Mode in Output Participation Factors for Linear Systems

Li Sheng, Eyad H. Abed, Munther A. Hassouneh, Huizhong Yang and Mohamed S. Saad

Abstract—The common definition of modal participation factors was intended to provide a measure of participation of modes in states and of states in modes for linear time-invariant (LTI) systems. In this paper, recent work by the authors that revisited this common definition is extended to yield definitions and calculations for quantifying the relative contribution of system modes in system outputs. When the system outputs are simply the system states, the mode-in-output participation factors are found to reduce to the original mode-in-state participation factors. A numerical example is given to illustrate the issues addressed and the results obtained.

I. INTRODUCTION

The concept of modal participation factors is an important component of the Selective Modal Analysis (SMA) framework introduced in [7] and [9]. Participation factors provide a mechanism for assessing the level of interaction between system modes and system state variables for linear systems. They have been used for stability analysis, sensor and actuator placement, order reduction, and coherency and clustering [7], [9], [8], [2], [3].

In their study of modal participation factors, the authors of [7] and [9] selected particular initial conditions and introduced definitions of mode in state and state in mode participation factors using those initial conditions. Based on their selected initial conditions, the authors of [7] and [9] gave a single formula for both mode in state and state in mode participation factors. Until recently, it has been routinely taken for granted that the measure of participation of mode in state is identical to that of participation of states in modes. However, the recent work [4] presented a fundamental approach to modal participation analysis by using averaging over an uncertain set of system initial conditions.

Based on this new approach, it has been shown in [4] that participation of modes in states and participation of states in modes should not be viewed as interchangeable and a new formula for measuring the participation of states in modes was proposed. Furthermore, the formula for participation factors measuring participation of modes in states based on the averaging approach taken in [4] was found to coincide with the commonly used formula originally introduced in [7] and [9].

Previous work on participation factors [7], [9], [1] and [4] has focused mainly on participations associated with the state variables of linear time-invariant systems. In many practical situations, not all system states are available for measurement and only system outputs can be measured. The goal of this paper is to develop a notion of participation factors measuring the participation of system modes in system outputs. Such a notion can be useful in applications such as system monitoring, actuator placement, clustering studies, detection of closeness to instability and local and coordinated control design.

Other related concepts have been considered in the literature. For example, the concept of “mode observability factor” and transfer function residues introduced by [5], were used to determine the most suitable location for installing power system stabilizers. In [6], a concept of “extended eigenvector” was introduced and used to identify the dominant output variable associated with a critical mode. However, a drawback of using these definitions is that they do not reduce to the conventional state participation factors when the output is simply a state variable.

Motivated by the discussion above, we focus on defining a notion of output participation factors for continuous-time LTI systems

\[ \dot{x}(t) = Ax(t), \]

\[ y(t) = Cx(t), \]

where \( x \in \mathbb{R}^n \) is a vector of states, \( y \in \mathbb{R}^m \) is the vector of outputs, \( A \) is a real \( n \times n \) matrix and \( C \) is an \( m \times n \) matrix.

This paper continues our previous work [4], and the approach taken here follows that in our previous work [1], [4]. In this approach, participation factors are formulated as the average relative contribution of modes in outputs over an uncertain set of system initial conditions. The new definition proposed here has the desirable property that it reduces to the original state participation factors definition when the outputs are simply the system states.

The paper proceeds as follows. In Section II, the original definition of participation factors [7], [9] and the new definitions of participation factors [1], [4] are recalled. In Section III, the definition and calculation of participation of modes in outputs are given. In Section IV, a mechanical system example is provided to illustrate the effectiveness of
the developed theory. Conclusions and suggestions for future work are collected in Section V.

II. BACKGROUND AND MOTIVATION

In this section, the definition of participation factors for linear systems are recalled from [7], [9], [1] and [4]. Consider the linear time-invariant continuous-time system
\[ \dot{x} = Ax(t), \]
where \( x \in \mathbb{R}^n \), and \( A \) is a real \( n \times n \) matrix. In the previous studies [1], [4], [7], [9] and in this paper, the blanket assumption is made that \( A \) has distinct eigenvalues \( (\lambda_1, \lambda_2, \ldots, \lambda_n) \). Let \((r_1, r_2, \ldots, r^n)\) be right (column) eigenvectors of the matrix \( A \) associated with the eigenvalues \((\lambda_1, \lambda_2, \ldots, \lambda_n)\), respectively. Let \((l^1, l^2, \ldots, l^n)\) denote left (row) eigenvectors of the matrix \( A \) associated with the eigenvalues \((\lambda_1, \lambda_2, \ldots, \lambda_n)\), respectively. The right and left eigenvectors are taken to satisfy the normalization,
\[ l^i r^j = \delta_{ij}, \tag{4} \]
where \( \delta_{ij} \) is the Kronecker delta:
\[ \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}. \]

The solution of the dynamic system (3) satisfying the initial condition \( x(0) = x^0 \) is
\[ x(t) = e^{At} x^0. \tag{5} \]
Since the eigenvalues of \( A \) are assumed distinct, \( A \) is similar to a diagonal matrix. Using this, (5) can be rewritten as
\[ x(t) = \sum_{i=1}^{n} (l^i x^0) e^{\lambda_i t} r_i. \tag{6} \]
Therefore, the \( k \)-th state \( x_k(t) \) is given by
\[ x_k(t) = \sum_{i=1}^{n} (l^i x^0) e^{\lambda_i t} r^i_k. \tag{7} \]

A. Original definition of participation factors

In order to determine the relative participation of the \( i \)-th mode in the \( k \)-th state, the authors of [7] and [9] selected an initial condition \( x^0 = e^k \), the unit vetor along the \( k \)-th coordinate axis. With this choice of \( x^0 \), the evolution of the \( k \)-th state becomes
\[ x_k(t) = \sum_{i=1}^{n} (l^i x^0) e^{\lambda_i t} r^i_k = \sum_{i=1}^{n} p_{ki} e^{\lambda_i t}. \tag{8} \]
The quantities
\[ p_{ki} := l^i_k r^i_k \] (9)
are taken in [7] and [9] as measures of relative participation of the \( i \)-th mode in the \( k \)-th state; \( p_{ki} \) is defined in [7] and [9] as the participation factor for the \( i \)-th mode in the \( k \)-th state.

On the other hand, the relative participation of the \( k \)-th state in the \( i \)-th mode is investigated in [7] and [9] by first applying the similarity transformation
\[ z := V^{-1} x, \tag{10} \]
to system (3), where \( V \) is the matrix of right eigenvectors of \( A \):
\[ V = [r^1 r^2 \cdots r^n] \tag{11} \]
and \( V^{-1} \) is the matrix of left eigenvectors of \( A \):
\[ V^{-1} = \begin{bmatrix} l^1 \\ l^2 \\ \vdots \\ l^n \end{bmatrix}. \tag{12} \]
Then \( z \) follows the dynamics
\[ \dot{z}(t) = V^{-1} AV z(t) = \Lambda z(t), \tag{13} \]
where \( \Lambda := \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \), with initial condition \( z^0 := V^{-1} x^0 \). Hence, the evolution of the new state vector components \( z_i, i = 1, 2, \ldots, n \) is given by
\[ z_i(t) = e^{\lambda_i t} = l^i x^0 e^{\lambda_i t}. \tag{14} \]
Obviously, \( z_i(t) \) represents the evolution of the \( i \)-th mode.

To determine the relative participation of the \( k \)-th state in the \( i \)-th mode, the authors of [7] and [9] selected an initial condition \( x^0 = r^i \), the right eigenvector associated with \( \lambda_i \). With this choice of initial condition, the evolution of the \( i \)-th mode becomes
\[ z_i(t) = l^i r^i e^{\lambda_i t} = \sum_{k=1}^{n} l^i_k r^i_k e^{\lambda_i t} = \sum_{k=1}^{n} p_{ki} e^{\lambda_i t}. \tag{15} \]
Based on (15), the authors of [7] and [9] proposed the formula
\[ p_{ki} = l^i_k r^i_k \] (16)
as a measure of the relative participation of the \( k \)-th state in the \( i \)-th mode. Note that this is the same formula as (9) which was proposed as a measure of participation of modes in states.

B. Recently proposed new definition of participation factors

The authors of [1] and [4] presented a new approach to defining modal participation factors for the linear system (3). This approach involves taking an average or a probabilistic expectation of a quantitative measure of relative modal participation. The average is performed with respect to all possible values of the initial state vector and is assumed to be unknown but to lie in a known set or satisfy a known probability distribution.

Since (9) and (16) provide identical formulas for participation of modes in states and participation of states in modes, respectively, the same notation \( p_{ki} \) was used for both types of participation factors and the participation of modes in states and the participation of states in modes have been viewed as interchangeable. However, the recent work of [4] demonstrated through simple examples that the participation of modes in states and the participation of states in modes should not in fact be viewed as interchangeable and the original state-in-mode participation factors formula is inadequate. Moreover, the authors of [4] proposed a new definition and
calculation to replace the existing state-in-mode participation factors formula, while the previously existing participation factors formula should be retained but viewed only in the sense of mode-in-state participation factors.

Next, we recall from the previous work in [1] and [4] the new definitions of mode-in-state and state-in-mode participation factors. Denote by “E” the expectation operator. The general definition for the participation factor $p_{ki}$ measuring participation of mode $i$ in state $x_k$ is

$$p_{ki} := E \left\{ \frac{(l^T x_0)^r_k}{x_k^r} \right\},$$  \quad (17)

where the expectation is evaluated using some assumed joint probability density function $f(x^{0})$ for the initial condition uncertainty. Under the assumption that the initial condition components $x_1^0, x_2^0, \ldots , x_n^0$ are independent with zero mean, definition (17) is shown to reduce to [1], [4]

$$p_{ki} = l_k^r r_k^i$$  \quad (18)

which is the same expression originally introduced by [7], [9] for the participation of the $i$-th mode in the $k$-th state.

For a complex eigenvalue $\lambda_i$, the mode can be viewed as consisting of the combined contributions from $\lambda_i$ and its complex conjugate eigenvalue $\lambda_i^*$ [4]. In this case, an alternative expression for the participation factor of a complex mode associated with $\lambda_i$ in state $x_k$ is given by [4]

$$p_{ki} = 2 \text{Re}\{l_k^r r_k^i\}.$$  \quad (19)

The participation factor measuring contribution of the $k$-th state in the $i$-th mode is defined in [4] as

$$\pi_{ki} := \begin{cases} E \left\{ \frac{z_i^0}{z_i^0} \right\}, & \text{if } \lambda_i \text{ is real} \\ E \left\{ \frac{z_i^0}{z_i^0 + z_i^0} \right\}, & \text{if } \lambda_i \text{ is complex} \end{cases}$$  \quad (20)

whenever this expectation exists. Note that in (20), the notation $z_i^0$ means $z_i(t = 0) = l^T x^0$ and the asterisk denotes complex conjugation. In order to obtain a simple formula from definition (20), the authors of [4] assumed that the probability density function $f(x^{0})$ is such that the components $x_1^0, x_2^0, \ldots , x_n^0$ are jointly uniformly distributed over the unit sphere in $\mathbb{R}^n$ centered at the origin. Under this assumption, a new formula (21), was based on definition (20) for the participation factor for the $k$-th state in the $i$-th mode

$$\pi_{ki} = l_k^r r_k^i + \sum_{j=1, j\neq i}^{n} l_k^r r_k^j \frac{l^T(l^T x_0)^T}{l^T(l^T x_0)^T}.$$  \quad (21)

Under the initial condition uncertainty assumption based on which (21) was obtained, the participation factor for the $i$-th mode in the $k$-th state is equal to the participation factor for the $k$-th state in the $i$-th mode (i.e., $\pi_{ki} = p_{ki}$) if the eigenvectors of the system matrix $A$ are mutually orthogonal, which is a very restrictive case.

### III. OUTPUT PARTICIPATION FACTORS

In this section, definitions and calculations are given for participation factors measuring the relative contribution of modes in outputs for the system (1)-(2). The solution to (1) which is given in (6) is repeated here for convenience:

$$x(t) = \sum_{i=1}^{n} (l^T x_0)^i e^{\lambda_i t} r^i.$$  \quad (22)

Using (22), (2) can be rewritten in the form

$$y(t) = C x(t) = C \sum_{i=1}^{n} (l^T x_0)^i e^{\lambda_i t} r^i.$$  \quad (23)

From (23), the $k$-th output $y_k(t)$ is given by

$$y_k(t) = C^k \sum_{i=1}^{n} (l^T x_0)^i e^{\lambda_i t} r^i,$$  \quad (24)

where $C^k$ is the $k$-th row of $C$.

The participation factor measuring participation of the $i$-th mode in the $k$-th output $y_k$ is defined as follows.

**Definition 1.** For the continuous LTI system (1)-(2), the participation factor for the $i$-th mode in the $k$-th output is

$$p_{ki}^{y} := \begin{cases} E \left\{ \frac{(l^T x_0)^C r^i}{y_k^r} \right\}, & \text{if } \lambda_i \text{ is real} \\ E \left\{ \frac{(l^T x_0)^C r^i}{y_k^r + (l^T x_0)^C r^i} \right\}, & \text{if } \lambda_i \text{ is complex} \end{cases}$$  \quad (25)

whenever this expectation exists.

Note that in (25), the notation $y_k^0$ means $C^k x^{0}$. To obtain a simple closed-form expression for the mode in output participation factors $p^{y}_{ki}$, using (25), we need to find an assumption on the probability density function $f(x^{0})$ governing the uncertainty in the initial condition $x^{0}$ that is intuitively appealing and allows us to explicitly evaluate the integrals inherent in the definition. In the reminder of this section, we make the following assumption.

**Assumption 1.** Assume that the units of the state variables have been scaled to ensure that the probability density function $f(x^{0})$ is such that the components $x_1^0, x_2^0, \ldots , x_n^0$ are jointly uniformly distributed over the unit sphere in $\mathbb{R}^n$ centered at the origin:

$$f(x^{0}) = \begin{cases} k, & \|x^{0}\| \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$  \quad (26)

The constant $k$ in Assumption 1 is chosen to ensure the normalization

$$\int_{\|x^{0}\| \leq 1} f(x^{0}) dx^{0} = 1.$$  \quad (27)

The value of the constant $k$ can be determined by evaluating the integral in (27) using $f(x^{0})$ in (26):

$$\int_{\|x^{0}\| \leq 1} f(x^{0}) dx^{0} = \int_{\|x^{0}\| \leq 1} k dx_1^0 dx_2^0 \ldots dx_n^0 = k V_n = 1,$$  \quad (28)
where $V_n$ is the volume of the unit sphere in $\mathbb{R}^n$. The constant $k$ is then given by

$$k = \frac{1}{V_n}. \tag{29}$$

The following Lemma will be used below.

**Lemma 1.** [4] For vectors $a \in \mathbb{C}^n$, $b \in \mathbb{R}^n$ with $b \neq 0$ we have

$$\int_{\|x\| \leq 1} \frac{a^T x}{b^T x} d_n x = \frac{a^T b}{b^T b} V_n,$$

where $d_n x$ denotes the differential volume element $dx_1 dx_2 \cdots dx_n$, and $V_n$ is the volume of a unit sphere in $\mathbb{R}^n$ which is given by

$$V_n = \begin{cases} 
2, & n = 1, \\
\pi, & n = 2, \\
\frac{2\pi}{\sqrt{n}} V_{n-2}, & n \geq 3.
\end{cases}$$

A. Real-mode-in-output participation factors

To determine the participation of a mode associated with a real eigenvalue $\lambda_i$ in the $i$-th output, substitute $y_i = C^k x^0$ in the first case of (25) and write $l^i = l^i_{C^k} + l^i_{C^k \perp}$, where $l^i_{C^k}$ is the projection of $l^i$ in the direction $C^k$ and $l^i_{C^k \perp}$ is the projection of $l^i$ in the direction orthogonal to $C^k$. Then

$$p^y_{ki} = C^k r_i E \left\{ \frac{(l^i_{C^k} + l^i_{C^k \perp}) x^0}{C^k x^0} \right\} = C^k r_i E \left\{ \frac{l^i_{C^k} x^0}{C^k x^0} + \frac{l^i_{C^k \perp} x^0}{C^k x^0} \right\}. \tag{30}$$

Moreover, $l^i_{C^k}$ can be written in terms of $l^i$ and $C^k$ as follows:

$$l^i_{C^k} = \frac{\|l^i\|}{\|l^i\| \|C^k\|} C^k = \frac{l^i (C^k)^T}{\|C^k\|^2} C^k. \tag{31}$$

Substituting (31) into (30) yields

$$p^y_{ki} = C^k r_i E \left\{ \frac{l^i (C^k)^T}{\|C^k\|^2} + \frac{l^i_{C^k \perp} x^0}{C^k x^0} \right\} = C^k r_i E \left\{ \frac{l^i_{C^k \perp} x^0}{C^k x^0} \right\} = C^k r_i E \left\{ \frac{l^i_{C^k \perp} x^0}{C^k x^0} \right\}. \tag{32}$$

In general, the second term in (32) does not vanish. This is true even in case the components $x^0_1, x^0_2, \ldots, x^0_n$ representing the initial conditions of the state are assumed to be independent. This is due to the fact that the second term involves the components which need not to be independent even under the assumption that the $x^0_i$ are independent, due to the transformations $l^i_{C^k \perp} x^0$ and $C^k x^0$.

We now use Lemma 1 to simplify the expression (32) for the participation factor for the $i$-th (real) mode in the $k$-th output. Denote $a = \left( \frac{l^i_{C^k}}{C^k} \right)^T$ and $b = (C^k)^T$ and note that $a$ is orthogonal to $b$. Using Lemma 1, the expectation term in (32) reduces to

$$E \left\{ \frac{l^i_{C^k} x^0}{C^k x^0} \right\} = \int_{\|x\| \leq 1} \frac{l^i_{C^k} x^0}{C^k x^0} f(x^0) dx^0 = k \int_{\|x\| \leq 1} \frac{a^T x^0}{b^T x^0} dx^0 = \frac{a^T b}{b^T b} k V_n = 0. \tag{33}$$

Therefore, the second term in (32) vanishes under Assumption 1 and the real-mode-in-output participation factors are given by

$$p^y_{ki} = \frac{C^k r_i l^i (C^k)^T}{\|C^k\|^2}. \tag{34}$$

Another way to obtain formula (34) under Assumption 1 is as follows. Recall from Definition 1, the general averaging-based definition for mode-in-output participation factors for the case of a real eigenvalue:

$$p^y_{ki} = E \left\{ \frac{(l^i x^0)^{C^k \perp}}{C^k x^0} \right\} = C^k r_i E \left\{ \frac{l^i x^0}{C^k x^0} \right\}. \tag{35}$$

Denote $a = (l^i)^T$ and $b = (C^k)^T$. Using Lemma 1 and the normalization $k V_n = 1$, (35) reduces to

$$p^y_{ki} = C^k r_i \frac{a^T b}{b^T b} = \frac{C^k r_i l^i (C^k)^T}{\|C^k\|^2}, \tag{36}$$

which is exactly (34).

B. Complex-mode-in-output participation factors

To determine the participation factor for a complex mode, i.e., a mode associated with a complex conjugate pair of nonreal eigenvalues $\lambda_i$ and $\lambda_i^*$, in an output, we use the second case of (25):

$$p^y_{ki} = E \left\{ \frac{(l^i x^0)^{C^k \perp} + (l^{i^*} x^0)^{C^k \perp}}{C^k x^0} \right\} = C^k r_i E \left\{ \frac{l^i x^0}{C^k x^0} + \frac{l^{i^*} x^0}{C^k x^0} \right\}. \tag{37}$$

Write $l^i = l^i_{C^k} + l^i_{C^k \perp}$ and $l^{i^*} = l^{i^*}_{C^k} + l^{i^*}_{C^k \perp}$, where $l^i_{C^k}$, $l^{i^*}_{C^k}$ are the projections of $l^i$ and $l^{i^*}$ in the direction $C^k$, respectively, and $l^i_{C^k \perp}$, $l^{i^*}_{C^k \perp}$ are the projections of $l^i$ and $l^{i^*}$ in the direction orthogonal to $C^k$, respectively. Substituting $l^i = l^i_{C^k} + l^i_{C^k \perp}$ and $l^{i^*} = l^{i^*}_{C^k} + l^{i^*}_{C^k \perp}$ in (37) gives

$$p^y_{ki} = C^k r_i E \left\{ \frac{l^i_{C^k} x^0}{C^k x^0} + \frac{l^{i^*}_{C^k} x^0}{C^k x^0} \right\} + C^k r_i E \left\{ \frac{l^i_{C^k \perp} x^0}{C^k x^0} + \frac{l^{i^*}_{C^k \perp} x^0}{C^k x^0} \right\}. \tag{38}$$

Furthermore, $l^i_{C^k}$ and $l^{i^*}_{C^k}$ can be written in terms of $l^i$, $l^{i^*}$ and $C^k$ as follows:

$$l^i_{C^k} = \frac{l^i (C^k)^T}{\|C^k\|^2} C^k, \tag{39}$$

$$l^{i^*}_{C^k} = \frac{l^{i^*} (C^k)^T}{\|C^k\|^2} C^k. \tag{40}$$
Substituting (39) and (40) into (38) yields
\[
p_k^y_{ki} = C_k r^i E \left\{ \begin{array}{l}
\frac{t^{x_0}(C_k^T)}{\|C_k\|^2} + \frac{t^{x_0^*}(C_k^T)}{C_k} \\
+ C_k r^i E \left\{ \frac{t^{x_0}(C_k^T)}{\|C_k\|^2} + \frac{t^{x_0^*}(C_k^T)}{C_k} \right\}
\end{array} \right\}
\]
\[
= \frac{C_k r^i t^{x_0}(C_k^T)}{\|C_k\|^2} + C_k r^i E \left\{ \frac{t^{x_0^*}(C_k^T)}{C_k} \right\}
\]
\[
+ \frac{C_k r^i t^{x_0^*}(C_k^T)}{\|C_k\|^2} + \frac{C_k r^i t^{x_0^*}(C_k^T)}{C_k}
\]
\[
= \frac{C_k r^i t^{x_0}(C_k^T)}{\|C_k\|^2} + \frac{C_k r^i t^{x_0^*}(C_k^T)}{\|C_k\|^2}
\]
\[
+ \frac{C_k r^i E \left\{ t^{x_0^*}(C_k^T) \right\}}{C_k} + C_k r^i E \left\{ \frac{t^{x_0^*}(C_k^T)}{C_k} \right\}.
\]

It is easy to show that the last two terms in (41) vanish as was done in (33). Hence, (41) reduces to
\[
p_k^y_{ki} = \frac{C_k r^i t^{x_0}(C_k^T)}{\|C_k\|^2} + \frac{C_k r^i t^{x_0^*}(C_k^T)}{\|C_k\|^2}
\]
\[
= \frac{2}{\|C_k\|^2} \text{Re} \left\{ C_k r^i t^{x_0}(C_k^T) \right\}.
\]

Next, we give another way to obtain formula (42) under Assumption 1. Recall from Definition 1, the general averaging-based definition for mode-in-output participation factors for the case of complex eigenvalues:
\[
p_k^y_{ki} = E \left\{ \frac{t^{x_0}(C_k^T)}{C_k} \right\} + C_k r^i E \left\{ \frac{t^{x_0^*}(C_k^T)}{C_k} \right\}.
\]

Using Lemma 1, it easy to show that
\[
E \left\{ \frac{t^{x_0}(C_k^T)}{C_k} \right\} = \frac{t^{x_0}(C_k^T)}{C_k}, \quad E \left\{ \frac{t^{x_0^*}(C_k^T)}{C_k} \right\} = \frac{t^{x_0^*}(C_k^T)}{C_k}.
\]

Thus, (43) can be rewritten as
\[
p_k^y_{ki} = \frac{C_k r^i t^{x_0}(C_k^T)}{\|C_k\|^2} + \frac{C_k r^i t^{x_0^*}(C_k^T)}{\|C_k\|^2}
\]
\[
= \frac{2}{\|C_k\|^2} \text{Re} \left\{ C_k r^i t^{x_0}(C_k^T) \right\},
\]

which is exactly (42).

Note that in formula (44), the participation factors for the \( k \)-th output depend on the matrix \( C \) only through its \( k \)-th row \( C_k \). An immediate implication of this is that when an output coincides with a state variable, the modal participations in the output equal the traditional modal participations in the state.

### IV. Numerical Example

The following example is borrowed from [10]. Consider the translation mechanical system depicted in Fig. 1, where \( d_1(t) \) and \( d_2(t) \) denote the displacements of mass 1 and mass 2, respectively, from the static equilibrium. The system parameters are the masses \( m_1 \) and \( m_2 \), viscous damping coefficients \( c_1 \) and \( c_2 \), and the spring constants \( k_1 \) and \( k_2 \).

A state space representation is obtained by defining the system states as
\[
\begin{align*}
x_1(t) &= d_1(t), \quad x_2(t) = \dot{d}_1 = \dot{x}_1, \\
x_3(t) &= d_2(t), \quad x_4(t) = \dot{d}_2 = \dot{x}_3.
\end{align*}
\]

According to [10], the system dynamics can be described by a linear time-invariant differential equation
\[
\dot{x}(t) = Ax(t),
\]

where
\[
A = \begin{bmatrix}
0 & 1 & 0 & 0 \\
-\frac{k_1 + k_2}{m_1} & -\frac{c_1 + c_2}{m_1} & \frac{k_2}{m_1} & \frac{c_2}{m_1} \\
0 & 0 & \frac{k_2}{m_2} & \frac{c_2}{m_2} \\
-14.6154 & -1.3333 & 5.0256 & 0.8462 \\
11.5294 & 1.9412 & -11.5294 & -1.9412
\end{bmatrix}.
\]

With the system parameters selected as \( m_1 = 39 \text{kg}, m_2 = 17 \text{kg}, c_1 = 19 \text{Ns/m}, c_2 = 33 \text{Ns/m}, k_1 = 374 \text{N/m}, k_2 = 196 \text{N/m} \), the system state dynamics matrix \( A \) becomes
\[
A = \begin{bmatrix}
0 & 1 & 0 & 0 \\
-0.0124 - j0.1938 & 0.4461 + j0.0708 & -0.0321 - j0.3425 & 0.7999 \\
0 & 0 & 0 & 0 \end{bmatrix}, \quad r^2 = \begin{bmatrix}
0.0716 + j0.1021 & -0.5492 + j0.1624 \\
-0.553 - j0.1673 & 0.7970
\end{bmatrix},
\]

respectively. Since \( \lambda_2 = \lambda_1^* \) and \( \lambda_4 = \lambda_3^* \), we have that \( r^2 = r_1^{x*} \) and \( r^4 = r_3^{x*} \), where asterisk represents complex conjugation. The left (row) eigenvectors of the system matrix \( A \) associated with \( \lambda_1 \) and \( \lambda_3 \) are
\[
\begin{align*}
l^1 &= \begin{bmatrix}
-0.3122 + j1.2059 \\
0.4806 + j0.0539 \\
0.2760 + j0.7884 \\
0.3730 + j0.0171
\end{bmatrix}^T, \\
l^3 &= \begin{bmatrix}
0.3134 - j2.4251 \\
-0.4824 - j0.1776 \\
-0.2771 + j1.3357 \\
0.2530 + j0.1903
\end{bmatrix}^T,
\end{align*}
\]

respectively, and \( l^2 = l_1^{x*} \) and \( l^4 = l_3^{x*} \).
TABLE I
MODE IN OUTPUT PARTICIPATION FACTORS FOR SYSTEM (45), (46)
BASED ON FORMULA (42)

<table>
<thead>
<tr>
<th>mode 1</th>
<th>mode 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_1$</td>
<td>$p_{11}^y = 0.4598$</td>
</tr>
<tr>
<td>$y_2$</td>
<td>$p_{21}^y = 0.4211$</td>
</tr>
<tr>
<td>$y_3$</td>
<td>$p_{31}^y = 1.0610$</td>
</tr>
</tbody>
</table>

TABLE II
MODE IN OUTPUT PARTICIPATION FACTORS FOR SYSTEM (45), (46)
BASED ON AVERAGING 100000 INITIAL CONDITIONS

<table>
<thead>
<tr>
<th>mode 1</th>
<th>mode 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_1$</td>
<td>$p_{11}^y = 0.4628$</td>
</tr>
<tr>
<td>$y_2$</td>
<td>$p_{21}^y = 0.4252$</td>
</tr>
<tr>
<td>$y_3$</td>
<td>$p_{31}^y = 1.1082$</td>
</tr>
</tbody>
</table>

Next, the output of the system (45) is chosen as

$$y(t) = Cx(t),$$

where

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$  

The mode in output participation factors $p_{ik}^y$ evaluated using formula (42) are given in Table I. To validate this formula numerically, a set of 100000 different initial conditions of the state vector $x_0$ that lie in the unit sphere is randomly generated and the relative contribution of the mode in the output for each of these initial conditions is evaluated. The average of all mode in output participation from all 100000 initial conditions is computed as shown in Table II. The values of the mode in output participation factors of Table II are in good agreement with the values of Table I obtained based on formula (42).

To verify that the formula of mode in output participation factors (42) reduces to the formula of mode in state participation factors (19), we consider the same system with outputs taken as system states

$$\dot{x}(t) = Ax(t),$$
$$\dot{y}(t) = \tilde{C}x(t),$$

where $A$ is the same matrix defined in (45) and

$$\tilde{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$  

For this output matrix $\tilde{C}$, the mode in output participation factors are given in Table III whereas the mode in state participation factors (19) are shown in Table IV. Clearly, the output participation factors reduce to the state participation factors when the output is simply a state variable.

V. CONCLUSION

In this paper, we have presented an approach to quantifying the participation of system modes in system outputs for linear time-invariant systems. We have proposed a definition and a formula for these output participation factors by using averaging over an uncertain set of system initial conditions. An example was given to demonstrate the usefulness of the results obtained. The case of participation of outputs in modes requires a different analysis and will be considered in future work.

REFERENCES