma^2 I \quad 0 \right] / 0 \quad I.
\]

(18)

Applying the S-Procedure Lemma to equations (8) and (17), we find \( V < 0 \) if and only if there exist \( \epsilon \geq 0 \) such that
\[
\left[ (A - LC)^T P + P(A - LC) + \epsilon^2 I \right] / P < 0.
\]

(19)

Hence the necessary and sufficient condition for observer design in this case is given by \( \exists \epsilon \geq 0 \) such that
\[
\left[ (A - LC)^T P + P(A - LC) + \epsilon^2 I \right] / P < 0.
\]

(20)

Less Conservative Lipschitz Condition

It is possible to write equation (14) in the matrix form as defined by
\[
\| \Phi(x,u) - \Phi(\hat{x},\hat{u}) \| \leq \mathbb{G}(x - \hat{x}), \forall x, \hat{x}
\]

(21)

Note that the matrix \( G \) in this case could be a sparsely populated matrix. Hence \( \| x - \hat{x} \| \) can be much smaller than the constant \( \gamma \| x - \hat{x} \| \) used earlier in equation (2) for the same nonlinear function.

Illustrative Example for Less Conservative Lipschitz Condition

Let \( \Phi(x,u) \) given by
\[
\Phi(x,u) = \begin{bmatrix} 2 \sin(\chi) \\ 0 \end{bmatrix}.
\]

Then, apply the traditional Lipschitz condition for equation (22).

\[
\begin{align*}
\| 2 \sin(\chi) - 2 \sin(\hat{\chi}) \| & \leq 2 \| x - \hat{x} \|, \forall x, \hat{x} \\
\sqrt{\| 2 \sin(\chi) - 2 \sin(\hat{\chi}) \|^2} & \leq 2 \sqrt{\| x - \hat{x} \|^2 + \| (x - \hat{x}) \|} \\
2 \sqrt{\| 2 \sin(\chi) - 2 \sin(\hat{\chi}) \|^2} & \leq 2 \sqrt{\| x - \hat{x} \|^2 + \| x - \hat{x} \|^2} \\
\end{align*}
\]

(23)

(24)

Next, apply the Less Conservative Lipschitz Condition for equation (22).

\[
\begin{align*}
\| 2 \sin(\chi) - 2 \sin(\hat{\chi}) \| & \leq \left[ \begin{array}{c} 2 \quad 0 \\ 0 \quad 0 \end{array} \right] \begin{bmatrix} x - \hat{x} \\ 0 \end{bmatrix}, \forall x, \hat{x} \\
\sqrt{\| 2 \sin(\chi) - 2 \sin(\hat{\chi}) \|^2} & \leq 2 \sqrt{\| x - \hat{x} \|^2, (x - \hat{x}) \|^2} \\
2 \sqrt{\| 2 \sin(\chi) - 2 \sin(\hat{\chi}) \|^2} & \leq 2 \sqrt{\| x - \hat{x} \|^2 + \| x - \hat{x} \|^2} \\
\end{align*}
\]

(25)

(26)

(27)

Thus the bound (21) is less conservative than the bound (2).
This result brings us to the corollary to theorem 1.

**Corollary to Theorem 1:** For the class of systems and observer forms described in equations (1), (4) and (21), if an observer gain matrix $L$ can be chosen such that

$$\begin{bmatrix} (A-LC)G + P(A-LC) + \varepsilon G^T G & \frac{1}{\varepsilon} PP - \beta^2 C^T C \end{bmatrix} < 0 \tag{28}$$

for some positive definite symmetric matrix $P$, then this choice of $L$ leads to asymptotically stable estimates by the observer (4) for the system (1).

4.2 Reformulation of Observer Design Using Riccati Equations

Equations (13) and (28) are LMIs involving two unknown matrices $L$ and $P$. The LMIs can be replaced by a Riccati inequality in just one variable $P$. A necessary and sufficient condition in terms of the existence of a solution to a Riccati inequality can then be obtained. The following theorem summarizes the result.

**Theorem 2:** There exists an observer of the type given by equation (4) for the system given by equations (1) and (21) such that the error dynamics are quadratically stabilized if and only if there exist $\varepsilon > 0$ and $\beta \in \mathbb{R}$ such that the following Riccati inequality has a symmetric positive definite solution $P$:

$$A^T P + PA + \varepsilon G^T G + \frac{1}{\varepsilon} PP - \beta^2 C^T C < 0 \tag{29}$$

The observer gain can then be chosen as

$$L = \frac{\beta^2}{2} P^{-1} C^T. \tag{30}$$

**Proof:** Applying the Schur Inequality (Lemma 2) to equation (28), the necessary and sufficient condition for observer design is

$$(A-LC)^T P + P(A-LC) + \varepsilon G^T G + \frac{1}{\varepsilon} PP < 0 \tag{31}$$

This can be rewritten as

$$A^T P + PA + \varepsilon G^T G + \frac{1}{\varepsilon} PP - \beta^2 C^T C - PLC < 0 \tag{32}$$

Note that equation (32) implies for all $\tilde{x}$ such that $C\tilde{x} = 0$, we must have $\tilde{x}^T (\tilde{x}^T P + PA + \varepsilon G^T G + \frac{1}{\varepsilon} PP)\tilde{x} < 0$. It also therefore follows that there must $\beta \in \mathbb{R}$ sufficiently large such that $\tilde{x}^T (\tilde{x}^T P + PA + \varepsilon G^T G + \frac{1}{\varepsilon} PP)\tilde{x} - \beta^2 \tilde{x}^T C^T C\tilde{x} < 0$ for all $\tilde{x}(t)$.

Equation (29) is therefore a necessary condition. The fact that equation (29) is a sufficient condition can be proved as follows:

Let there exist a positive definite solution $P$ to equation (29). Let the observer gain be chosen as

$$L = \frac{\beta^2}{2} P^{-1} C^T. \tag{33}$$

Then equation (29) can be rewritten as

$$A^T P + PA + \varepsilon G^T G + \frac{1}{\varepsilon} PP - \beta^2 C^T C - PLC < 0 \tag{34}$$

which is the same as equation (28), thus proving the result.

V. NOTES ON COMPUTATION OF $P$ AND $L$

5.1 Solution of LMI Inequality for $P$ and $L$

The previous section has shown the necessary and sufficient conditions for observer design. The observer design by using theorems 1-2 needs to search for $P$ and $L$ to satisfy inequality (13) or (28). Solving the inequality by a numerical method would be an obvious approach. However, the inequalities (13) and (28) seem to be nonconvex because each involves the product of the two variables $P$ and $L$. Therefore, a simple change of variables that separates $L$ from $P$ needs to be made. Expand (13) to obtain

$$\begin{bmatrix} A^T P + PA - C^T L^2 P - PLC + \varepsilon \gamma^2 I & P \\ \varepsilon I & P \end{bmatrix} < 0, P > 0 \tag{35}$$

Now let $Y = PL$. Then the inequality (13) becomes

$$\begin{bmatrix} A^T P + PA - C^T Y^T -YC + \varepsilon \gamma^2 I & P \\ \varepsilon I & P \end{bmatrix} < 0, P > 0. \tag{36}$$

Searching for $P$ and $L$ satisfying (13) is equivalent to searching for $P$ and $Y$ satisfying (36). Once a feasible set of $P$ and $Y$ is found, $L$ can be computed as $L = P^{-1} Y$. Note that $P > 0$ (or equivalently $P$ is invertible) and that there is always a one to one mapping from $Y$ to $L$ for a given $P$.

In the same way, we can apply the change of variables to inequality (28). Then the inequality (28) becomes

$$\begin{bmatrix} A^T P + PA - C^T Y^T -YC + \varepsilon G^T G & P \\ \varepsilon I & P \end{bmatrix} < 0, P > 0 \tag{37}$$

Hence, for a given $\varepsilon$ and $\gamma$ or $G$, the inequality (36), or (37), is affine with respect to its two variables $P$ and $Y$. The corresponding feasibility problem can be solved using many standard convex optimization techniques or the MATLAB LMI control toolbox.

5.2 Solution of Algebraic Riccati Equation for $P$ and $L$

The observer design by using theorem 2 involves solving for one unknown matrix $P$, by providing $\varepsilon, G$, and $\beta$. Since $P$ is the only unknown, it is easier to solve the Riccati inequality than to solve a LMI inequality.

Equation (29) can be modified as follows. Assume

$$A^T P + PA + \varepsilon G^T G + \frac{1}{\varepsilon} PP - \mu C^T C = -\mu I < 0 \tag{38}$$

where $\mu > 0$, $\mu$ is a small scalar.

$$A^T P + PA + \frac{1}{\varepsilon} PP + \varepsilon G^T G - \beta^2 C^T C + \mu I = 0 \tag{39}$$

For given $\varepsilon, G$, $\beta$, and $\mu$, the Riccati equality (39) can be easily solved. This equation may be solved with the MATLAB command “ARE”:

$$X = ARE(a,b,c) \tag{40}$$

This returns the stabilizing solution (if it exists) to the
continuous-time Riccati equation:
\[ a'X + Xa - XbX + c = 0 \]  \( \text{(41)} \)
with \( b \) being symmetric and nonnegative definite and \( c \) being symmetric.

Rearrange the equation (39) to appropriate equation (41) as follows:
\[ (-A') P + P(-A) - P(1) P + (-eG^T G + \beta C^T C - \mu I) = 0 \]  \( \text{(42)} \)
The Matlab function ‘ARE’ can now be used to find the solution.

VI. ILLUSTRATIVE EXAMPLE

6.1 Dynamic model for a flexible link robot

Consider a one link manipulator with revolute joints actuated by a DC motor. The elasticity of the joint can be well-modeled by a linear tensional spring [12]. The elastic coupling of the motor shaft to the link introduces an additional degree of freedom. The states of this system are motor position and velocity, and the link position and velocity.

The corresponding state-space model is
\[ \dot{x} = Ax + Bu + \Phi(x) \]
\[ y = Cx \]
with \( x = [\theta_m \, \omega_m \, \theta_l \, \omega_l]^T \).

\[ A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -48.6 & -1.25 & 48.6 & 0 \\ 0 & 0 & 0 & 1 \\ 19.5 & 0 & -19.5 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 21.6 \\ 0 \\ 0 \end{bmatrix} \]
\[ \Phi(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \]

The value of the Lipschitz constant for this system is \( \gamma = 3.33 \). The above parameters for the system are typical and have been taken from Spong [12].

[12] presents a nonlinear I/O linearizing control law for this system. The control law guarantees closed-loop stability and tracking of any desired trajectory by the robotic link.

However, this control law requires measurement of all the states. Physically, one can measure the motor position and velocity, but the measurement of the other states is non-trivial.

6.2 Observer design for a flexible link robot

In this section we present an observer for the above robotic system with guaranteed convergence of the state estimates.

Using Corollary to Theorem 1 on LMI based solution with \( e = 1 \), an observer gain is found to be
\[ L = \begin{bmatrix} 4.8710 & 2.7729 \\ 0.9774 & 60.2442 \\ 5.4689 & 12.8094 \\ 4.1360 & -13.1102 \end{bmatrix} \]

The eigenvalues of \( (A - LC) \) are
\[ \text{eig}(A - LC) = \begin{bmatrix} -50.7854 \\ -9.5417 \\ -5.0855 \\ -0.9525 \end{bmatrix} \]

Thus, the eigenvalues are well damped in this case. The response will show no transient oscillations. However, there is one small eigenvalue. So the estimated state will converge slowly to the actual state.

The robotic system with the above observer gain matrix was simulated under an open-loop excitation with \( u \) being a sinusoid at 1 Hz. Fig. 1 shows a comparison of actual and estimated link angular position. The actual system has an initial condition equal to 1 radian while the observer starts from an initial value of zero. Fig. 2 shows actual and observer estimated link angular velocities. Both the estimated states converge to the correct values.

![Fig. 1 Actual and observer estimated link angular position (Corollary to Theorem 1)](image-url)
Using theorem 2 on the Riccati based solution with $\varepsilon = 1$ and $\beta = 180$, an observer gain matrix is found to be
\[
L = \begin{bmatrix}
78.9814 & -9.2259 \\
-9.2259 & 112.9897 \\
10.0309 & 91.3947 \\
14.2679 & 51.5085
\end{bmatrix}. \quad (48)
\]

The eigenvalues of $(A - LC)$ are
\[
eig(A - LC) = \begin{bmatrix}
-71.6615 \\
-65.4807 + 34.5212i \\
-65.4807 - 34.5212i \\
-1.0914
\end{bmatrix}. \quad (49)
\]

Thus, the eigenvalues have lower damping in this case. However, the overall eigenvalues are large. So the estimated states rapidly converge to the actual state and transient oscillations are not present. The transient performance is better than the previous LMI solution.

6.3 Comparison with previous observer design techniques

This section will compare observer design techniques in term of how conservative they are. We will examine which observer design techniques can deal with a larger Lipschitz constant. The problem in section 6.1 is reconsidered and we will keep the same plant equations. However, the nonlinearity will be scaled to continuously increase the value of the Lipschitz constant for this system until the observer can no longer solve the problem when the Lipschitz constant is too large.

First, we find an observer gain by Theorem 1 which needs the solution of LMI Inequalities. Then, we use Theorem 2 to find an observer gain matrix by solving the corresponding Algebraic Riccati equation. Finally, the traditional standard LMI observer [16] as described below is used to find an observer gain matrix for comparison.

**Standard LMI Observer**

For the class of systems and observer forms described in equations (1) and (4), if an observer gain matrix $L$ can be chosen such that
\[
\begin{bmatrix}
(A - LC)^T P + P(A - LC) + I & P \\
-1 / \gamma^2
\end{bmatrix} < 0
\]
for some positive definite symmetric matrix $P$, then this choice of $L$ leads to asymptotically stable estimates by the observer (4) for the system (1).

<table>
<thead>
<tr>
<th>Method</th>
<th>Corollary to Theorem 1</th>
<th>Theorem 2</th>
<th>Standard LMI Observer</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma_{\text{max}}$</td>
<td>48.5</td>
<td>48.4</td>
<td>0.99</td>
</tr>
</tbody>
</table>

Table 1 shows that the observer design techniques from corollary to theorem 1, and from theorem 2 can deal with a larger Lipschitz constant. They can solve the problem with a Lipschitz constant up to 48.4. On the other hand, the standard LMI observer can deal with the problem only for values of Lipschitz constant up to 0.99. Thus, the new observer design

![Fig. 2 Actual and observer estimated link angular velocity](image1)

![Fig. 4 Actual and observer estimated link angular velocity (theorem 2)](image2)

![Fig. 3 Actual and observer estimated link angular position (theorem 2)](image3)
techniques are significantly less conservative than the traditional LMI observer design technique.

VII. CONCLUSIONS

This paper presented a new observer design technique for Lipschitz nonlinear systems. Necessary and sufficient conditions for existence of a stable observer gain were developed using a S-Procedure Lemma. The developed condition was then expressed in terms of the existence of a solution to an Algebraic Riccati Equation in one variable. Thus, the need to solve Linear Matrix Inequalities in multiple variables was eliminated. The advantage of the developed approach is that it is significantly less conservative than other previously published results for Lipschitz systems. Using an illustrative example of a flexible joint robot, the developed observer design technique was found to yield a stable observer for much larger Lipschitz constants than a traditional LMI design technique.

References