Signal Complexity in Cyclic Consensus Systems

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Abstract—In this paper, we deal with classes of LTI cyclic systems from the view point of the information theoretic signal complexity in the networks. The classes contain simple one-layered cyclic systems and hierarchical multi-layered cyclic systems. We firstly give the transfer functions between state variables in the former case and show the entropy of each signal. This result is also discussed in the view point of the channel capacity necessary for the one-layered cyclic systems. We next extend the results to the hierarchical multi-layered cyclic systems. The situation is more complicated and partial results are given. We illustrate the locations of zeros in such cyclic systems from the view point of the structure of the networks.

I. INTRODUCTION

According to the enormous development of the recent computer networks, control systems with signal transmission through exclusive or general-purpose networks have been realized. The corresponding theoretical research has been also actively investigated in the last decade.

One of the important subjects in the networked control is the relationship between the channel capacity and the characteristics of the systems. The necessary channel capacity for stabilization in a simple closed loop system has been investigated and rigorous results were provided in [20], [10], [17], [19]. The different research approach such as the optimal quantization of signals for control systems has also been discussed [3], [5], [6]. Furthermore, recently, the research focus moves on the control systems with more complex or particular structure of the network [9], [2], [1].

There is another approach for reducing transmission of signals in which the focus is on the complexity of the network structure rather than the signal complexity. The consensus problem is one of such research topics where each state variable is regarded as an agent and it exchanges the state variable with the other agents in order to achieve a consensus. The signal exchange can be represented by the corresponding incidence matrix between the agents and the relationship between the structure and the stability of the whole system has been discussed [12], [11], [2], [1].

One of simple structures of such signal exchange is a cyclic network system. It originates from the formation control of vehicles or the analysis on biological swarms. In a cyclic network system, the state variables of an agent are supposed to be transferred only to the neighborhoods in a same manner. In [7], with such simple structure, it is shown that the convergence to a stable formation can be realized.

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On the other hand, as a more abstract and general theory, the research on the stability of hierarchical cyclic network systems was reported by Smith et al. [16]. The system is composed of a recursive and hierarchical cyclic structure of the incidence matrix where clusters are composed of a cyclic system of agents and they also recursively connect to the other clusters in a cyclic way. It was shown that the pole locations are strictly represented recursively and the stability of the whole system was discussed. Recently, this result was extended to more general cases [15].

These results on consensus systems with simple structures motivate us to return to the original purpose of reducing the signal transmission, and our natural interest moves to the evaluation of the signal complexity of such systems. Up to our knowledge, although the discussion on the structure of the incidence matrices has been reported, however, only few results of the complexity of the signals are reported [2], [1], in which the coarseness of signal quantization or the number of connections in the network is regarded as the signal complexity.

Contrary to those results [2], [1], firstly deal with the signal complexity of consensus systems via information theoretic approach, in which the signal complexity can be dealt with via entropy. Entropy of signals is defined by the probability distribution and it can be represented by its spectral density function when the signal is the output of a linear time-invariant system with an input of white Gaussian. This quantity can be also reduced to the stationary gain and unstable zeros of the transfer function.

With the above discussion, in this paper, we deal with LTI cyclic systems and try to derive the transfer function between state variables. The main focus of the discussion is on the stability of the zeros. With it we give the corresponding entropy and analyze the property of the cyclic consensus systems from the view point of the signal complexity.

The organization of this paper is as follows: In Section II, we derive transfer functions of simple one-layered consensus systems with single-reference for the cases of continuous time systems and discrete time systems, respectively. We then also examine a case of multi-reference consensus systems. Section III is the main part of this paper, where we strictly give the differential entropy of the state variables in the one-layered consensus systems. We also discuss the properties of the cyclic consensus networks from the view point of the derived entropy. In Section IV, we try to extend the results to the multi-layered cyclic consensus systems. We give a partial result and show several numerical simulations which give an insight on the signal complexity and the network structure. Section V is the conclusion of this paper.
II. CYCLIC CONSENSUS SYSTEM

In this paper, we deal with the following continuous time linear systems:
\[ \dot{x}(t) = Ax(t), \quad (1) \]
or the corresponding discrete time systems:
\[ \frac{x[t + 1] - x[t]}{\delta} = Ax[t], \quad 0 < \delta < 1, \quad (2) \]
where \( \delta \) represents an intensity of the feedback. The matrix \( A \)
is in a class of circulant matrices defined below: In general, an \( n \times n \)-dimensional matrix \( M \) is called circulant when it has the following structure of the elements:

\[ M = \begin{bmatrix}
m_0 & m_1 & m_2 & \cdots & m_{n-2} & m_{n-1} \\
m_{n-1} & m_0 & m_1 & \cdots & m_{n-3} & m_{n-2} \\
m_{n-2} & m_{n-1} & m_0 & \cdots & m_{n-4} & m_{n-3} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
m_2 & m_3 & m_4 & \cdots & m_0 & m_1 \\
m_1 & m_2 & m_3 & \cdots & m_{n-1} & m_0 \\
\end{bmatrix} =: \text{circ} (m_0, m_1, \ldots, m_{n-1}). \quad (3) \]

We call (1) or (2) as a cyclic consensus system, which are composed of state variables regarded as agents and only the differences between the value of each agent and the other state variables are returned to update the agent as a feedback in order to converge to the identical value of all agents.

The advantage of such system is that the structure is simple and it does not require any additional centralized system to compute some specific value such as the average of the whole system. This structure attracts interests of several research groups and is applied to control problems such as cyclic pursuit [7] or a kind of formation control [16], [15].

The eigenvalues and the eigenvectors of the matrix (3) are well investigated [18]. At first, define a unit complex number \( \omega \) by \( \omega := e^{\pi i/2} \). Then, the eigenvalue \( \lambda_i \) and the corresponding eigenvector \( \phi_i \) of the matrix \( M \) are represented by

\[ \lambda_i = m_0 (\omega^i)^0 + m_1 (\omega^i)^1 + \cdots + m_{n-1} (\omega^i)^{n-1}, \quad (5) \]

\[ \phi_i = [(\omega^i)^0, (\omega^i)^1, \ldots, (\omega^i)^{n-1}]^T, \quad (6) \]

\( i = 0, 1, \ldots, n - 1, \)

respectively.

Contrast to the case of the poles of the system (1) or (2), there is not enough results on the zeros of the transfer functions, although they are essential for finding the complexity of signals in the systems. This fact motivates our interest and discussion in the following of this paper.

In the following of this section, we derive transfer functions of some typical, however important cyclic consensus systems, which are preliminaries for the main results in Section III and IV.

A. Continuous time system

The most simple structure of the exchange of signals between agents for the consensus systems is given as

\[ A = \text{circ} (-1, 1, 0, \ldots, 0). \quad (7) \]

This system is simple, however essential as discussed in [7], [16], [15]. We call this as a one-layered consensus system with single-reference. In this subsection, we deal with the system (1) with (7) and find the transfer functions \( F_i(s) \) from the state variable \( x_j \) to \( x_i \).

The transfer function matrix from \( x \) to \( x \), i.e. \( (F_i(s)) \), is also cyclic, therefore, in order to find \( (F_i(s)) \), it is enough to find the transfer functions from \( x_j \) to \( x_i \), \( i = 1, \ldots, n \). Moreover, for convenience, we define a matrix \( A' \):

\[ A' = \text{circ} (0, 1, 0, \ldots, 0) \quad (8) \]

for a substitution of \( A \) in the system (1). Then, the transfer functions \( F_{i1}'(s) \) from \( x_j \) to \( x_i \) are represented as

\[ F_{i1}'(s) = \frac{c_i \text{adj}(sI - A')b_1}{\text{det}(sI - A')} \quad (9) \]

where \( b_1 := [1 \ 0 \ \cdots \ 0]^T \) and \( c_i \) represents a vector where only \( i \)th element is 1.

The following is Faddeev’s formula for a general matrix \( V \) in order to calculate the denominator and the numerator of \( (sI-V)^{-1} \): \( \text{det}(sI-V) = s^n + \alpha_1 s^{n-1} + \cdots + \alpha_n \) and \( \text{adj}(sI-V) = s^{n-1}I + s^{n-2}W_1 + \cdots + sW_{n-2} + W_{n-1} \) where \( Y_1 = VW_{n-1}, \quad Y_i = W_i + \alpha_i I, \quad i \geq 1, \quad W_0 = I. \)

By the substitution \( V = A' = \text{circ} (0, 1, 0, \ldots, 0) \), then, we get \( (Y_1 = A', \quad \alpha_1 = 0, \quad W_1 = A'), \quad (Y_2 = A'^2, \quad \alpha_2 = 0, \quad W_2 = A'^2), \ldots, \quad (Y_{n-1} = A'^{n-1}, \quad \alpha_{n-1} = 0, \quad W_{n-2} = A'^{n-1}), \quad (Y_n = A'^n = I, \quad \alpha_n = -1, \quad W_{n-1} = 0) \), therefore,

\[ \text{det}(sI - A') = s^n - 1 \quad (10) \]

\[ \text{adj}(sI - A') = s^{n-1}I + s^{n-2}A' + s^{n-3}A'^2 + \cdots + sA'^{n-2} + A'^{n-1}. \quad (11) \]

This implies \( F_{11}'(s) = \frac{s^{n-1}}{s^n - 1}, \quad F_{21}'(s) = \frac{1}{s^n - 1}, \quad F_{n1}'(s) = \frac{s^{n-2}}{s^n - 1} \).

With the relation: \( A = A' - I \), which implies \( s \) in \( F_{i1}'(s) \) is replaced by \( s + 1 \) to derive \( F_{i1}(s) \), and the symmetric structure of the cyclic system, we get

\[ F(s) = \frac{1}{p(s)} \text{circ} \ (q^{n-1}(s), q^{n-2}(s), \ldots, q^1(s), q^0(s)) \quad (12) \]

\[ q(s) = s + 1, \quad p(s) = (s + 1)^n - 1. \quad (13) \]

From (12) we know that the zeros are located only at -1 or at the infinity. This implies that the all transfer functions between any state variables in the cyclic system (1) with the matrix \( A \) of (7) are minimum phase.

B. Discrete time system

With the discussion of the previous subsection, we next deal with the corresponding discrete time systems:

\[ \frac{x[t+1] - x[t]}{\delta} = Ax[t], \quad 0 < \delta < 1, \quad (14) \]

where \( A \) is given by (7) and \( \delta \) represents a feedback intensity. Let \( z \) denote the time advance operator, then we get

\[ zx[t] = (I + \delta A)x[t] \quad (15) \]
and the transfer function from \( x[t] \) to itself is represented by
\[
F(z) = (zI - (I + \delta A))^{-1}. \tag{16}
\]
Therefore, the individual transfer function from \( x_1 \) to \( x_i \) is given as
\[
F_{1i}(z) = \frac{c_1 \text{adj}(zI - (I + \delta A))b_1}{\det(zI - (I + \delta A))}. \tag{17}
\]
By employing the results of the previous subsection, we get
\[
F(z) = \frac{1}{p(z)} \text{circ}(q^{n-1}(z), q^{n-2}(z), \ldots, q^1(z), q^0(z)) \tag{18}
\]
\[
q(z) = z - (1 - \delta), \quad p(z) = (z - (1 - \delta))^n - \delta^n \tag{19}
\]
This implies that the zeros are located at \( 1 - \delta \), therefore, the all transfer functions from any state variables to them are also minimum phase. This fact is significant in the following discussion of this paper.

C. Example of multi-reference consensus systems

In the previous subsections, we deal with a basic case of consensus systems; single-layered consensus systems with single-reference. An optional situation is to feed back the ratios between the quantities \( m_i \) and \( m_2 \), however, unfortunately, we cannot derive the solution for the general case. In order to give an insight for this problem, in this subsection, we focus on an example; however it is suggestive, as
\[
A = \text{circ} \left( -\sum_{i=1}^{n-1} m_i, m_1, m_2, \ldots, m_{n-1} \right), \tag{20}
\]
where \( m_i \) are any positive numbers. We call it a multi-reference consensus system.

Our interest in this subsection is to find the relationship between the transfer functions or the stability of the zeros and the ratios between the quantities \( m_i \), however, unfortunately, we cannot derive the solution for the general case. In order to give an insight for this problem, in this subsection, we focus on an example; however it is suggestive, as
\[
A = \text{circ} \left( -(m_1 + m_2), m_1, m_2, 0 \right), \tag{21}
\]
where \( m_1 \) and \( m_2 \) are any positive numbers.

From (5) and (6), the transfer function from \( x_1 \) to \( x_1 \) is represented as follows:
\[
F_{11}(s) = c_1(sI - A)^{-1}b_1 = \frac{1}{4}c_1\Psi^*(sI - \Lambda)^{-1}\Psi b_1
\]
\[
= \frac{1}{4} \left( \frac{1}{s - \lambda_1} + \frac{(\omega^*)^{-1}}{s - \lambda_2} + \frac{((\omega^*)^{-1})^2}{s - \lambda_3} + \frac{((\omega^*)^{-1})^3}{s - \lambda_4} \right). \tag{23}
\]
Hereafter we examine each case of \( F_{1i}(s), i = 1, 2, 3, 4 \).

Case of \( F_{11}(s) \): In this case, the direct calculation with (25) gives the transfer function:
\[
F_{11}(s) = \frac{1}{4} \left( \frac{1}{s - \lambda_1} + \frac{1}{s - \lambda_2} + \frac{1}{s - \lambda_3} + \frac{1}{s - \lambda_4} \right). \tag{24}
\]
From the fact that \( \text{Re}[\lambda_i] \leq 0, \text{Re}[s - \lambda_i] > 0 \) for \( s \) (\( \text{Re}[] \) means the real part of \( \cdot \) in the open right half plane, therefore, \( \text{Re}[\frac{1}{s - \lambda_i}] < 0, \forall i \). This implies that the numerator of (24) is stable. Moreover, we can also see the numerator is the 3rd order monic polynomial.

Case of \( F_{21}(s) \): The direct calculation with (23) gives the transfer function:
\[
F_{21}(s) = \frac{1}{4} \left( 2m_1((s + m_1 + 2m_2)^2 + m_1^2) + 2m_1s(s + 2m_1) \right), \tag{25}
\]
We can observe that the numerator is a 2nd order polynomial and the all coefficients are positive. This implies that the numerator of (25) is stable. Moreover, we can also see the leading coefficient of the numerator is \( m_1 \).

Case of \( F_{31}(s) \): The direct calculation with (23) gives the transfer function:
\[
F_{31}(s) = \frac{m_2s^2 + Ms + m_1M}{s(s + 2m_1)((s + m_1 + 2m_2)^2 + m_1^2)}. \tag{26}
\]
We can observe that the numerator is a 2nd order polynomial and the all coefficients are also positive. This implies that the numerator of (26) is stable. In this case, we can also see the leading coefficient of the numerator is \( m_2 \).

Case of \( F_{41}(s) \): The direct calculation with (23) gives the transfer function:
\[
F_{41}(s) = \frac{2m_1m_2s + m_1(m_1 + m_2)^2}{s(s + 2m_1)((s + m_1 + 2m_2)^2 + m_1^2)}. \tag{27}
\]
We can observe that the numerator is a 1st order polynomial and the coefficients are also positive. This implies that the numerator of (27) is stable. In this case, we can also see the leading coefficient of the numerator is \( 2m_1m_2 \).

From the above calculations, the multi-reference consensus system with (21) is always minimum phase for any \( m_1 > 0 \) and \( m_2 > 0 \). In the case of the discrete time systems, which can be given in the form of (14), the similar situation also holds by setting an appropriate positive number \( \delta \); depending on the any parameter \( m_1 > 0 \) and \( m_2 > 0 \), an
enough small $\delta > 0$ always gives (marginally) stable and minimum phase discrete time consensus systems.

The ratio of these parameters is related to the signal complexity in the consensus networks and it is explained in Section III.

III. SIGNAL COMPLEXITY IN CYCLIC CONSENSUS SYSTEMS

A. Case of single-reference consensus systems

In this section, we find signal complexity in the discrete time single-reference consensus system (14) with (7) shown in the previous section. Hereafter we suppose that artificial Gaussian noises $v_i$ are generated by $v_i(t) = \frac{z_i}{\delta} v_i(t)$, where $w_i$ are mutually independent white Gaussian noises with a unit variance, and added to the corresponding state variables $x_i$ in the system (14) such as in Fig. 1. These noises are a model to describe finite channel capacity in the signal transmission between the agents, finite bit-length of memory for representing the state variable, or exogenous noise (Fig. 1 represents an example of a block diagram for the situation).

Fig. 1: An example of cyclic consensus systems with exogenous signals

In general, entropy $H(g)$ for a discretized stochastic variable $g \in \{ g_1, g_2, \ldots, g_M \}$ is defined by $H(g) = \sum_{i=1}^{M} -P(g_i) \log P(g_i)$, where $P(g_i)$ represents the probability of $g_i$. The corresponding notion for a continuous variable $g$ is the following differential entropy: $h(g) := \int -p(g) \log p(g) d\omega$, where $p(g)$ represents the probability density of $g$. When a discrete time signal $g[t]$ is Gaussian and the power spectrum is given by $|G(e^{j\omega})|^2$, its differential entropy $h(g)$ is equal to the following [4], [13]:

$$h(g) = \frac{1}{2 \pi} \int_{-\pi}^{\pi} \log |G(e^{j\omega})| d\omega + \frac{1}{2} \log(2\pi e).$$  (28)

The differential entropy is known to represent the stochastic complexity of continuous signals [13].

In the followings, we show the stochastic complexity of the state variables $x_i$ under the situation of Fig. 1. We get the next theorem:

**Theorem 3.1:** For the system (14) with (7), denote the differential entropy of the state variables by $h(x_i)$. Then, $h(x_i)$ is given by

$$h(x_i) = \frac{1}{2} (\log n + \log(2\pi e)), \ \forall i.$$  (29)

**Proof:** In general, the differential entropy (28) is equal to the following [14]:

$$h(g) = \log |G'(\infty)| - \sum_{i} \log |\alpha_i| + \frac{1}{2} \log(2\pi e),$$  (30)

where $G'(z) = z G(z)$, $r$: relative degree of $G(z)$, $\alpha_i$: unstable zeros (outside of the unit circle) of $G(z)$.

In the case of (18) with $w_1, w_2, \ldots$, which are mutually independent white Gaussian noises with a unit variance, we get

$$h(x_i) = \frac{1}{2 \pi} \int_{-\pi}^{\pi} \log \left( \sum_{j=1}^{n} \left| F_{ij}(e^{j\omega}) \frac{e^{j\omega} - 1}{e^{j\omega}} \right|^2 \right)^{\frac{1}{2}} d\omega + \frac{1}{2} \log(2\pi e).$$  (31)

Note that $F_{ij}(z) \frac{z-1}{z}$ is asymptotically stable and minimum phase, therefore there always exists a factorization [8]:

$$\left( \sum_{i=1}^{n} \left| F_{ij}(e^{j\omega}) \frac{e^{j\omega} - 1}{e^{j\omega}} \right|^2 \right)^{\frac{1}{2}} = |\tilde{f}_j(e^{j\omega})|,$$  (32)

where $\tilde{f}_j(z)$ is also minimum phase. In the case of (18), we can also calculate $|\tilde{f}_j(\infty)| = n^{\frac{1}{2}}$.

Finally, from (28)-(30), we can get $h(x_i) = \log |\tilde{f}_j(\infty)| + \frac{1}{2} \log(2\pi e) = \frac{1}{2} (\log n + \log(2\pi e))$.

From this theorem, when the number of agents $n$ increases, the complexity of each signal increases in the order of $\log n$. This fact tells two opposite phases of this simple cyclic consensus systems. In general, cyclic consensus systems are regarded such as the number of connections of agents is small and then, the quantity of signals is also so. However, in the information theoretic sense, the complexity of a signal increases in monotone of the order $\log n$ and too large-scaled systems should be avoided. On the other hand, the all transfer functions of this cyclic consensus system are minimum phase, and therefore, the increase of the signal complexity is fully restrained compared to the other possible systems. The reason is explained as follows:

In general, let a signal $g[t]$ be a sum of $g_i[t], \ i = 1, 2, \ldots, n$, which are generated by source signals $w_i[t]$ with the corresponding transfer functions $f_i(z)$ as

$$g[t] = \sum_{i=1}^{n} g_i[t], \ g_i = f_i(z)w_i, \ i = 1, 2, \ldots, n,$$

where $w_i[t], \ i = 1, \ldots, n$, are mutually independent white Gaussian noises with a unit variance and $f_i(z)$ are not necessarily minimum phase. Then, we can derive the following formula: $h(\sum_{i=1}^{n} g_i[t]) = \frac{1}{n} \int_{-\pi}^{\pi} \log \left( \sum_{i=1}^{n} |f_i(e^{j\omega})|^2 \right)^{\frac{1}{2}} d\omega + \frac{1}{2} \log(2\pi e)$. By applying the following inequality: $\log \left( \sum_{i=1}^{n} |f_i(\infty)|^2 \right)^{\frac{1}{2}} \geq \frac{1}{n} \sum_{i=1}^{n} \log |f_i(\infty)| + \frac{1}{2} \log n$, we get

$$h \left( \sum_{i=1}^{n} g_i[t] \right) \geq \frac{1}{n} \sum_{i=1}^{n} \log |f_i(\infty)| - \sum_{j} \log |\alpha_{ij}| + \frac{1}{2} \log n + \frac{1}{2} \log(2\pi e),$$  (33)
where $\alpha_{ij}$ are unstable zeros of $f'_i(s)$. In the case that the number of the unstable zeros of $f_i$ obeys $O(n)$, we can derive the following bound:

$$h(g[t]) = h\left(\sum_{i=1}^{n} g_i[t]\right) \geq O(0) + O(n) + \frac{1}{2} \log n + \frac{1}{2} \log(2\pi e). \quad (34)$$

Compared to this bound, the differential entropy (29) for the system (14) with (7) does not have a term $O(n)$ and the signal complexity is fully restrained.

B. Case of multi-reference consensus systems

Now we turn to the case of (21). In Section II-C, we showed that the transfer functions $F_{i1}(s)$, $i = 1, 2$, are minimum phase and the corresponding discrete time systems are also so by setting $\delta$ enough small. In the same setting in Section III-A, we can get the differential entropy as

$$h(x_i) = \log |f'_i(\infty)| + \frac{1}{2} \log(2\pi e) = \frac{1}{2} \left(\log(1 + \delta^2 m_1^2 + \delta^2 m_2^2 + 4\delta^4 m_1^2 m_2^2) + \log(2\pi e)\right). \quad (35)$$

Under the condition: $\delta^2 m_1^2 + \delta^2 m_2^2 = \text{constant} =: K$, the minimizer $m_1$ and $m_2$ for (35) is given as $\delta m_1 = K^{\frac{1}{2}}$, $m_2 = 0$ or $m_1 = 0$, $\delta m_2 = K^{\frac{1}{2}}$. Moreover, the maximizer is $\delta m_1 = \delta m_2 = (\frac{1}{2} K)^{\frac{1}{2}}$. In the latter case, $h(x_i) = 2n \times \frac{1}{2} \left(\log(1 + K) + \log(2\pi e)\right) \geq \log n$ because of the number of paths of signals in the system is $n$, however, in the latter case, $h(x_i) = 2n \times \frac{1}{2} \left(\log(1 + K + K^2) + \log(2\pi e)\right)$ because of the number of paths of signals in the system is $2n$. From these observations, we can conclude that the signal complexity can be restrained in the single-reference consensus systems given with (7) compared to the multi-reference consensus systems.

In the above discussion, note that the convergence-rate of the state variables of the systems is ignored which may affect on the total signal complexity under some appropriate problem settings. This problem is left for the future work.

IV. A CASE OF MULTI-LAYERED CYCLIC CONSENSUS SYSTEMS

The following of this paper is devoted to an analysis on the stability of zeros in the multi-layered cyclic consensus systems, which are regarded as a model for a flocking of biological swarms. The research focus of the signal complexity of such systems is the relationships between the complexity and the hierarchical structure of the consensus networks.

Smith et al. defined multi-layered cyclic consensus systems [16], [15] (we call them as multi-layered cyclic consensus systems):

$$\dot{x} = A_L x, \quad A_1 = -I + P_{n_1}, \quad n = \sum_{i=1}^{L} n_i$$

$$A_i = I \otimes (A_{i-1} - I) + P_{n_i} \otimes I, \quad i = 2, 3, \ldots, L. \quad (36)$$

The matrix $A_L$ has a property of block-circulant [18]. In [16], [15], the poles of this system are rigorously derived, however, even if $P$, is of a simple case (8), the corresponding transfer function $F(z)$ and the numerator polynomials are in a comparatively complex form and then, it is a hard problem to find the locations of the zeros.

Hereafter, we explain the partial results on the locations of the zeros of $F(z)$. We can derive the following proposition:

Proposition 4.1: Let the matrix $P$ be a circulant matrix of a form (8). Then, the transfer functions $F_i(s)$, $i = 1, 2, \ldots, n$ of (36) are minimum phase.

Proof: From the symmetric structure of the system, it is enough to show the case of $F_{11}$. From [18], the eigenvalues of $A_L$ are $\lambda_i = -L + \omega_{n_i}^2 + \omega_{n_i}^4 + \cdots + \omega_{n_i}^{2L}$, $\omega_{n_i} = e^{i\frac{2\pi}{n_i}}$, $\lambda_i = 0, 1, \ldots, n_i - 1$, $i = (t_1, t_2, \ldots, t_L)$. This shows that the all eigenvalues $\lambda_i$ are located on a disc centered at $-L$ with a radius $L$. The eigenvectors corresponding to (6) in this case of block-circulant matrices are in a form of recursive blocked-power [18] and

$$\phi(0,0,\ldots,0) = [1 \quad 1 \quad \cdots \quad 1]^T. \quad (37)$$

The transfer function $F_{11}(s)$ is given by

$$F_{11}(s) = c_1(sI - A_L)^{-1}b_1 = c_1U^*(sI - \Lambda)^{-1}Ub_1 \quad (38)$$

where

$$V = \text{diag}(\lambda(0,0,\ldots,0), \lambda(0,0,\ldots,1), \ldots, \lambda(n_L - 1, n_L - 1, \ldots, n_L - 1)),$$

$$U = \left[\phi(0,0,\ldots,0), \phantom{0} \phi(0,0,\ldots,1), \cdots, \phi(0,0,\ldots,n_L - 1, \ldots, n_L - 1)\right]. \quad (39)$$

therefore, with (37), we can get

$$F_{11}(z) = \sum_{i=1}^{L} \frac{1}{s - \lambda_i}. \quad (40)$$

For $s$ satisfying $\text{Re}(s) > 0$, $\text{Re}(\delta - \lambda_i) > 0$, $\forall i$, and $\text{Re}((\hat{\delta} - \lambda_i)^{-1}) > 0$, $\forall i$. Therefore, we get $\text{Re}(F_{11}(s)) > 0$. This implies that $F_{11}(s) \neq 0$ in the right half plane.

We can also derive the corresponding result in a discrete time system:

Corollary 4.1: Let the matrix $P$ be a circulant matrix of a form (8). Then, in the discrete time system (14) with $A = A_L$, the transfer functions $F_i(z)$, $i = 1, 2, \ldots, n$ are minimum phase.

Unfortunately, we cannot derive any formula on the locations of the zeros in the off-diagonal transfer functions. However, a numerical example suggests insight on the locations. Fig. 2.1–6 show the locations of the poles and the zeros of $F_{11}(s)$ calculated by computer, where $F(s)$ is a continuous time two-layered cyclic system (36) with

$$P_1 = P_2 = \text{circ} (0, 1, 0, 0, 0, 0, 0). \quad (41)$$

Fig. 2.1 shows that the locations of the zeros of $F_{11}(s)$ are limited in the left half plane and it supports Proposition 4.1. We also find the zeros are located inside the disc. Fig. 2.2 shows that some of the zeros of $F_{21}$ are located outside the disc in a radial pattern and two zeros are located
in the right half plane. On the other hand, in Fig. 2.(3), all zeros of $F_{61}$ are again located inside the disc. We can explain this fact as follows: The variable $x_2$ is most apart from $x_1$ inside the cluster of the lower layer, in which $x_1$, $x_2$ and $x_6$ are included, and this causes phase lag. On the other hand, $x_6$ is most close to $x_1$. Therefore, the complexity of $x_2$ caused by $x_1$ becomes higher than that of $x_6$.

![Fig. 2.1](image1)

![Fig. 2.2](image2)

![Fig. 2.3](image3)

![Fig. 2.4](image4)

![Fig. 2.5](image5)

![Fig. 2.6](image6)

Fig. 2.(1)–(6): The locations of poles and zeros of $F_{11}$, $F_{21}$, $F_{61}$, $F_{71}$, $F_{31 1}$, and $F_{11 i}$, $i = 1, \ldots, 36$ with (41) (poles: marked by ‘o’, zeros: marked by ‘+’).

Fig. 2.(4) shows that some of the zeros of $F_{71}$ are located outside the disc and two zeros are located in the right half plane. The variable $x_2$ is included in the most apart cluster from that of $x_1$ and then, we can interpret that the signal complexity becomes high. On the other hand, in Fig. 2.(5), the all zeros of $F_{31 1}$ are again located inside the disc. The cluster in which $x_{31}$ is included is close to that of $x_1$ and we can interpret the signal complexity is low in this case.

Finally, Fig. 2.(6) shows the all locations of zeros of $F_{11 i}$, $i = 1, 2, \ldots, 36$. The locations are like a simple radial shape, however, they are actually more complex.

As explained in Section III, unstable zeros in the transfer functions increase the signal complexity. In 1-layered cyclic systems, the signal connection is not partially uniform, therefore, there is a case that the signal complexity cause by the other variables is different as in the example.

V. CONCLUSION

In this paper, we dealt with some classes of cyclic consensus systems and then investigated their transfer functions or the locations of the zeros. By employing this result, we discussed the information theoretic signal complexity in the network of the systems and succeeded to give a lower bound on their differential entropy which increases of order $\log n$ in 1-layered cyclic consensus systems. We also discuss the signal complexity of multi-layered cyclic consensus systems.

REFERENCES


