H\(_\infty\) estimates for discrete-time Markovian jump linear systems

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Abstract—This paper deals with the problem of H\(_\infty\) filtering for discrete-time Markovian jump linear systems. Predicted and filtered recursive estimates are obtained based on the game theory in this paper, it is assumed that the jump parameter is not accessible. A numerical example is provided in order to show the effectiveness of the approach proposed.

Index Terms—H\(_\infty\) filtering, discrete-time, Markovian jump, robust filtering.

I. INTRODUCTION

The filtering of Markovian jump linear systems (MJLS) has been subject of intensive study in the last years. Different and creative approaches to deal with this class of filter have been considered in the literature. One of the main alternatives to solve this kind of problem is based on Kalman filter algorithms. When the robustness is relevant to the filtering process and demands an extra performance of the filtering approach, H\(_\infty\) techniques are always considered as one of the best solutions to be adopted.

In this way a closed-loop transfer function from the unknown disturbances to the estimation error is designed in order to satisfy a prescribed H\(_\infty\)-norm constraint. In general, the algorithms developed to deduce this class of filter are based on linear matrix inequalities (LMIs), see for instance [2], [3], [7], [9], [10], [11], [12], [13], [14], and [15]. An important feature of this kind of filter is when it assumes that the jump parameter of the Markov chain is not accessible. A numerical example is provided in order to see for instance the filters proposed in [3] and [9].

In this paper we propose H\(_\infty\) filters for discrete-time MJLS which can be calculated in terms of recursive algorithms based on Riccati equations. Following the arguments proposed in [6], this paper develops predicted and filtered estimates based on the two-players game theory. The first player can be interpreted as the maximizer of the estimation cost, whereas the second player tries to find an estimate that brings the quadratic cost to a minimum. A solution exists for a specified γ-level if the resulting cost is positive.

We assume in these Markovian filtering problems that the jump parameter is not accessible. The recursiveness of the approaches we are proposing is the main advantage of these Markovian filters if compared with the filters aforementioned. We provide necessary conditions for the existence of them based on only known parameters of the Markovian system.

This paper is organized as follows: In Section II the filtering problems we are dealing with are defined, in Section III the H\(_\infty\) filters for discrete-time MJLS are presented, and in Section IV a comparative study, based on numerical example, between the approach we are proposing in this paper and the filter developed in [3] is shown.

II. PROBLEM DEFINITION

The H\(_\infty\) recursive filters developed in this paper are based on the following discrete-time MJLS

\[
\begin{align*}
\dot{x}_i &= F_i \theta_i x_i + G_i \theta_i u_i, & i = 0, 1, \ldots \nonumber \\
y_i &= H_i \theta_i x_i + D_i \theta_i w_i, \\
s_l &= L_i \theta_i x_i + R_i \theta_i v_i
\end{align*}
\]

where \(x_i \in \mathbb{R}^m\) is the valued state, \(y_i \in \mathbb{R}^m\) is the valued output sequence, \(s_i \in \mathbb{R}^p\) is the valued signal to be estimated, \(u_i \in \mathbb{R}^p\), \(w_i \in \mathbb{R}^q\) and \(v_i \in \mathbb{R}^q\) are random disturbances; \(\Theta_i\) is a discrete-time Markov chain with finite state-space \(\{1, \ldots, N\}\) and transition probability matrix \(P = [p_{ij}]\). We set \(\pi_{i,j} := P(\Theta_i = j)|F_{i,k} \in \mathbb{R}^{m \times n}, G_{i,k} \in \mathbb{R}^{m \times p}, H_{i,k} \in \mathbb{R}^{m \times n}, D_{i,k} \in \mathbb{R}^{m \times q}, L_{i,k} \in \mathbb{R}^{p \times n}\), and \(R_{i,k} \in \mathbb{R}^{p \times q}\), \(k \in \{1, \ldots, N\}\) and \(i \geq 0\). The random disturbances \(\{u_i\}\), \(\{w_i\}\) and \(\{v_i\}\) are assumed to be null mean second-order independent random variables. The \(\{u_i\}\), \(\{w_i\}\) and \(\{v_i\}\) are assumed to be null mean second-order independent random variables.

The H\(_\infty\) filtering and prediction estimation problems we are solving in this paper are based on the following augmented state variable

\[
\begin{align*}
z_i &:= [z_{i,1} \ldots z_{i,N}]^T \in \mathbb{R}^{Nn}, \\
z_{i,k} &:= x_i1_{\{\Theta_i = k\}} \in \mathbb{R}^n
\end{align*}
\]

whose parameter matrices will be defined in the following. The filtered version is stated based on a scalar \(\gamma > 0\) and on a sequence of sets of observations

\[
\{y_0\}, \{y_0, y_1\}, \ldots, \{y_0, \ldots, y_l\}, \ldots
\]

which aims to find at each \(l\), an estimate \(s_{lj}\) of \(s_{lj}\), if it exists, in terms of the measurements \(\{y_0, \ldots, y_l\}\), such that

\[
\sup_{z_0} \left\| s_{00} - L_0 z_0 \right\|_{L_0^{-1}}^2 + \left\| y_0 - H_0 z_0 \right\|_{P_0^{-1}}^2 < \gamma^2
\]

is satisfied for \(l = 0\), and (4) is satisfied for \(l > 0\). The sequence of solutions \(s_{lj}\) is the output of the H\(_\infty\) filter. For

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the predicted version, given a scalar \( \gamma > 0 \) and a sequence of observations \( \{y_0, y_1, \ldots, y_l, \ldots\} \), the problem is to find at each \( l \) a prediction \( s_{t+1|l} \) of \( s_{t+1} \), if it exists, such that

\[
\sup_{z_0} \left| \frac{\parallel s_{l-1} - L_0 z_0 \parallel_{\Lambda_0}^2}{\parallel z_0 - F_0 z_0 \parallel_{\Lambda_0}^2} \right| < \gamma^2
\]

(4)

\[
\sup_{z_{l+1}} \left| \frac{\parallel s_{l|l} - L_i z_i \parallel_{\Lambda_i}^2}{\parallel z_i - F_i z_i \parallel_{\Lambda_i}^2} \right| < \gamma^2
\]

(5)

Lemma 2.1: Let \( \psi_i \) be given by the following equation, for \( i \geq 0 \) and \( j, k \in \{1, \ldots, N\} \)

\[
\psi_i := M_{i+1} z_i + \theta_i,
\]

(13)

where

\[
M_{i+1} := \begin{bmatrix} M_{i+1,1} & \cdots & M_{i+1,N} \end{bmatrix},
\]

\[
M_{i+1,k} := \begin{bmatrix} (1_{\theta_{i+1} = k} - p_{ik}) F_{i,k} 1_{\theta_i = 1} & \cdots & (1_{\theta_{i+1} = k} - p_{Nk}) F_{i,N} 1_{\theta_i = N} \end{bmatrix},
\]

\[
\theta_i := \begin{bmatrix} \psi_i \end{bmatrix},
\]

(14)

The second moment of \( \psi_i \) can be obtained by the following recursive equation

\[
\Gamma_{i} := \text{diag} \left[ \sum_{j=1}^{N} p_{jk} F_{i,j} Z_{i,j}^T F_{i,j}^T \right] - F_i Z_i F_i^T + \text{diag} \left[ \sum_{j=1}^{N} p_{jk} \pi_{i,j} G_{i,j} U_{i,j} G_{i,j}^T \right].
\]

(15)

Lemma 2.2: Let \( \psi_i \) and \( \sigma_i \) of (7) be given by

\[
\psi_i := D_i \theta_i, w_i,
\]

\[
\sigma_i := R_i \theta_i, v_i,
\]

(16)

The variances of \( \psi_i \) and \( \sigma_i \) are defined as

\[
\Pi_i := D_i D_i^T,
\]

\[
\Lambda_i := R_i R_i^T,
\]

(17)

(18)

respectively, where

\[
D_i := \begin{bmatrix} D_i,1, \ldots, D_i, N \end{bmatrix},
\]

\[
R_i := \begin{bmatrix} R_i,1, \ldots, R_i, N \end{bmatrix},
\]

(19)

(20)

Proof: The proof follows the arguments proposed in [8], Chapter 3.

III. \( \mathcal{H}_\infty \) ESTIMATES FOR DMJLS

In this section, we propose recursive \( \mathcal{H}_\infty \) estimates for DMJLS based on the game theory. It is known that for standard state-space systems this kind of filtering approach is difficult to be implemented online due to the fact that we do not know the minimum of \( \gamma \) for each step of the recursion.
In general, it depends on the estimate error variance matrix which should be calculated at the same instant of time. In this sense, the existence condition of this filter is not known a priori. To ameliorate this limitation we provide, for this Markovian problem, necessary conditions to adjust this parameter depending on the variances of the random disturbances \( \psi_i, \varphi_i \) and \( \sigma_i \) and on the known parameter matrices of the augmented model (1) given in (7).

### A. \( \mathcal{H}_\infty \) filter

The \( \mathcal{H}_\infty \) filtering problem (3)-(4) for DMJLS can be related to the following optimization problem

\[
\min_{\{z_{i|l}\}_{i=0}^\infty} \quad \max_{\{s_{i|l}\}_{i=0}^\infty} \quad J_f^I \left( \{y_i\}_{i=0}^\infty, \{s_{i|l}\}_{i=0}^\infty, \{z_{i|l}\}_{i=0}^\infty \right) > 0
\]

where

\[
J_f^I := \|z_{0|0} - \mathcal{F}_{-1} \hat{z}_0\|_{\mathcal{Z}_0^{-1}}^2 + \|y_0 - \mathcal{H}_0 z_{0|0}\|_{\mathcal{Z}_{1|0}}^2
\]

\[
- \gamma^{-2} \|z_{0|0} - \mathcal{L}_0 z_{0|0}\|_{\mathcal{Z}_{0|0}}^2, \quad l = 0 \tag{21}
\]

and

\[
J_f^I := \|z_{0|l} - \mathcal{F}_{-1} \hat{z}_0\|_{\mathcal{Z}_0^{-1}}^2 + \sum_{i=0}^{l-1} \|y_i - \mathcal{H}_i z_{i|l}\|_{\mathcal{Z}_{1|l}}^2
\]

\[
+ \sum_{i=0}^{l-1} \|z_{i+1|l} - \mathcal{F}_{i+1} z_{i|l}\|_{\mathcal{Z}_{2|l}}^2
\]

\[
- \gamma^{-2} \sum_{i=0}^{l-1} \|s_{i|l} - \mathcal{K}_i z_{i|l}\|_{\mathcal{Z}_{3|l}}^2, \quad l > 0. \tag{22}
\]

In order to solve the original problem (3)-(4), observe that it is not necessary to maximize \( J_f^I \) over \( \{s_{i|l}\}_{i=0}^\infty \) in the optimization problem (21). The solution of the \( \mathcal{H}_\infty \) filter is guaranteed if, and only if, there exists a \( \{z_{i|l}\}_{i=0}^\infty \) for which \( J_f^I \left( \{y_i\}_{i=0}^\infty, \{s_{i|l}\}_{i=0}^\infty, \{z_{i|l}\}_{i=0}^\infty \right) > 0 \) has a minimum \( \{z_{i|l}\}_{i=0}^\infty \). Therefore, in order to deal with only the minimization problem (21), for each \( l \geq 0 \) it is easy to show that (23) can be rewritten as

\[
J_f^I := (\mathcal{U}_l \hat{X}_{l|l} - \mathcal{B}_l)^T \mathcal{R}_l (\mathcal{U}_l \hat{X}_{l|l} - \mathcal{B}_l) \tag{24}
\]

where

\[
\hat{X}_{l|l} := \begin{bmatrix} z_{l|l} \\ \vdots \\ z_{1|l} \\ z_0|l \end{bmatrix}, \quad \mathcal{B}_l := \begin{bmatrix} \hat{Z}_l \\ \vdots \\ \hat{Z}_1 \\ \hat{Z}_0 \end{bmatrix}, \quad \mathcal{R}_l := \begin{bmatrix} R_l & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & R_0 \end{bmatrix}, 
\]

\[
\begin{bmatrix} \mathcal{E}_l & \mathcal{A}_{l-1} & 0 & 0 \\ 0 & \mathcal{E}_{l-1} & \ddots & 0 \\ 0 & 0 & \mathcal{A}_i & 0 \\ 0 & 0 & 0 & \mathcal{E}_0 \end{bmatrix}, \quad \mathcal{A}_0 := \begin{bmatrix} \mathcal{F}_{-1} \hat{z}_0 \\ \mathcal{H}_0 z_{0|0} \\ \mathcal{L}_0 z_{0|0} \end{bmatrix}, \quad \gamma^{-2} \mathcal{L}_l^{-1} \mathcal{Z}_{-1|l} > 0. \tag{25}
\]

For all \( l \geq 0 \), it is easy to conclude that the following recurrent relations are valid

\[
\begin{align*}
\mathcal{R}_l & := \begin{bmatrix} R_{l-1} & 0 \end{bmatrix}, \quad \mathcal{B}_l := \begin{bmatrix} \hat{Z}_l \end{bmatrix}, \\
\hat{X}_{l|l} & := \begin{bmatrix} z_{l|l} \\ \vdots \\ z_{1|l} \\ z_0|l \end{bmatrix}, \quad \mathcal{U}_l := \begin{bmatrix} \mathcal{E}_l & \mathcal{A}_{l-1} \\ 0 & \mathcal{U}_{l-1} \end{bmatrix}, \\
\mathcal{A}_{l-1} & := \begin{bmatrix} \mathcal{F}_{-1} \hat{z}_0 \\ \mathcal{H}_0 z_{0|0} \\ \mathcal{L}_0 z_{0|0} \end{bmatrix}.
\end{align*} \tag{26}
\]

It is important to note that is possible to formulate this optimization problem for filtering of the DMJLS (1), which is similar for the formulations that appear in the literature for standard state-space systems, thanks to the augmented Markovian model aforementioned. According to [5], there exists a minimum for (24) if and only if the positiveness of \( \mathcal{U}_l^T \mathcal{R}_l \mathcal{U}_l \) is guaranteed.

**Lemma 3.1:** Consider matrices \( \mathcal{U}_l \) and \( \mathcal{R}_l \) and column vectors \( \mathcal{B}_l \) and \( \hat{X}_{l|l} \) of appropriate dimensions with \( \mathcal{R}_l \) symmetric. For any \( \mathcal{B}_l \) we have

\[
\inf_{\mathcal{X}_l \in \mathcal{H}_\infty} J_f^I > -\infty \tag{27}
\]

if and only if \( \mathcal{U}_l^T \mathcal{R}_l \mathcal{U}_l \geq 0 \), and \( \text{Ker} (\mathcal{U}_l^T \mathcal{R}_l \mathcal{U}_l) \subset \text{Ker} (\mathcal{R}_l) \). If the minimum is attained, it is unique if and only if \( \mathcal{U}_l^T \mathcal{R}_l \mathcal{U}_l > 0 \), and the optimal solution is given by

\[
\hat{X}_{l|l} = (\mathcal{U}_l^T \mathcal{R}_l \mathcal{U}_l)^{-1} \mathcal{U}_l^T \mathcal{R}_l \mathcal{B}_l. \tag{28}
\]

With this fundamental Lemma in mind, we can obtain an existence condition for the filter we are proposing in the following, based on the known parameter matrices of the Markovian model. For \( l > 0 \), the term \( \mathcal{U}_l^T \mathcal{R}_l \mathcal{U}_l \) of \( \hat{X}_{l|l} \) can be written as

\[
\begin{bmatrix} \mathcal{E}_l^T \mathcal{R}_l \mathcal{E}_l & \mathcal{E}_l^T \mathcal{R}_l \mathcal{A}_{l-1} \\ \mathcal{A}_{l-1}^T \mathcal{R}_{l-1} \mathcal{E}_l & \mathcal{A}_{l-1}^T \mathcal{R}_{l-1} \mathcal{A}_{l-1} \end{bmatrix} \tag{29}
\]

and a necessary condition for \( \mathcal{U}_l^T \mathcal{R}_l \mathcal{U}_l > 0 \) is the positiveness of \( \mathcal{E}_l^T \mathcal{R}_l \mathcal{E}_l \), which in terms of the augmented Markovian model of (1) can be established as follows

\[
\Gamma_{l-1}^{-1} + \mathcal{H}_l^T \Pi_{-1} \mathcal{H}_l - \gamma^{-2} \mathcal{L}_l^T \mathcal{L}_l > 0. \tag{30}
\]

**Remark 3.1:** A lower bound for \( \gamma \) can be calculated through (30) based on the augmented model (7) and on the Lemmas 2.1 and 2.2. Note that (30) is a natural and interesting extension from a necessary condition for the filtering of standard state-space systems [6], without jumps, to the DMJLS we are dealing with in this paper.

Now we are in position to deduce the \( \mathcal{H}_\infty \) recursive filtered estimate for DMJLS. The next theorem provides the
solution for this problem based on the augmented model (7) and on the sequence of measurements (2) of the original Markovian system (1).

**Theorem 3.1:** Consider the augmented Markovian model (7). There exists a recursive $H_\infty$ filter for this system, defined by the following recursive equations

\[
\begin{align*}
\tilde{Z}_{0|0}^{-1} & := \tilde{Z}_0^{-1} + H_0^T \Pi_{0}^{-1} H_0 - \gamma^{-2} L_0^T \Lambda_0^{-1} L_0, \\
\tilde{Z}_{t|t}^{-1} & := \left( \Gamma_{t-1} + F_{t-1} \tilde{Z}_{t-1|t-1} F_{t-1}^T \right)^{-1} + H_{t}^T \Pi_{t}^{-1} H_{t} \\
& \quad - \gamma^{-2} L_{t}^T \Lambda_{t}^{-1} L_{t},
\end{align*}
\]

(31)

\[
\begin{align*}
\tilde{z}_{0|0} & := \left( P_{0|0}^{-1} + \gamma^{-2} L_0^T \Lambda_0^{-1} L_0 \right)^{-1} \times \left( Z_0^{-1} F_0^T \tilde{z}_0 + H_0^T \Pi_{0}^{-1} \gamma_0 \right), \\
\tilde{z}_{t|t} & := \left( \tilde{Z}_{t|t}^{-1} + \gamma^{-2} L_{t}^T \Lambda_{t}^{-1} L_{t} \right)^{-1} \times \left( X^{-1} F_{t-1} \tilde{z}_{t-1|t-1} + H^T_{t-1} \Pi_{t}^{-1} \gamma_t \right), \\
X_{t-1} & := \Gamma_{t-1} + F_{t-1} \tilde{Z}_{t-1|t-1} F_{t-1}^T,
\end{align*}
\]

(32)

\[
\tilde{S}_{t|t} := \mathcal{L} \tilde{z}_{t|t}
\]

(33)

if, and only if, $\tilde{Z}_{t|t} > 0$, $l = 0, 1, \ldots$, and the attenuation level $\gamma$ is positive.

**Proof:** This proof follows standard arguments used to deduce Kalman filters and basically consists in checking the positiveness of (29) and in finding the equation of the filter through the Lemma (3.1). In order to have $\Pi_{t}^T \mathcal{R}_{t} \mathcal{U}_t > 0$, we can rewrite this term as (29). The (2, 2) sub-block of (29) must be positive definite. Assuming that the positiveness of $\Pi_{t-1}^T \mathcal{R}_{t-1} \mathcal{U}_{t-1}$ is guaranteed and knowing that

\[
\alpha_{t-1}^T \mathcal{R}_{t} \alpha_{t-1} = \left[ \begin{array}{c} F_{t-1} \Gamma_{t-1}^T F_{t-1} \end{array} \right]^{-1} \geq 0,
\]

(34)

the (2, 2) term of (29) is positive definite and $\Pi_{t}^T \mathcal{R}_{t} \mathcal{U}_t > 0$ if and only if the Schur complement of the (2, 2) block

\[
M_{t|t}^{-1} := \mathcal{E}_t^T \left( \mathcal{R}_{t}^{-1} + \alpha_{t-1} \left( \Pi_{t-1}^T \mathcal{R}_{t-1} \mathcal{U}_{t-1} \right)^{-1} \alpha_{t-1}^T \right)^{-1} \mathcal{E}_t
\]

is positive definite. For $l = 1$, $M_{l|l-1}$ is the (1, 1) block of

\[
\Pi_{t}^T \mathcal{R}_{t} \mathcal{U}_t\Pi_{t-1} \mathcal{U}_{t-1}^{-1} \Pi_{t-1} \mathcal{U}_{t-1}^{-1} \Pi_{t}^T \mathcal{R}_{t} \mathcal{U}_t
\]

(27), we obtain

\[
M_{t|t}^{-1} := \mathcal{E}_t^T \left( \mathcal{R}_{t}^{-1} + A_{t-1} M_{t-1|l-1} A_{t-1}^T \right)^{-1} \mathcal{E}_t.
\]

(35)

Considering $\tilde{Z}_{t|t} := M_{t|t}$ in terms of the original data (25), we obtain (31). From Lemma (3.1) we obtain the solution for the minimization of (24). Based on the recurrent relations (25) and (27), considering $\alpha_{l-1} \mathcal{X}_{l-1|l} = \mathcal{A}_{l-1} x_{l-1|l}$ for $j \geq l - 1$, and introducing

\[
\begin{align*}
\tilde{Z}_{t|t} & := \mathcal{E}_t^T \mathcal{R}_t \mathcal{E}_t \\
\mathcal{P}_{t|t} & := \mathcal{E}_t^T \mathcal{R}_t \mathcal{E}_t, \\
\mathcal{Q}_{t|t} & := \mathcal{E}_t^T \mathcal{R}_t \mathcal{E}_t \alpha_{t-1}^T \mathcal{R}_t \alpha_{t-1}, \\
\mathcal{R}_t & := \mathcal{E}_t^T \mathcal{R}_t \mathcal{E}_t \alpha_{t}^T \mathcal{R}_t \alpha_{t}^{-1}
\end{align*}
\]

(36)

we obtain the $H_\infty$ estimate (32), taking into account that

\[
\tilde{S}_{t|t} := \mathcal{S}_{t|t} := \mathcal{L} \tilde{z}_{t|t}.
\]

**B. $H_\infty$ Predictor**

In this subsection we present the recursive $H_\infty$ predictor filter for DMJLS. The original $H_\infty$ problem (6)-(5) can be rewritten in terms of the following optimization problem

\[
\begin{align*}
\min \{ z_{0|0} \} \quad \max \{ z_{1|1} \} > 0 \quad \text{such that}
\end{align*}
\]

\[
J^P_1 \left( \{ y_{i} \}_{i=0}, \{ s_{i|1} \}_{i=0} \right) \geq 0
\]

(37)

where

\[
J^P_1 := \| z_{0|0} - F_0 \tilde{z}_0 \|_0^2 + \sum_{i=0}^{1} \| y_{i} - H_{i} z_{i|1} \|_{\Pi_{i}}^2
\]

(38)

\[
J^P_2 := \sum_{i=0}^{1} \| s_{i|1} - L_{i} z_{i|1} \|_{\Lambda_{i}}^2, \quad l = 0
\]

(39)

The $H_\infty$ predictor filter exists at the instant $l$ if, and only if, there exist a sequence $\{ \tilde{z}_{i|1} \}_{i=0}^{1}$ such that $J^P_1 \left( \{ y_{i} \}_{i=0}, \{ \tilde{z}_{i|1} \}_{i=0}^{1} \right)$ has a minimum $\{ \tilde{z}_{i|1} \}_{i=0}^{1}$ for which $J^P_2 \left( \{ y_{i} \}_{i=0}, \{ \tilde{z}_{i|1} \}_{i=0}^{1} \right) > 0$. Note that the prediction optimization problem is equivalent in nature to the filter problem aforementioned. Therefore, following the solution considered which turns the min-max approach on only a minimization problem, we can deduce the recursive Markovian predictor filter. For each $l \geq 0$, we can rewrite (39) as

\[
J^P_2 := \left( \Pi_{l} \mathcal{X}_{l+1|l} - \mathcal{B}_l \right)^T \mathcal{R}_l \left( \Pi_{l} \mathcal{X}_{l+1|l} - \mathcal{B}_l \right)
\]

(40)

where

\[
\begin{align*}
\mathcal{X}_{l+1|l} := \left[ \begin{array}{c} \mathcal{P}_{l+1|l} \\
\mathcal{S}_{l+1|l} \end{array} \right], \\
\mathcal{R}_l := \left[ \begin{array}{c} \mathcal{P}_{l+1|l} \\
\mathcal{S}_{l+1|l} \end{array} \right], \\
\mathcal{A}_l := \left[ \begin{array}{c} \mathcal{S}_{l+1|l} \\
\mathcal{S}_{l+1|l} \end{array} \right], \\
\mathcal{E}_l := \left[ \begin{array}{c} \mathcal{S}_{l+1|l} \\
\mathcal{S}_{l+1|l} \end{array} \right], \\
\mathcal{U}_l := \left[ \begin{array}{c} \mathcal{S}_{l+1|l} \\
\mathcal{S}_{l+1|l} \end{array} \right]
\end{align*}
\]

(41)
\[
E_i := \begin{bmatrix}
I_L \\
0
\end{bmatrix},
U_{i-1} := E_0 := \begin{bmatrix}
I_L \\
0
\end{bmatrix},
R_i := \begin{bmatrix}
\Gamma_i^{-1} & 0 & 0 \\
0 & \Pi_i^{-1} & 0 \\
0 & 0 & -\gamma^{-2}L_{t+i}^{-1}
\end{bmatrix},
\]

\[
Z_i := \begin{bmatrix}
0 & y_i \\
1 & s_{i+1|t}
\end{bmatrix}, i \geq 0,
\]

\[
Z_{i-1} := \begin{bmatrix}
\mathcal{F}_{l-1} z_0 \\
0
\end{bmatrix}.
\] (42)

Considering the above variables, it is easy to see that we can rewrite these arrays as follows

\[
R_i := \begin{bmatrix}
\mathcal{R}_l & 0 \\
0 & \mathcal{R}_{i-1}
\end{bmatrix},
\]

\[
X_{i+1|t} := \begin{bmatrix}
\hat{z}_{i+1|t} \\
\hat{x}_{i+1|t}
\end{bmatrix},
\]

\[
U_l := \begin{bmatrix}
\mathcal{E}_{l+1} \\
0
\end{bmatrix},
\]

\[
\alpha_l := \begin{bmatrix}
A_l \\
0 \\
\vdots \\
0
\end{bmatrix}.
\] (43)

Similar to the filtered case, we can also find a necessary existence condition to the Markovian \(H_\infty\) prediction filter. For a fixed \(l \geq -1\), consider the minimization problem (40). There exists a unique minimal solution \(\{\hat{z}_{i|t}\}_{t=0}^{\infty}\) if, and only if, \(\mathcal{R}_l \mathcal{R}_u \mathcal{U}_l > 0\) where \(\mathcal{U}_l\) and \(\mathcal{R}_l\) are defined in (41). A necessary condition for the minimum uniqueness is that

\[
\hat{Z}_{l-1}^{-1} - \gamma^{-2}L_{t+i}^{-1}L_0 > 0, \quad l = -1
\] (44)

\[
\Gamma_i^{-1} - \gamma^{-2}L_{t+i}^{-1}L_{t+i+1} > 0, \quad l \geq 0.
\] (45)

In the next theorem we present the \(H_\infty\) predictor filter whose proof follows the list of the filtered version.

**Theorem 3.2:** The Markovian \(H_\infty\) prediction problem (37) is solvable if and only if \(\hat{Z}_{l+1|t}^{-1} - \gamma^{-2}L_{t+i}^{-1}L_{t+i} > 0\), where the sequence \(\{\hat{Z}_{l+1|t}\}\) and the predictor filter are calculated by the recursions

\[
\hat{Z}_{0|t} := \hat{Z}_0,
\]

\[
\hat{Z}_{l+1|t} := \Gamma_l + \mathcal{F}_l \hat{Z}_{l|t-1} \mathcal{F}_l^T - \mathcal{F}_l \hat{Z}_{l|t-1} \mathcal{H}_l^T W_{e\infty, l}^{-1} \mathcal{H}_l \hat{Z}_{l|t-1} \mathcal{F}_l,
\]

\[
W_{e\infty, l} := \begin{bmatrix}
\Pi_l & 0 \\
0 & -\gamma^{-2}L_l
\end{bmatrix} + \begin{bmatrix}
\mathcal{H}_l \\
\mathcal{L}_l
\end{bmatrix} \hat{Z}_{l|t-1} \begin{bmatrix}
\mathcal{H}_l \\
\mathcal{L}_l
\end{bmatrix} \hat{Z}_{l|t-1} \begin{bmatrix}
\mathcal{H}_l \\
\mathcal{L}_l
\end{bmatrix} \hat{Z}_{l|t-1},
\]

\[
\hat{s}_{l+1|t} := \mathcal{L}_{l+1} \hat{z}_{l+1|t}.
\] (46)

\[
\hat{z}_{l+1|t} := \mathcal{F}_l \hat{z}_{l|t-1} + \mathcal{F}_l M_{l|t-1} \mathcal{H}_l^T (\Pi_l + \mathcal{H}_l M_{l|t-1} \mathcal{H}_l^T)^{-1} \mathcal{H}_l
\]

\[
M_{l|t-1} := \hat{Z}_{l|t-1}^{-1} - \gamma^{-2}L_{t+i}^{-1}L_l,
\] (47)

\[
\hat{s}_{l+1|t} := \mathcal{L}_{l+1} \hat{z}_{l+1|t}.
\] (48)

**Remark 3.2:** To establish the stability of the stationary \(H_\infty\) prediction filter, it is assumed that all matrices of (1) and the transition probabilities \(p_{jk}\) are time-invariant, the System (1) is mean square stable (MSS) and its Markov chain \(\{\Theta_i\}\) is ergodic. There exists a unique positive-semidefinite solution \(\hat{Z}\) (with \(i \to \infty\)) to the algebraic Riccati equation

\[
\hat{Z} := \Gamma + \mathcal{F} \hat{Z} \mathcal{F}^T - \mathcal{F} \hat{Z} \begin{bmatrix}
\mathcal{H} \\
\mathcal{L}
\end{bmatrix} W_{e\infty}^{-1} \begin{bmatrix}
\mathcal{H} \\
\mathcal{L}
\end{bmatrix} \hat{Z} \mathcal{F},
\]

\[
W_{e\infty} := \begin{bmatrix}
\Pi & 0 \\
0 & -\gamma^{-2}L
\end{bmatrix} + \begin{bmatrix}
\mathcal{H} \\
\mathcal{L}
\end{bmatrix} \hat{Z} \begin{bmatrix}
\mathcal{H}^T \\
\mathcal{L}^T
\end{bmatrix}
\] (49)

provide that \(\hat{Z}_{l+1}^{-1} - \gamma^{-2}L_{t+i}^{-1}L > 0\), for any \(\gamma\) fixed, which guarantee that \(W_{e\infty} > 0\) and thus the above inverse is well defined. Following the guidelines defined in [1] and [4], the stability of the predictor filter is assured with

\[
r_{\sigma}(\mathcal{F} - \mathcal{F} M \mathcal{H}^T (\Pi + \mathcal{H} M \mathcal{H}^T)^{-1} \mathcal{H}) < 1
\]

where \(r_{\sigma}(\cdot)\) denotes spectral radius of the dynamic matrix of the filter with stationary gain. The asymptotic stability of the filtered version can also be assured following this line of argumentation.

**Remark 3.3:** For the case with no jumps \((N = 1)\) the predicted and the filtered \(H_\infty\) estimates reduce to the \(H_\infty\) filters for state-space systems given in [4].

**IV. Numerical Example**

In this section we compare the filtering approach we are proposing in this paper with the filter developed in [3], based on numerical example with two Markovian states. The probability matrix and the parameters of the Markovian model (1) are defined as

\[
P = \begin{bmatrix}
0.9 & 0.1 \\
0.9 & 0.1
\end{bmatrix},
\]

\[
F_1 = \begin{bmatrix}
0.7 & 0 \\
0.1 & 0.1
\end{bmatrix},
\]

\[
F_2 = \begin{bmatrix}
0.6 & 0 \\
0.1 & 0.2
\end{bmatrix},
\]

\[
G_1 = G_2 = \begin{bmatrix}
0.8731 & 0 \\
0 & 0.2089
\end{bmatrix},
\]

\[
H_1 = H_2 = \begin{bmatrix}
0.1 & 0 \\
\end{bmatrix},
\]

\[
D_1 = D_2 = \begin{bmatrix}
0.008 & 0 \\
\end{bmatrix},
\]

\[
L_1 = L_2 = \begin{bmatrix}
0.5 & 0 \\
\end{bmatrix},
\]

\[
R_1 = R_2 = \begin{bmatrix}
0.1 & 0.3
\end{bmatrix}.
\]

We compute both filters in the prediction form. We can write the predicted filter we are proposing as

\[
\hat{z}_{l+1|t} := A_l \hat{z}_{l|t-1} + B_l y_l,
\]

\[
\hat{s}_{l+1|t} := \mathcal{L}_{l+1} \hat{z}_{l+1|t},
\] (50)

where

\[
A_l = \mathcal{F}_l - \mathcal{F}_l M_{l|t-1} \mathcal{H}_l^T (\Pi_l + \mathcal{H}_l M_{l|t-1} \mathcal{H}_l^T)^{-1} \mathcal{H}_l
\]

\[
B_l = \mathcal{F}_l M_{l|t-1} \mathcal{H}_l^T (\Pi_l + \mathcal{H}_l M_{l|t-1} \mathcal{H}_l^T)^{-1}.
\]

The filter of [3] we consider in the strictly proper form as

\[
x_f(k + 1) = A_f x_f(k) + B_f y(k)
\]

\[
z_f(k) = C_f x_f(k)
\] (51)

whose solution is given in terms of linear matrix inequalities.

We show in Figure 1 the square-root of the mean square errors (rms) of both filters. We performed 1000 Monte Carlo simulations from \(i = 0, \ldots, 6\) with the values of
Fig. 1. Square-root mean square errors for $\mathcal{H}_\infty$ filters based on Riccati recursive equation and linear matrix inequalities.

The $\Theta_i$'s, generated randomly. The initial condition $x_0$ is considered Gaussian with mean $[0.196 \ 0.295]^T$ and variance $[0.0384 \ 0.0578 \ 0.0578 \ 0.870]$, $\Theta_i \in \{1, 2\}$, $u_i$, $w_i$ and $v_i$ are independent sequence of noises and $\pi_1(0) = 0.05$ and $\pi_2(0) = 0.95$. We obtained $\gamma = 1.9523$ for the filter proposed in this paper and $\gamma = 0.2886$ for the filter proposed in [3]. The parameter matrices of our filter were computed as

$$
A_t = \begin{bmatrix}
0.0116 & 0 & -0.0784 & 0 \\
0.0003 & 0.09 & 0.0003 & 0.18 \\
0.0013 & 0 & -0.0087 & 0 \\
0 & 0.01 & 0 & 0.02
\end{bmatrix},
$$

$$
B_t = \begin{bmatrix}
6.1843 \\
0.8967 \\
0.6871 \\
0.0996
\end{bmatrix}, \quad \mathcal{L}_{t+1} = \begin{bmatrix}
0.5 & 0 & 0.5 & 0
\end{bmatrix} \tag{52}
$$

and for the filter of [3] were computed as

$$
A_f = \begin{bmatrix}
-0.2154 & -0.0122 \\
-0.0342 & -0.3654
\end{bmatrix},
$$

$$
B_f = \begin{bmatrix}
0.4196 \\
2.4656
\end{bmatrix}, \quad C_f = \begin{bmatrix}
0.4790 & 0.0962
\end{bmatrix}. \tag{53}
$$

Note that in spite of the smaller $\gamma$ provided by the filter of [3], the rms of both filters are equivalent with a little advantage to the filter of this paper. We can explain the difference between both $\gamma$’s, as a reasonable hypothesis, due to the fact that our filter was deduced for an augmented system whose dimension is greater than the filter of [3]. It is worth emphasizing that our filter is based on recursive approach where, provided that the existence conditions are guaranteed, the solutions for each instant of time always exist.

V. CONCLUSION

This paper developed $\mathcal{H}_\infty$ predicted and filtered recursive estimates for DMJLS. They were deduced based on the assumption that jump parameter is not accessible. The numerical example proposed showed the effectiveness of this approach. An important feature of the filter of [3] is that it was developed as a proper filter. As future work we intend to extend the filters of this paper to a proper form.

REFERENCES