Inventory Control and LQG: Connections and Extensions

Wai Kit Ong\textsuperscript{1}, Elizabeth J. Durango-Cohen\textsuperscript{2}, and Donald J. Chmielewski\textsuperscript{1,}\textsuperscript{*}

ABSTRACT

This paper presents a review of two classical business approaches to inventory control. This is followed by a discussion of an engineering approach using Linear Quadratic Gaussian (LQG) control. The main result is to show that under certain conditions all three methods generate the same policy. Using the state-space model of the LQG formulation, we explore the multi-echelon system and model uncertainty cases.

1. INTRODUCTION

Inventory is the heart and soul of most retail companies. Profits are made when they are able to turn these inventories into sales. Companies hold inventories for several reasons: to meet customer demand, to maintain desired level of customer service, to reduce purchase cost due to economies of scale, and to act as a buffer in case there are unexpected increase in demand or reduction in supply. Expectedly, there are costs associated with holding inventories. Inventories require insurance and financing, a warehouse to store, and the cost of foregone return of alternative investments.

The objective of this paper is to find the link between the management science approach to inventory control and the engineering focused control theory. A general description of applying control methods to management science can be found in Sethi & Thompson (2000). We will focus on inventory control (see; Silver et al.; 1998; Magee; 1967, Ortega & Lin; 2004 and Jillson et al.; 2008). Application of classic control methods can be found in Perea et al. (2000); Lin et al. (2004); and Hoberg et al. (2007). MPC methods include: Bose & Pekny (2000); Perea-Lopez et al. (2003); Braun et al. (2003) and Wang & Rivera (2008).

In the next section, we provide a review of several classical approaches to inventory control as described in the management science literature. In Section 3, we propose a Linear Quadratic Gaussian (LQG) approach to inventory control. Then, connections between the two are discussed. Finally, we propose some extensions of the classic inventory control problem based on application of standard LQG and robust control methods.

\textsuperscript{1} Department of Chemical and Biological Engineering, Illinois Institute of Technology, Chicago, Illinois 60616.
\textsuperscript{2} Stuart School of Business, Illinois Institute of Technology, Chicago, Illinois 60616.
\textsuperscript{*} Corresponding author: 312-567-3537. Fax: 312-567-8874, chmielewski@iit.edu

2. REVIEW OF CLASSICAL APPROACHES TO INVENTORY CONTROL

When it comes to inventory control, there are two basic questions – order quantity and order timing. This leads to two extreme cases, the Fixed Order Quantity System, and the Fixed Interval System, see Silver et al. (1998) for a detailed discussion. In this work, we will describe only the Fixed Interval System.

Consider the simplest possible example of a single retail location selling a single product. Given a fixed amount of inventory space and a known delivery delay (also known as order lead time, L), the retailer must decide on a reorder policy that avoids lost sales, due to out-of-stock situations, but also does not overflow the stockroom. A fundamental concept of inventory control is that of inventory position. What we call actual inventory can be described as the stock-on-hand capable of meeting demand immediately. Inventory position is the sum of the actual inventory and all the orders that have yet to arrive.

2.1 (R,S) Approach with Deterministic Demand

The Fixed Interval (R,S) System is also known as the Periodic Review System. Here, R represents the review period while S represents the order-up-to level. In the context of figure 1, the S can be thought of as the inventory set-point. (This is not exactly true but it is a good place to start.) In this system, the inventory position is reviewed at fixed intervals. An order would then be placed for the difference between the order-up-to level and the inventory position. In the context of figure 1, the placement of orders is denoted as starts. As shown in figure 2, an order is placed to return the inventory position back to the order-up-to level at time \( t \). When the stocks arrive at time \( t + L \), the actual inventory increases by the order amount and the actual inventory is equal to the inventory position until the next review period at \( t + R \).

The optimal review period, \( R \) is determined as follows. Define \( a \), \( b \), and \( c \) as the setup, unit, and holding costs, respectively. If \( D \) is the expected annual demand and \( R \) is the review period, then the total annual cost is:
\[ C = a/R + cDR/2 + bD \]

The first term represents the total annual setup cost since the reciprocal of \( R \) would give the number of orders placed in a year; the second term represents the total annual holding cost since \( DR/2 \) is the average amount of inventory held; the third term represents the total cost of inventory purchased in a year. If one differentiates with respect to the review period, the optimal review period is determined as:

\[ R = \sqrt{2a/cD} \]

As indicated by figure 2, the order-up-to level should be defined such that the actual inventory is exactly depleted at time \( t+R+L \).

\[ S = (L + R)D \]

**Example 1:** Consider an \((R, S)\) system with a setup cost of $10/order, an annual holding cost of $0.3244/item, and an annual demand of 36500 items. This results in a review period of 15 days. If the lead time is 5 days, then the order-up-to level should be 2000 items.

**Example 2:** Reconsider the scenario of example 1, but assume the standard deviation of demand over one day is 10 items. A service level of 16% stock-outs is desired, then \( SS = 45 \) items and \( S = 2045 \) items.

**Example 3:** Due to advances in information technology, setup costs have dropped significantly. Reconsider the scenario in example 2, but change \( a \) to $0.0444/order. This results in \( R = 1 \) day, \( SS = 25 \) items and \( S = 625 \) items.

### 2.2 \((R, S)\) Approach with Stochastic Demand

We shall now discuss the notion of safety stock for the scenario of stochastic demand. To illustrate, consider figure 3. The figure shows that after an order is placed at a given review period, the actual inventory should be equal to the safety stock (SS) level at \( t+L \). The safety stock level is determined by considering the uncertainty in demand, represented by the shaded regions in figure 3. In many cases, the demand is assumed to be a Gaussian process, with an average, \( \bar{D} \), and a standard deviation, \( \sigma \). Thus, the shaded region of figure 3 can be interpreted as one standard deviation of inventory (or inventory position). Given a known initial condition at time \( t_0 \), the variance of inventory position (and eventually inventory) at time \( t_0 + t_k \) is calculated as \( t_0^2 \). Thus, the standard deviation of inventory at time \( t+L+R \) is given as \( \sqrt{L+R} \sigma \).

**Remark:** Strictly speaking, \( D \) should be considered a white noise process, and given the continuous-time framework of this initial discussion, should have an infinite variance. In such a case, the variability of \( D \) should be indicated by its power spectral density, \( S_D \). Unfortunately, demand statistics are rarely reported with respect to \( S_D \). They are usually reported as a variance in demand over a specified interval. For example, one may find that the demand reported on a daily basis has a standard deviation of 10 items. Since this statistic represents the sum of items sold over a day, one would model the statistic as \( dQ = d\xi \), where \( Q \) is the sum of items sold, \( Q(0) = 0 \) and \( \xi = dD \). Thus, the variability of \( Q \) is calculated as \( \Sigma_D(t) = \Sigma_D \). Thus, \( \Sigma_D = \Sigma_D/t \) and for the above example would be equal to 100 items^2/day. The proper calculation of inventory variance is nearly identical: the system is \( d\bar{Q} = d\bar{\xi} \) where \( \bar{\xi} = t \xi \) and then \( \Sigma_\xi(t) = \Sigma_D \). This gives a standard deviation at \( t_1 = R+L \) equal to \( \sqrt{L+R} \sqrt{\Sigma_D} \). It is additionally noted that the units of this formula work out correctly to be items. However, rather than abandon the \( \sqrt{L+R} \sigma \) formula, simply interpret \( \sigma \) as the variance of cumulative demand over a given interval and require \( L \) and \( R \) to have units equal to one interval.

In many cases, the use of one standard deviation will not give the desired service level. Therefore, to create additional freedom in selecting the SS level, a parameter \( \alpha \) is added to allow for the selection of desired service levels. If \( \alpha = 1 \), then the probability of stock-outs will be 16%. If a stock-out probability of 10% is desired, then \( \alpha \) should be set to 1.28 (see Silver et al. (1998) for additional discussion on service level selection). Thus, the order-up-to level for the stochastic demand case is:

\[ S = (L + R)\bar{D} + SS \]

\[ SS = \alpha \sqrt{(L+R)\sigma} \]

**Example 2:** Reconsider the scenario of example 1, but assume the standard deviation of demand over one day is 10 items. A service level of 16% stock-outs is desired, then \( SS = 45 \) items and \( S = 2045 \) items.

**Example 3:** Due to advances in information technology, setup costs have dropped significantly. Reconsider the scenario in example 2, but change \( a \) to $0.0444/order. This results in \( R = 1 \) day, \( SS = 25 \) items and \( S = 625 \) items.

### 2.3 The Warehouse Rule (WR)

In the case of \( L \) being an integer multiple of \( R \), one can define a recursive equation to represent the inventory system.

\[ I_{k+1} = I_k + q_{k-\theta} + d_k \]

where \( I_k \) is the actual inventory at the end of interval \( k \), \( d_k \) is the total demand during interval \( k \), \( \theta = LR \), \( q_{k-\theta} \) is amount ordered at the end of interval \( k-\theta \) and arriving at the end of interval \( k \). The * points in figure 5 indicate the sequence \( I_k \).

Given this recursive model, a very popular control strategy is the Warehouse Rule (Vassian, 1955).

\[ q_k = \sum_{i=1}^{\theta+1} \hat{d}_{k+i} - \sum_{i=1}^{\theta} q_{k-i} - (I_k - \bar{I}) \]

where \( \hat{d}_{k+i} \) is the forecasted demand for interval \( k \) and \( \bar{I} \) is the target (or set-point) value for actual inventory at the end of each interval (The correct interpretation of figure 1).
position, the (R,S) approach. Example 3 based on (6) and (7) is given in figure 6.

3. LQG AND INVENTORY CONTROL

Fundamental to the development of an LQG controller is a state-space version of the process.

\[ x_{k+1} = Ax_k + Bu_k + Gw_k \]  \hspace{1cm} (10)

\[ z_k = Dx_k + Du_k \]  \hspace{1cm} (11)

**Example 4:** Convert (6) to state-space form, assuming \( \theta = 3 \). Define the state, control, disturbance, and performance vectors \( (x_k, u_k, w_k, z_k) \) respectively, as follows:

\[ x_k = [\tilde{I}_k \tilde{d}_k (3) \tilde{q}_k (2) \tilde{q}_k (1) \]  \hspace{1cm} (12)

\[ w_k = [\tilde{d}_k], z_k = [\tilde{I}_k \tilde{q}_k (0)] \]  \hspace{1cm} (13)

The superscript index on \( q \) shows the number of days the order has been placed. The tilde \((-\) sign indicates that they are deviation variables. Thus, \( \tilde{I}_k = I_k - \bar{I} \), where \( \bar{I} \) is the set-point of \( I_k \), which will become the average of \( I_k \). The other deviation variables have a similar definition, but all will be with respect to \( \bar{d} \) the average of \( d_k \). The performance vector is defined for the user’s convenience and we select inventory and starts. Thus,

\[ A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, G = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \]  \hspace{1cm} (14)

\[ D_x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, D_u = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \]  \hspace{1cm} (15)

The constrained minimum variance (CMV) control problem, defined in Chmielewski & Manthanwar (2004) is:

\[ \min_{\sum, x, L, \xi} \left\{ \sum_{i=1}^{p} g_i \xi_i \right\} \]  \hspace{1cm} (16)

\[ (A + BK)\Sigma_x (A + BK)^T + G\Sigma_w G^T = \Sigma_x \]  \hspace{1cm} (17)

\[ \phi_i (D_x + D_u K)\Sigma_x (D_x + D_u K)^T \phi_i^T = \xi_i \]  \hspace{1cm} (18)

\[ \xi_i < \bar{z}_i^2, i = 1, ..., p \]  \hspace{1cm} (19)

where \( K \) is the feedback matrix \( (u_k = Kx_k) \), \( \Sigma_w \) is the covariance of \( x \), \( \phi_i \) is the \( i \)th row of an identity matrix, \( \xi_i \) is the variance of \( z_0 \), and \( \bar{z}_i \) is a bound on standard deviation.

The CMV problem also has the following convex form.

\[ \min_{\sum, x, L, \xi} \left\{ \sum_{i=1}^{p} g_i \xi_i \right\} \]  \hspace{1cm} (20)

\[ X - G\Sigma_w G^T \begin{bmatrix} AX + BY \\ (AX + BY)^T \end{bmatrix} > 0 \]  \hspace{1cm} (21)

\[ \begin{bmatrix} \xi_i \phi_i (D_x X + D_u Y) \\ (D_x X + D_u Y)^T \phi_i X \end{bmatrix} > 0 \]  \hspace{1cm} (22)
The optimal controller is then defined as $K^* = Y^*X^{-1}$ where $Y^*$ and $X^*$ are the solution matrices of the LMI-constrained minimization problem. In Chmielewski and Manthanwar (2004), it is shown that all controllers generated by the CMV problem are equal to some LQG controller.

Example 5: Solve the CMV problem for the scenario of example 3. In figure 7, point A is when the weights are $g = [1 0]$, point B is $g = [1 5]$, and point C is $g = [1 100]$. The curve in figure 7 is a Pareto frontier, any point above the curve is possible and any point on the curve is generated by a CMV controller.

Figure 7: Standard deviation of inventory versus starts

Figure 8: Operating regions with safety stock back-off

Figure 9: Two-echelon system decentralized controllers (top) and a centralized controller (bottom)

Example 6: Consider a 2-echelon system with parameters similar to the previous examples. The delivery delay for each is 3 and 5 days, and the starts of the second tank are equal to the demand for the first tank. Since there are two units, the system can have a centralized controller or a decentralized controller (see figure 9). In the decentralized case assume the (R,S) approach is applied to both units separately. For the centralized case, a CMV problem is solved for the aggregate system. Centralized Case A has weights $g = [1 0 1 0]$, and Centralized Case B is similar but with more weight placed on the inventory of the first unit (or tank). The point to notice is that a small inventory increase at tank 2 (the retailer) can result in large inventory reductions at tank 1 (the warehouse).

Figure 10: Inventory standard deviations for 2-echelon system

\[ u_k = q_k - \bar{d} = -\bar{T}_k - \sum_{i=1}^{\theta} \bar{q}_k^{(i)} = Kx_k \]  

where $K = [-1 -1 -1 -1 -1 -1]$, if $\theta = 5$. This happens to be the exact controller found for case A in example 5. Thus, the WR is recovered using the LQG approach with $g = [1 0]$.  

4. CONNECTIONS AND EXTENSIONS

We now show that LQG can generate the WR. If $\sum_{i=1}^{\theta+1} d_i = (\theta + 1)\bar{d}$, then equation 7 can be converted to
4.2 Extension to Robust Formulation

Next, we consider the impact of uncertain parameters. Two cases will be explored, uncertain yield and delivery delay.

In previous examples it was assumed that 100% of the items ordered would arrive. However, for some systems (for example production systems), such an assumption is likely poor, and the notion of yield is used. If the yield is 90%, then an order of 100 items will result in only 90 items being produced. As expected, the yield of each order is uncertain.

Example 7: Consider the scenario of example 5 but with uncertainty in the yield. The addition of this uncertain parameter to the CMV formulation is achieved by simply replacing equation 21 with the following:

\[ A(\beta) = \begin{bmatrix} 1 & \beta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, A_1 = A(0.8), A_2 = A(1.0) \]  

(25)

\[ X - G\Sigma_w G^T (A_j X + BY) > 0, \ j = 1, 2 \]  

(26)

Three controllers were designed. The first assumed yield of 100%, the second 80% and the third assumed yield could be anywhere between 80 and 100% (the robust controller). These three controllers were then analyzed at all possible values of the yield (see figure 13). As expected, the 80% controller results in the lowest inventory standard deviation when the yield is in fact at 80%. However, when the yield turns out to be 100%, the controller becomes ineffective, resulting in a high standard deviation of inventory. A similar situation occurs for the 100% controller. The robust controller, however, provides the best overall result since the standard deviation of inventory remains low over the entire range. It is also worth noting that although there is a clear indication of which controller may be better at a certain condition, the difference in standard deviation is only about 0.5 units.
5. CONCLUSION

A review of classic inventory control was presented. Two methods were discussed (the fixed interval (R,S) approach and the warehouse rule) and conditions were given for the two to generate the same policy. Then an LQG approach to the problem was presented, and it was conclude that the classic inventory control methods are actually special cases of the LQG approach. Finally, we illustrated that the LQG approach extends quite naturally to the multi-echelon and uncertain parameter scenarios.

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7. REFERENCES


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