A Decoupled Inversion-based Iterative Control Approach to Multi-axis Precision Positioning: 3-D Nanopositioning Example

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Abstract—This article proposes the multi-axis inversion-based (MAIIC) approach. System inverse provides a nature avenue to utilize the priori knowledge of system dynamics in iterative learning control, resulting in rapid convergence as well as exact tracking (for nonminimum-phase systems). The benefits of system inverse, however, are not fully exploited in time-domain ILCs due to the lack of uncertainty quantification. This critical limit was removed in the inversion-based iterative control (IIC) techniques through a frequency-domain formulation. The existing IIC techniques, however, is limited to single-input-single-output (SISO) systems, and the time-domain properties of the IIC techniques are not clear. The contribution of this article is: First, the IIC technique is extended from SISO systems to multi-input-multi-output systems, and the convergence condition is analyzed. Secondly, the time-domain properties of the MAIIC law are discussed. It is shown that the set of tractable frequencies is characterized by the bounds of system uncertainty that are quantifiable in practices, and the truncated MAIIC input-output convergences to the neighborhood of the desired input-output truncated in time, where arbitrarily small tracking error can be obtained by having a large enough truncation time. The proposed MAIIC technique is illustrated in a 3-D nanopositioning experiment using piezoelectric actuators.

I. INTRODUCTION

This paper proposes an inversion-based iterative control (IIC) approach for multi-axis precision positioning. In the last decade, the role of system inverse in iterative learning control (ILC) has been unveiled and recognized [1], [2]. Particularly, the system inverse is utilized in a frequency-domain framework in the recently developed IIC techniques [3]-[6], which provides a straightforward and systematic avenue to exploit the a priori knowledge of system dynamics on one hand, and account for the dynamics uncertainty on the other. The efficacy of the IIC techniques has been demonstrated in various high-speed precision positioning applications [3], [7]. However, currently the IIC techniques are limited to single-input-single-output (SISO) systems. Moreover, the time-domain properties of the IIC techniques—with their frequency-domain representation—have not been clearly understood. Thus, this paper aims to (1) extend the IIC technique proposed in [3] to multi-input-multi-output (MIMO) systems that have a dominant input to each output channel (i.e., the input that is more influential than the combined cross-coupling inputs), and (2) characterize the time-domain properties of the IIC algorithms.

It is advantageous to incorporate system inverse in the ILC framework. We note that in many ILC applications, the a priori knowledge of the system dynamics is available, thereby can be utilized for iterative control. System inversion is a nature choice because the use of system inverse in ILC law results in rapid convergence [1], [6], very much desirable in practical implementations [8]. Moreover, system inverse becomes particularly critical to output tracking of nonminimum-phase systems, for which noncausal ILC law is necessary to achieve precision tracking [2], [1], [9]. The advantages of system inversion, however, are not fully exploited in time-domain based ILC approaches [1], [10], where the system dynamics uncertainty is not quantified. Hence, the iteration coefficient has to be chosen small enough, and thereby, rather conservative, resulting in slow convergence. These system uncertainty related constraints are removed through the development of the IIC techniques [3]-[6]. By formulating the IIC law in frequency-domain, the bound of system dynamics uncertainty can be quantified in a systematic and straightforward manner [3]-[5] resulting in a rapid convergence. Thus, the utilization of system inverse in the frequency-domain IIC techniques provides features desirable in practical implementations.

Limits, however, exist in the existing IIC techniques. The IIC techniques [3]-[6], as presented succinctly in frequency domain, are developed for SISO systems only. The direct extension of the IIC techniques to MIMO systems, however, can results in loss of performance, because of the quantification of the dynamics uncertainty of MIMO systems, however, tends to be over conservative, resulting in not only slow convergence, but also a small set of tractable frequencies. Moreover, although the development of the IIC techniques provides unique insights into the ILC framework by determining the tractable frequencies in terms of system dynamics properties, the set of tractable frequencies as well as the time-domain properties of the IIC techniques are not characterized yet. The effect of such frequency-time truncation on the IIC input as well as the tracking performance is yet to be clarified. Therefore, there exists a need to further extend the IIC techniques.

The main contribution of this paper is to extend the IIC technique [3] to square MIMO systems in a straightforward and simple manner. Rather than inverting the transfer matrix of the entire MIMO system, a diagonal matrix consisting of the inverse of the dominant dynamics for each output is used in the proposed multi-axis IIC (MAIIC) law. It is

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shown that the proposed MAIIC converges at frequencies where the diagonal subdynamics has (1) a gain larger than the combined gain of other cross-coupling inputs, and (2) a phase uncertainty smaller than a bound given by the combined cross-coupling dynamics effect. The allowable cross-coupling effect and the set of tractable frequencies are characterized and quantified, and the optimal iteration coefficient is obtained that minimizes the number of iterations for given bound of system uncertainties. Secondly, the time-domain properties of the MAIIC law are addressed for practical implementations. We show that the set of tractable frequencies is compact, thereby time-domain truncation is needed in practical implementation of the MAIIC law. The difference between the MAIIC input-output truncated in time and the desired input-output is bounded by the length of the truncation time window, hence, can be rendered arbitrarily small by having a large enough truncation window. Finally, the proposed MAIIC technique is illustrated in experiments by a 3-D nanopositioning application using piezoelectric actuators. The experimental results obtained demonstrate the efficacy of the proposed approach.

II. MULTI-AXIS INVERSION-BASED ITERATIVE CONTROL

In this section, we extend the IIC technique originally developed for SISO systems in Ref. [3] to MIMO systems.

A. Multi-axis Inversion-based Iterative Control

Considering a square MIMO system $G(j\omega): \mathbb{C} \rightarrow \mathbb{C}^{n \times n}$,

$$\dot{\hat{Y}}(j\omega) = G(j\omega)\hat{U}(j\omega),$$

where

$$G(j\omega) = \begin{bmatrix} G_{11}(j\omega) & G_{12}(j\omega) & \cdots & G_{1n}(j\omega) \\ G_{21}(j\omega) & G_{22}(j\omega) & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ G_{n1}(j\omega) & G_{n2}(j\omega) & \cdots & G_{nn}(j\omega) \end{bmatrix}$$

represents the actual linear dynamics of the entire MIMO system, and

$$\dot{\hat{Y}}(j\omega) = \begin{bmatrix} \hat{y}_1(j\omega) \\ \hat{y}_2(j\omega) \\ \vdots \\ \hat{y}_n(j\omega) \end{bmatrix}^T,$$

$$\hat{U}(j\omega) = \begin{bmatrix} \hat{u}_1(j\omega) \\ \hat{u}_2(j\omega) \\ \vdots \\ \hat{u}_n(j\omega) \end{bmatrix}^T.$$  \tag{3}

The proposed MAIIC law is described in frequency domain as follows,

$$\hat{U}_{0}(j\omega) = 0$$

$$\hat{U}_k(j\omega) = \hat{U}_{k-1}(j\omega) + \rho(j\omega)G^{-1}_{I,m}(j\omega)(\hat{Y}_d(j\omega) - \hat{Y}_{k-1}(j\omega))$$

where $G_{I,m}$ is a diagonal matrix with the diagonal elements being the model of the diagonal subsystems of system $G(j\omega)$, obtained in the $m^{th}$ modeling process, i.e., for $m \in \mathbb{N}$ ($\mathbb{N} = \{1, 2, \ldots, n\}$),

$$G_{I,m}(j\omega) = \text{diag} \{ G_{11,m}(j\omega), \ldots, G_{nn,m}(j\omega) \},$$

$$\rho(j\omega) = \text{diag} \{ \rho_1(j\omega), \rho_2(j\omega), \ldots, \rho_n(j\omega) \}.$$  \tag{5}

is the frequency-dependent iteration coefficient matrix where the diagonal element $\rho_p(j\omega) \in \mathbb{R}^+$ for each $p \in \mathbb{N}$. In (4), $\hat{U}_k(j\omega), \hat{Y}_k(j\omega)$ denote the input and the output of the system in the $k^{th}$ iteration, respectively,

$$\hat{U}_k(j\omega) = \begin{bmatrix} \hat{u}_{1,k}(j\omega) \\ \hat{u}_{2,k}(j\omega) \\ \vdots \\ \hat{u}_{n,k}(j\omega) \end{bmatrix}^T,$$

$$\hat{Y}_k(j\omega) = \begin{bmatrix} \hat{y}_{1,k}(j\omega) \\ \hat{y}_{2,k}(j\omega) \\ \vdots \\ \hat{y}_{n,k}(j\omega) \end{bmatrix}^T.$$  \tag{7}

Assumption 1: The desired trajectory $Y_d(\cdot)$ is in $L^2$, i.e., $Y_d(\cdot) \in L^2$, and for system (1), exact output tracking of the desired trajectory $\hat{Y}_d(j\omega)$ exists at any given frequency $\omega$, i.e., there exists a desired input $U_d(\cdot)$ such that

$$\hat{Y}_d(j\omega) = G(j\omega)U_d(j\omega).$$  \tag{8}

Assumption 2: System (1) and its model are both proper, stable, and hyperbolic, i.e., for all $p, q \in \mathbb{N}$ and $m \in \mathbb{N}$, both $G_{pq}(j\omega)$ and $G_{pq,m}(j\omega)$ are proper, stable, and hyperbolic (i.e., have no pure imaginary zeros).

Assumption 3: During each iteration, the iterative control input $U_k(\cdot)$ is applied to system (1) under the same initial condition as that for the desired input $U_d(\cdot)$.

B. Convergence Analysis of the MAIIC Law

We shall present the convergence of the proposed MAIIC law in steps. We start with clarifying the convergence condition of the MAIIC law at one given frequency $\omega$, and with no truncation of both the iterative input and the output to finite time window (The truncation constraint will be relaxed later).

Lemma 1: Let Assumptions 1 to 3 be satisfied. Moreover, assume that no truncation is applied to either the iterative control input nor the corresponding output.

Then, at any given frequency $\omega$, the system output $Y_k(\omega)$ converges to the desired output $Y_d(\omega)$, i.e. $\lim_{k \rightarrow \infty} Y_k(j\omega) = Y_d(j\omega)$, if and only if

$$|\Lambda(j\omega)|_\infty < 1,$$  \tag{9}

where $\Lambda(j\omega) \triangleq \mathcal{E} - \rho(j\omega)G_{I,m}^{-1}(j\omega)G(j\omega)$.

The MAIIC law (4) is decoupled, which substantially simplifies the implementation of the MAIIC law. Next, we define the “tractable” set $E_G$ as below.

Definition 1: The “tractable” set $E_G$ is defined as

$$E_G \triangleq \bigcap_{p=1}^{n} E_p$$  \tag{10}

where $E_p$ denotes the set of frequencies given by

$$E_p = \{ \omega \in \mathbb{R}^1 | C_p(j\omega) < 1, \quad |\angle G_{pq}(j\omega)| < \arccos(C_p(j\omega)) \},$$  \tag{11}

with $C_p(j\omega)$ the relative combined cross-axis coupling effect with respect to the gain of the diagonal subsystem in the $p^{th}$ axis, $G_{pp}(j\omega)$ and $\angle G_{pq}(j\omega)$ (for $p = q$) the phase part of the system uncertainty for subsystem $G_{pq}(j\omega)$,

$$C_p(j\omega) \triangleq \frac{1}{G_{pp}(j\omega)} \sum_{q=1, q \neq p}^{n} |G_{pq}(j\omega)|,$$  \tag{12}

$$\Delta G_{pq}(j\omega) \triangleq \frac{G_{pq}(j\omega)}{G_{pp}(j\omega)} = |\Delta G_{pq}(j\omega)| e^{j\angle \Delta G_{pq}(j\omega)}.$$

Furthermore, we assume that

Assumption 4: For the square system $G(j\omega)$, the measure of the “tractable” set $E_G$ is not zero, i.e.,

$$\mu(E_G) \neq 0,$$  \tag{13}

where $\mu(E)$ is the Lebesgue measure of set $E$ in $\mathbb{R}^1$ [11].

The above definition and assumption allow us to quantify the iteration coefficient $\rho(j\omega)$ by the system dynamics uncertainties. We quantify, next, the iteration coefficient for each individual $p$-axis.
**Lemma 2:** Let Assumption 4 and conditions in Lemma 1 be satisfied, then for any frequency in the “tractable” set, \( \omega \in E_G \), the MAIIC law (4) converges if and only if for any given \( p \in \mathbb{N} \), the iteration coefficient \( \rho_p(j\omega) \) is chosen as

\[
0 < \rho_p(j\omega) < \rho_{p,ab}(j\omega), \quad \text{where} \quad \rho_{p,ab}(j\omega) = \frac{2}{\Delta G_{pp}(j\omega)} \left[ \cos\angle \Delta G_{pp}(j\omega) - C_p(j\omega) \right],
\]

\( \Delta G_{pp}(j\omega) \) is the total cross-axis coupling dynamics effect on the possible to quantify the boundary functions (16-20).

**C. Characterization of the “Tractable” Set \( E_G \)**

First, we define the set of admissible models for any given subsystem dynamics \( G_{pq}(j\omega) \)

\[
\mathbf{S}_{pq} \triangleq \{ G_{pq,k}(j\omega) \};
\]

Thus, in practical implementations, \( G_{pq,k}(j\omega) \) represents the dynamics model from the \( q \)th input to the \( p \)th output, obtained from the \( k \)th modeling process Then, the supremum of the magnitude uncertainty and that of the phase uncertainty for the models in set \( \mathbf{S}_{pq} \), \( \Delta \rho_{pq,\text{sup}}(j\omega) \) and \( \Delta \theta_{pq,\text{sup}}(j\omega) \), respectively, are given by

\[
\Delta \rho_{pq,\text{sup}}(j\omega) \triangleq \sup_{r,k \in \mathbb{N}} \left\{ \frac{|G_{pq,r}(j\omega)|}{G_{pq,k}(j\omega)} \right\},
\]

\[
\Delta \theta_{pq,\text{sup}}(j\omega) \triangleq \sup_{r,k \in \mathbb{N}} \left\{ \angle \left( \frac{|G_{pq,r}(j\omega)|}{G_{pq,k}(j\omega)} \right) \right\};
\]

Hence, the infimum for the magnitude uncertainty among the models in set \( \mathbf{S}_{pq} \), \( \Delta \rho_{pq,\text{inf}}(j\omega) \), is given by

\[
\Delta \rho_{pq,\text{inf}}(j\omega) \triangleq \frac{1}{\Delta \rho_{pq,\text{sup}}(j\omega)}
\]

The above definitions imply that the “true” system dynamics uncertainty, as given by (12), can be bounded as

\[
\Delta \rho_{pq,\text{inf}}(j\omega) \leq |\Delta G_{pq}(j\omega)| \leq \Delta \rho_{pq,\text{sup}}(j\omega),
\]

\[
|\angle \Delta G_{pq}(j\omega)| \leq \Delta \theta_{pq,\text{sup}}(j\omega).
\]

By using the defined supremum and infimum functions, the total cross-axis coupling dynamics effect on the \( p \)th axis (12) can be bounded as

\[
C_{\text{inf}}(j\omega) \leq C_p(j\omega) \leq C_{\text{sup}}(j\omega),
\]

where for any \( r,k \in \mathbb{N} \),

\[
C_{\text{inf}}(j\omega) \triangleq \Delta \rho_{pp,\text{inf}}(j\omega) \sum_{q=1}^{n} \Delta \rho_{pq,\text{inf}}(j\omega) \inf_{s_{pq}, s_{pp}} \left| \frac{G_{pq,r}(j\omega)}{G_{pq,k}(j\omega)} \right|
\]

\[
C_{\text{sup}}(j\omega) \triangleq \Delta \rho_{pp,\text{sup}}(j\omega) \sum_{q=1}^{n} \Delta \rho_{pq,\text{sup}}(j\omega) \sup_{s_{pq}, s_{pp}} \left| \frac{G_{pq,r}(j\omega)}{G_{pq,k}(j\omega)} \right|
\]

**Remark 1:** Note that in practical implementations, it is possible to quantify the boundary functions (16-20).

**Lemma 3:** Let Assumption 2 be satisfied, then the bound functions of System (1), \( \Delta \rho_{pq,\text{sup}}(j\omega) \), \( \Delta \theta_{pq,\text{sup}}(j\omega) \), and \( C_{\text{sup}}(j\omega) \), are lower semi-continuous functions of \( j\omega \), and the bound functions \( \Delta \rho_{pq,\text{inf}}(j\omega) \) and \( C_{\text{inf}}(j\omega) \) are upper semi-continuous functions of \( j\omega \).

**Definition 2:** The practically tractable set \( \Omega_G \) is

\[
\Omega_G = \bigcap_{p=1}^{n} \Omega_p
\]

where for any given arbitrarily small positive constants \( \epsilon_{p,k} \in \mathbb{R}^+ \) with \( k = 1,2 \),

\[
\Omega_p \triangleq \{ \omega \mid C_{\text{sup}}(j\omega) \leq 1 - \epsilon_{p,1}, \quad \angle \Delta \theta_{pp,\text{sup}}(j\omega) \leq \arccos \left( C_{\text{sup}}(j\omega) \right) - \epsilon_{p,2} \}.
\]

**Lemma 4:** The practically tractable set \( \Omega_G \) is a subset of the tractable set \( E_G \), \( \Omega_G \subset E_G \), and provided that the set \( \Omega_G \) is nonempty, \( \Omega_G \) is compact.

Using the above boundary functions, we can estimate the range of the iteration coefficient \( \rho_p(j\omega) \) for each \( p \in \mathbb{N} \) that can be quantified in practical implementations (see Remark 1).

**Lemma 5:** Let Conditions in Lemma 2 be satisfied, then for each \( p \)th axis, the iteration coefficient \( \rho_p(j\omega) \) can be chosen as

\[
\rho_p(j\omega) \in \{ (0, \kappa_p(j\omega)], \omega \in \Omega_p \}
\]

where

\[
\kappa_p(j\omega) = \frac{2}{\Delta \rho_{pp,\text{sup}}(j\omega)} \cos \angle \Delta \theta_{pp,\text{sup}}(j\omega) - C_{\text{sup}}(j\omega).
\]

**Corollary 1:** Let the iteration coefficient \( \rho(j\omega) = [\rho_1(j\omega), \rho_2(j\omega), \cdots, \rho_p(j\omega)] \) be chosen according to (22) for each \( p \in \mathbb{N} \), the MAIIC input converges to the desired input \( \hat{U}_d(\cdot) \) restricted to the set \( \Omega_G \), and tracking of the desired output \( \hat{Y}_d(\cdot) \) restricted to the set \( \Omega_G \) is guaranteed,

\[
\lim_{k \to \infty} \hat{U}_d(j\omega) = \hat{U}_d(j\omega) \chi_{\Omega_G}(j\omega) \triangleq \hat{U}^*_d(j\omega),
\]

\[
\lim_{k \to \infty} \hat{Y}_d(j\omega) = \hat{Y}_d(j\omega) \chi_{\Omega_G}(j\omega) \triangleq \hat{Y}^*_d(j\omega)
\]

where \( \chi_{\Omega_G} \) denotes the characteristic function of the set \( \Omega_G \). In the following, the restricted desired input and output, \( \hat{U}^*_d(\cdot) \) and \( \hat{Y}^*_d(\cdot) \), are referred as the modified desired input and the modified desired output, respectively.

The quantification of the practically tractable set \( \Omega_G \) also allows us to seek an optimal iteration coefficient \( \rho^*_p(j\omega) \) for each \( p \in \mathbb{N} \), that solves the following minimax problem,

\[
\inf_{p} \sup_{\Delta \rho_{pp}} J(p, \Delta G_{pp}, C_p)
\]

\[
= \inf_{p} \sup_{\Delta \rho_{pp}} |1 - \rho_p |\Delta G_{pp}| + \rho_p |\Delta G_{pp}| C_p.
\]

The optimal coefficient \( \rho^*_p(j\omega) \) is given by the positive solution to the following algebraic equation that falls in (0, 2);

\[
J(p, \Delta G_{pp,sup}, C_{sup}) \triangleq J^*(p, \Delta G_{pp,sup}, C_{sup}) \triangleq \hat{j}^*(26)
\]

where the dependence on ‘j’ is omitted and \( \Delta G_{pp,sup} = \{ \Delta \rho_{pp,sup}, \Delta \theta_{sup} \} \), \( \Delta G_{pp,inf} = \{ \Delta \rho_{pp,inf}, \Delta \theta_{sup} \} \).

**Remark 2:** When there is no cross-coupling effect, \( C_p = 0 \), condition (26) reduces to the SISO case as in [12].

**D. Time-Domain Analysis of Truncated MAIIC Law**

Representing the proposed MAIIC in frequency-domain leads naturally to the frequency-domain implementation—the time-domain MAIIC input is obtained via inverse Fourier transform. Such a frequency-domain scheme prompts the characterization of the MAIIC law in time-domain.
Lemma 7: Let the iteration coefficient \( \rho(j\omega) \) be chosen as in Lemma 5, then the MAIIC input in time-domain, \( U_k(t) \), 1) does not have a compact support; 2) is continuous, bounded above by its \( L^1 \) norm,
\[
\|U_k(t)\|_\infty \leq \|U_k(t)\|_1, \tag{28}
\]
and vanishes at infinity, i.e.,
\[
\lim_{t \to \infty} |U_k(t)| = 0 \tag{29}
\]
Combining Corollary 1 and Lemma 7, it is now clear that truncation of the iterative control input \( U_k(t) \) in time-domain is necessary (and guaranteed also) in implementations. Such a time-domain truncation needs to be applied to the control input in each iteration. Thus, we define

**Definition 3:** The truncated MAIIC law
\[
\hat{U}_k^d(j\omega) = \begin{cases} 
\hat{U}_k^d(j\omega) + \rho(j\omega)G_{r,m}(j\omega) & \omega \in \Omega_0 \\
0 & \text{otherwise}
\end{cases} \tag{30}
\]
where \( U_k^d(t) \) denotes the iterative input truncated in time when it is applied to the system, \( W_T(t) : \mathbb{R} \to [0, \infty) \) denotes a window function with a support of interval \( I_T \equiv [-T/2, T/2] \) (e.g., \( W_T(t) \) can be chosen as the characteristic function of interval \( I_T \), \( W_T(t) = \chi_T(t) \)), \( \star \) denotes the convolution operation, \( Y^d_k(t) \) is the output of the system for the truncated iterative input \( U_k^d(t) \), i.e.,
\[
Y^d_k(t) = G(j\omega)\hat{U}^d_k(j\omega) \tag{31}
\]
Moreover, in (30), \( Y^d_k(t) \) is the modified desired output \( Y^d_k(t) \) truncated in time directly.
\[
Y^d_k(t) = Y^d_k(t)W_T(t). \tag{32}
\]
We call \( Y^d_k(t) \) the directly-truncated desired output, and the corresponding desired input, \( U^d_k(t) \), the directly-truncated desired input:
\[
\hat{Y}^d_k(j\omega) = G(j\omega)\hat{U}^d_k(j\omega) \tag{33}
\]
To show the convergence of the truncated MAIIC law, we introduce the modified desired input \( U^d_k(t) \) (in (24)) truncated in time, \( U^d_k(t) \), and the corresponding desired output, \( Y^d_k(t) \), as follows,
\[
\hat{U}^d_k(j\omega) \triangleq \hat{U}^d_k(j\omega) \ast \hat{W}_T(t), \text{ and } \hat{Y}^d_k(j\omega) \triangleq G(j\omega)\hat{U}^d_k(j\omega). \tag{34}
\]
Note that in general, the above \( Y^d_k(t) \) is unknown (because the desired input, \( U^d_k \), is unknown, so is the modified one truncated in time, \( U^d_k(t) \)), and not equal to \( Y^d_k(t) \) (which is known, see (31)), because
\[
\hat{Y}^d_k(j\omega) = (G \hat{U}_k) \ast \hat{W}_T(j\omega), \text{ and } \hat{Y}^d_k(j\omega) = G(j\omega) \hat{U}_k(j\omega). \tag{35}
\]

We shall first quantify the truncation effect on the modified desired input in the following Lemma. The discussion is based on the properties of frequency-domain Dirac delta function \( \delta(\cdot) \). To simplify the discussion \( \int_{\mathbb{R}} \delta(j\omega) d\omega = 1 \) is used in the following.

**Lemma 8:** Assume that the window function \( W_T(t) \) converges to the Dirac delta function \( \delta(\cdot) \) in frequency domain as the window length \( T \to \infty \), i.e.,
\[
\lim_{T \to \infty} \hat{W}_T(j\omega) = \delta(j\omega). \tag{36}
\]
Then for any given constants \( \epsilon > 0 \), there exists a finite truncation time window, \( T^* \in (0, \infty) \), such that when the truncation window is larger than \( T^* \),
\[
\|E_{U,d}(\cdot)\|_2, \|E_{U,d}(\cdot)\|_2, \|E_{U,d}(\cdot)\|_2 \leq \epsilon, \tag{37}
\]
where
\[
E_{U,d}(t) \triangleq U^d(t) - U^d(t), \text{ and } E_{U,d}(t) \triangleq U^d(t) - U^d(t). \tag{38}
\]
With the above Lemma 8, the convergence of the truncated MAIIC law is addressed in the following Theorem.

**Theorem 1:** Let Assumptions 1 to 4 be satisfied, and the iteration coefficient \( \rho_p(j\omega) \) be chosen as in Lemma 5. Then for any given positive constant \( \epsilon > 0 \), there exists a finite truncation time \( T \), such that the iterative input-output obtained from the truncated MAIIC law (30) converges to the neighborhood of \( \epsilon \) of the directly-truncated desired input \( U^d(t) \) and that of the directly-truncated desired output \( U^d(t) \) in \( L^2 \)-space, i.e.,
\[
\lim_{k \to \infty} \|U^d_k(t)\|_2 \leq \lim_{k \to \infty} \|U^d_k(t) - U^d(t)\|_2 < \epsilon, \tag{39}
\]
\[
\lim_{k \to \infty} \|Y^d_k(t) - Y^d(t)\|_2 < \epsilon. \tag{40}
\]

**Remark 3:** Design of the Window Function \( W_T(t) \) The above time-domain analysis of the truncated MAIIC law implies that the design of the truncation window function \( W_T(t) \) can be important for applications where the implementation time is stringent. For example, the Kaiser window filter technique \([13]\) that trades side-lobe amplitude for main-lobe width can be utilized.

III. Example: Nanopositioning in 3-D
In this section, we illustrate the proposed MAIIC technique by 3-D nanopositioning using piezo actuators.

A. Experiment Scheme
The two piezotube actuators for positioning in 3-D (one for the lateral \( x-y \) axes, and the other for the vertical \( z \)-axis direction) on a scanning probe microscope (SPM, Dimension 3100, Veeco Inc.) were used in the experiments. A pyramid pattern was chosen as the desired trajectory. Tracking of such a pattern required equally-well positioning precision in all \( x-y-z \) axes. Three different pattern-tracking rates (2 Hz, 10 Hz and 20 Hz) were tested in the experiment, where the pattern-tracking rate was defined as the rate to traverse the four side triangles once by the order denoted in Fig. 1. With the \( x-y \)-axes displacement range both at 20 \( \mu \)m and the vertical one at 2 \( \mu \)m, the average speeds for these three rates were at 0.39 mm/sec, 1.94 mm/sec and 3.88 mm/sec, respectively.

B. Dynamics of the Piezo Actuators for 3-D Positioning
To implement the proposed MAIIC technique, the diagonal subdynamics of the \( x \) and \( z \)-axes piezo actuators (i.e., the frequency response) in 3-D (e.g., \( x \)-to-\( x \)) were experimentally measured along with the cross-axis coupling dynamics, as shown in Figs. 2 and 3. To quantify the dynamics uncertainty, frequency responses under different input conditions were acquired for, and the supremum and the infimum of the magnitude uncertainty and the phase
uncertainty were estimated according to (16), then used to quantify the upper bound of the iteration coefficient for each axis \( \kappa_x(j\omega), \kappa_y(j\omega), \kappa_z(j\omega) \) (see (23)). The obtained upper bounds of the iteration coefficients for all the three axes and the iteration coefficients are shown in Fig. 4).

Experimental results show that significant cross-axis coupling effect existed in the piezo actuators used in this experiment. The lateral to vertical \( x-y \)-to-\( z \) coupling effects were pronounced (see Figs. 2 and 3). Such a relatively large coupling effect might be caused by the slightly misalignment of the \( z \)-axis displacement sensor. In this experiment, the lateral trajectories for the desired pyramid pattern were relatively slow compared to the coupling dynamics between \( x \)- and \( y \)-axes. Thereby, the lateral coupling caused disturbances between \( x \)- and \( y \)-axes were expected to be small and negligible in the experiments. Hence, the experiment results presented next are focused on the compensation for the lateral to vertical coupling effect.

C. Tracking results and discussion

The MAIIC technique was applied to track the pyramid pattern by implementing the MAIIC law (4) to all three axes simultaneously. For comparison, the desired pyramid pattern was also tracked by using the DC-Gain method and the IIC technique [3]. In the IIC technique, the IIC law was applied to each axis separately (not simultaneously!) to obtain the control input for each axis individually, and then the control inputs for all three axes were applied. The tracking results obtained by using the MAIIC technique and the DC-Gain method are compared in Fig. 5 for the \( x \)-axis (\( y \)-axis tracking results are omitted), and the vertical \( z \)-axis tracking results are compared in Fig. 6 for the MAIIC, the IIC, and the DC-gain methods. Finally, the 3-D pyramid pattern tracking results are also compared for the above three methods with respect to the desired pyramid pattern in Fig. 7.

The experiment results demonstrate that multi-axis (3-D) precision tracking can be achieved by using the proposed method. In the lateral \( x-y \)-axes tracking at low speed, the dynamics effect was relatively small (as the lateral speed was relatively small), so was the cross-axis coupling effect between \( x-y \)-axes (as explained above). Hence, the tracking error was mainly caused by the hysteresis effect of piezo actuators (see Fig. 5). We note, however, as the pattern-tracking rate increased to 20 Hz, the dynamics effect became pronounced and augmented to the hysteresis effect, resulting in larger tracking error at \( \sim 20\% \), as shown in Fig. 5 (c). We note that although the vibrational dynamics effect of the \( z \)-axis dynamics itself was significantly removed by using the IIC technique (see Fig. 6 (a2) to (c2)), the cross-axis coupling effect still existed and became oscillatory as the pattern tracking increased to 20 Hz. On the contrary, by using the proposed MAIIC technique, such large \( x-y \)-to-\( z \) coupling effect was successfully removed. We also note that rapid convergence was achieved in the experiments: With no more
than three iterations, the RMS-tracking error was reduced to ~1% in both lateral $x-y$ axes tracking and vertical $z$-axes tracking. Therefore, the combined tracking precision of the 3-D pyramid pattern illustrated the advantages of the MAIIC technique in 3-D nanopositioning control (see Fig. 7))

IV. CONCLUSION

In this article, the multi-axis inversion-based iterative control (MAIIC) technique was proposed by extending the IIC technique from SISO systems to MIMO systems. It was shown that by only using the dominant dynamics for each output in the iterative control algorithm, the control of each output channel was decoupled in the MAIIC. The convergence condition of the MAIIC was characterized in frequency domain, and the set of tractable frequencies was quantified by the bounds of system uncertainties. Moreover, the optimal iteration coefficient in the MAIIC law was obtained, and the analysis of the time domain properties of the MAIIC law showed that the MAIIC law with time-domain truncation convergences to the truncated desired input-output. The proposed MAIIC technique was illustrated through 3-D nanopositioning using two piezo actuators in experiments.

REFERENCES