On the observability of continuous time linear systems with Markov jump parameters.

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Abstract—This paper studies observability of a class of Markov systems with jumping parameters, and an associated set of observability matrices. We explore some invariance results in order to demonstrate a certain property involving those matrices. This property is known in the literature of this class of systems, but there is no available proof. The obtained results are technically involving and important, as they validate many existing results that rely on that property.

I. INTRODUCTION

The concept of observability plays an important role in the theory of dynamical systems. For instance, it is related to the capability to determine the state at some time instant from output observations during an arbitrary time interval. In addition, it is also important in characterizing the behavior of optimal control, e.g. ensuring positivity of the cost functional and stability of the controlled system. There is a great deal of attention to the study of observability, and there are a number of available results in literature, see e.g. [1], [7], [9], [11] just to mention a few papers in different contexts. In particular, for Markov Jump Linear Systems (MJLSs), which comprise systems whose parameters are governed by a subjacent Markov chain \( \Theta = \{ \theta(t), t \geq 0 \} \), with \( \theta(t) \) taking values on \( S = \{ 1, 2, \ldots, N \} \) (please see [10] for an operator theoretical approach for MJLS and the numerous references therein, and [8], [12], [13] for MJLS applications), the results and properties concerning observability have reached a fairly complete parallel with the one for Linear Deterministic Systems (LDS). For instance, we mention the following properties for MJLS that have a direct counterpart for LDS:

(I) Non-observed subspaces are invariant, that is, the state trajectory remains in the non-observed space when the initial conditions is in that subspace.

(II) A system is observable if and only if certain observability matrices \( \Theta_i \) are of full rank.

(III) Weak observability (41)) is a sufficient condition for the LQ-optimal controls to be stabilizing.

See [3], [4], [5] for further results on weak observability and the related notion of weak detectability for MJLSs. Most of available results (including (I)–(III)) are based on the property that, for each \( v \in \mathbb{R}^n \), \( i \in S \) and \( k = 0, \ldots, n^2N - 1 \)

\[
(P) : \quad \nu' \Theta_i(k) \nu = 0 \iff \Theta_i(k) \nu = 0,
\]

The property (P) is easy to demonstrate in the discrete-time scenario since \( O_i(k) \) is a positive semi-definite matrix for each \( k \) [3]. However, this is not the case in continuous-time, as we shall see later.

In this paper, we provide a proof for property (P) for continuous-time MJLS. We consider MJLSs defined in a fundamental probability space \( (\Omega, \mathcal{F}, \{ \mathcal{F}_t \}, \mathbb{P}) \) as

\[
\begin{align*}
\Phi: & \quad x(t) = A_{\theta(t)} x(t), \quad x(t_0) = x_0, \quad \theta(t_0) \sim \mu_0 \nonumber \\
y(t) = C_{\theta(t)} x(t) & \quad \text{with } t \geq 0, x \in \mathbb{R}^n \text{ and } y \in \mathbb{R}^r. \nonumber
\end{align*}
\]

At each time instant \( t \), we have \( A_{\theta(t)} = A_j \) whenever \( \theta(t) = i \), where \( A_j \) is a matrix of appropriate dimensions taken from a known collection of matrices \( A = (A_1, \ldots, A_N) \), and similarly for \( C \). The transition rate matrix for \( \Theta \) is denoted by \( \Lambda = [\lambda_{ij}] \) (see [2] for details on Markov chains).

The approach is as follows. We start showing that \( x(t), t \geq 0 \), is in the null space of \( C_j \) almost surely (a.s.) whenever \( j \) is accessible from \( \theta(0) \) and the average square norm of the output is zero (or, as we shall see, \( \dot{x}' O_{\theta(t)}(k)x = 0, k = 0, \ldots, n^2N - 1 \), see Corollary 2). Next we explore a similar invariance result for \( A_{\theta(t)}^p x(t), p \geq 0 \), that is, we show that the derivatives of \( x(t) \) are in the null space of \( C_j \), see Lemma 4. In Lemma 5 we show that this invariance still holds if we replace \( \theta(t) \) with \( i \), assuming \( i \) is accessible from \( \theta(t) \), formally,

\[
C_j A_j^p v = 0.
\]

This allows us to evaluate, for each Markov state sequence \( i, i_1, \ldots, i_m \) such that \( \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_{m-1}i_m} \neq 0 \),

\[
C_{i_m} A_{i_m-1}^{p_1} A_{i_{m-2}}^{p_2} \cdots A_i^{p_m} v = 0,
\]

whenever \( \nu' O_j(k) \nu = 0, k = 0, \ldots, n^2N - 1 \). The definition of \( \Theta_i \) involves terms as in the left hand side of (1), which can be used to show that \( \Theta_i \nu = 0 \) by direct inspection, hence completing the proof of necessity of (P) (the sufficiency is trivial).

The paper is organized as follows. In section II we provide notation and some preliminaries. Section III presents some invariance properties that are employed in the main result. Section IV contains illustrative numerical examples. Finally, section V presents some conclusions.

II. PRELIMINARY RESULTS AND NOTATION

In this section, we present some notation and results for future reference. Let \( \mathbb{R}^n \) (respectively \( \mathbb{R}^m \)) be the linear space formed by all matrices of size \( n \times q \) (respectively \( n \times n \)) and \( \mathbb{R}^{q \times r} \) (the closed convex cone of symmetric semidefinite...
positive matrices \( \{ U \in \mathbb{R}^r : U = U' \geq 0 \} \), (the open cone of symmetric definite positive matrices \( \{ U \in \mathbb{R}^r : U = U' > 0 \} \)). Here \( U' \) denotes the transposed of \( U \); \( U \geq V (U > V) \) means that \( U - V \in \mathbb{R}^r \) \( (U - V \in \mathbb{R}^r^+) \). For \( U \in \mathbb{R}^{m,n}, \mathcal{N}(U) \) represents the null space of \( U \). Also, if \( U \in \mathbb{R}^m \) and \( V \in \mathbb{R}^{m,n} \) then 
\[ \bigtriangledown(U, V) = \{ V', \cdots : U^{m-1}V' \} \]
is the observability matrix of pair \( (U, V) \). The operator \( 1 \{ \cdot \} \) is the indicator function (or characteristic function) and \( \text{tr}\{ \cdot \} \) denotes the trace. Let \( \mathcal{M} = \mathbb{R}^r \) be the linear space formed by a number \( N \) of matrices such that \( \mathcal{M} = \{ U = (U_1, \cdots, U_N) : U_i \in \mathbb{R}^{r \times n}, i = 1, \cdots, N \} \); also, \( \mathcal{M} = \mathbb{R}^{m \times n} \). We denote by \( \mathcal{M}^0 (\mathcal{M}^T) \) the set \( \mathcal{M}' \) when it is formed by \( U_i \in \mathbb{R}^0 (U_i \in \mathbb{R}^T) \) for all \( i = 1, \cdots, N \). \( \mathcal{M} \) defined as before with the inner product given by 
\[ \langle U, V \rangle = \sum_{j=1}^{N} \text{tr}(U_j'V_j) \]
is a Hilbert space [6]. Furthermore, define the norm \( \| U \| = \langle U, U \rangle \) in \( \mathcal{M}^0 \), and, regarding the system \( \Phi \) with \( t_0 \geq 0 \), for \( i = 1, \cdots, N \) define:

\[ X_i(t) = \mathbb{E}\{ x(t)\mid \Theta(t) = \theta \} \big|_{t = t_0} \] 

Let the operator \( \mathcal{L} : \mathcal{M} \longrightarrow \mathcal{M} \) defined as:

\[ \mathcal{L}_i(U) = A_i U_1 + U_i A_i + \sum_{j=1}^{N} \lambda_{ij} U_j, \tag{2} \]

for \( i = 1, \cdots, N \). Denote \( \mathcal{M}^0(U) = U \) and, for \( k \geq 1 \), \( \mathcal{L}^k(U) = \mathcal{L}(\mathcal{L}^{k-1}(U)) \). Also, define \( L(t), t \geq 0 \), by the linear differential equation:

\[ L(t) = \mathcal{L}(L(t)) + C'_t C_t, \quad L(0) = 0, \quad t \geq 0 \tag{3} \]

for each \( i \in \mathcal{T} \). Now let us consider the functional:

\[ W^{t, \theta}(x, \theta) = \mathbb{E} \left\{ \int_{0}^{t} x(t)^' C'_t C_t x(t) d\tau \mid \mathcal{F}_0 \right\} \tag{4} \]
defined whenever \( x(t_0) = x \) and \( \theta(t_0) = \theta \).

It will be used the following additional notation. For \( V \in \mathbb{R}^n \),
the columns of \( V \) will be identified as \( V = [v_1; \cdots; v_n] \).
and, for \( U = (U_1, U_2, \cdots, U_N) \) we introduce the known linear and invertible operator \( \Phi : \mathcal{M} \longrightarrow \mathbb{R}^{n \times n} \),

\[ \Phi(U) = \begin{bmatrix} \varphi(U_1) \\ \varphi(U_2) \\ \vdots \\ \varphi(U_N) \end{bmatrix}, \quad \text{where} \quad \varphi(V) = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}. \]

With this, let \( \ell(t) \in \mathbb{R}^{n \times n} \) be defined as:

\[ \ell(t) = \Phi(L(t)). \]

The previous proposition allows to generalize the definition of the functional \( W \) as follows:

\[ W'^{t, \ell}(U) = \int_{0}^{t} \langle U(\tau), C'C \rangle d\tau = \langle U, L(t) \rangle, \quad \text{for } U \in \mathcal{M}. \tag{6} \]

Also, we introduce the following representation for \( \langle U, L(t) \rangle \):

\[ \langle U, L(t) \rangle = \Phi(U)' \ell(t). \tag{7} \]

**Lemma 1:** If \( W^{t, \theta}(x, \theta) = 0 \) then \( W^t(x, \theta) = 0 \) almost surely (a.s.) for all \( t \geq 0 \).

**Proof:** Let us suppose \( W^t(x, \theta) > 0 \) with positive probability. Then we have that \( \mathbb{E}\{ W^t(x, \theta) \mid \mathcal{F}_0 \} > 0 \), in order to evaluate

\[ W'^{t, \theta}(x, \theta) = \mathbb{E} \left\{ \int_{0}^{t} x(t)' C'_t C_t x(t) d\tau \mid \mathcal{F}_0 \right\} + \mathbb{E} \left\{ \int_{0}^{t} x(t)' C'_t C_t x(t) d\tau \mid \mathcal{F}_0 \right\} \geq \mathbb{E}\{ W^t(x, \theta) \mid \mathcal{F}_0 \} > 0. \]

An important preliminary result is stated in the following proposition [4]. It can be interpreted as an evaluation for the maximal expansion of the trajectory \( x(t) \) around a given state \( x_0 \), and it follows from the facts that the trajectory is continuous and has bounded derivatives (\( A_{\theta(t)} \) are taken form the known, finite collection \( \mathcal{A} \)).

**Proposition 2:** For each scalar \( M > 0 \), exists \( t_M > 0 \) such that \( \| x(t) - x_0 \| \leq M \| x_0 \| \) (a.s.), \( 0 \leq t \leq t_M \).

Let us define the sequence of collections of matrices \( O_i(k) \in \mathcal{M} \) for each \( i \in \mathcal{T} \) as:

\[ O_i(k) = \frac{d^{k+1}L_i}{dt^{k+1}}(0) \tag{8} \]

with \( L(t) \in \mathcal{M}^0, t \geq 0 \), defined by linear equations given in (3) where \( O_i(0) = C'_t C_t \) for each \( i \in \mathcal{T} \). Thus, we introduce and study the collection of matrices \( O_i \in \mathcal{M}^{n^2N} \), defined for each \( i = 1, \cdots, N \) as:

\[ O_i = (O_i(0) \quad O_i(1) \quad \cdots \quad O_i(n^2N - 1)) \]

which is called the set of observability matrices of the system \( \Phi \). \( O_i(k) \) might not be positive because, otherwise (if this were positive for \( t > 0 \)) \( L_i(t) \) would tend to infinity and then, \( W^t(x, i) = \infty \). The following is an extract from [4].

**Lemma 2:** For \( x \in \mathbb{R}^n \), \( i \in \mathcal{T} \), \( X \in \mathcal{M}^0 \) defined as \( X_i = x \) and \( X_j = 0 \), \( \forall j \neq i \) and \( w \in \mathbb{R}^{n^2N} \) defined as \( w = \Phi(X) \). The following assertions are equivalent:

(i) \( \frac{dw}{dt^{k+1}}(0) = 0 \), for \( k = 0, \cdots, n^2N - 1 \).

(ii) \( w \in \mathcal{N}(L_i(t)) \), or, equivalently, \( w' \ell(t) = 0 \) for all \( t \geq 0 \).

**III. MAIN RESULT**

The purpose here is to show that the statements (i) and (ii) of Lemma 2 are equivalent to the fact that \( x \) belongs to \( \mathcal{N}(O_i) \), as proposed in [4]. However, to this end, we shall need some preliminary results. The first one is an adaptation of [4, Lemma 8].
Corollary 1: $x'O_i(k)x = 0$ for all $k = 0, \ldots, n^2 N - 1$, if and only if $W_t(x, i) = 0, t \geq 0$.

Proof: Let $X \in M_n$ be defined as $X_i = x\ell'$ and $X_j = 0, \forall j \neq i$ and $w \in \mathbb{R}^{nN}$ as $w = \phi(X)$. The following implications are obtained in a straightforward manner for each $k = 0, \ldots, n^2 N - 1$,

$$x'O_i(k)x = 0 \iff x'\frac{d^{k+1}L}{dt^{k+1}}(0)x = 0 \iff \langle X, \frac{d^{k+1}L}{dt^{k+1}}(0) \rangle = 0 \iff w'\frac{d^{k+1}L}{dt^{k+1}}(0) = 0.$$  \hspace{1cm} (10)

From (10) and Lemma 2 we have that $x'O_i(k)x = 0, k = 0, \ldots, n^2 N - 1$, is equivalent to $w'\ell(t) = 0, t \geq 0$. (6) and (7) complete the proof.

Next we explore an invariance result for the trajectory $x(t)$ and the null spaces associated to $C$.

Lemma 3: If there exists $T$ such that $x(T) \not\in \mathcal{N}(C_j)$ with positive probability, $1 \leq j \leq N$, then there exists $I \geq 0$ such that $W^{i+r}(x(T), \theta(T)) > 0$ (a.s.), $s \geq 0$, whenever $j$ is accessible from $\theta(T)$.

Proof: In this proof we will define a time interval $[\tau, t_e]$ to allow $\theta(T)$ to reach $j$ with positive probability and a time interval $[t_e, t_e + s]$ for integrating the term $x(t)'C_jx(t)$, and we obtain evaluations for the minimal distance between the trajectory of $x(t)$ and the null space of $C_j$ along these time intervals, see Figure 1. If $x(T) \not\in \mathcal{N}(C_j)$, we have that

$$|x - x(T)| \geq m, \forall x \in \mathcal{N}(C_j),$$

where $0 < m = d(x(T), \mathcal{N}(C_j))$. Now, for $\epsilon = \frac{m}{2||x(T)||}$, exists $\delta_\epsilon > 0$ such that

$$x(t) \in B\left(x(T), \frac{m}{2}\right)$$

(a.s.), with $T \leq t \leq t_e = T + \delta_\epsilon$, (see Proposition 2). Let $\tilde{x}(t_e)$ and $\hat{x}(t_e)$ be the orthogonal projections of $x(t_e)$ in $\mathcal{N}(C_j)$ and $\mathcal{N}(C_j)^\perp$ respectively.

Thus, $x(t_e) = \tilde{x}(t_e) + \hat{x}(t_e)$ and, therefore, $\exists \mu > 0$ such that:

$$x(t_e)'C_jx(t_e) = \tilde{x}(t_e)'C_j\tilde{x}(t_e) \geq \mu ||\tilde{x}(t_e)||^2 \geq \mu(||x(t_e) - x(T)|| - ||x(t_e) - x(T)||)^2 \geq \mu \frac{m^2}{4}$$

(a.s.).

In a similar fashion we obtain for $s > 0$,

$$x(t)'C_jx(t) \geq \mu \frac{m^2}{16}$$

(a.s.), $t_e \leq \tau \leq t_e + s$, which yields

$$\int_{t_e}^{t_e + s} x(t)'C_jx(t)d\tau \geq \frac{m^2}{16}s$$

(a.s.), i.e.,

$$W^x(x(t_e), j) \geq \mu \frac{m^2}{16}$$

(a.s.).

Now, we evaluate

$$E\{W^x(x(t_e), \theta(t_e))|\mathcal{F}_T\} \geq E\{W^\epsilon(x(t_e), j)1_{\{\theta(t_e) = j\}}|\mathcal{F}_T\} \geq \frac{m^2}{16}sE\{1_{\{\theta(t_e) = j\}}|\mathcal{F}_T\} > 0,$$

where the last inequality follows from the fact that $j$ is accessible from $\theta(T)$. Finally,

$$\int_T^{t_e + s} x(t)'C_jx(t)d\tau = \int_T^{t_e} x(t)'C_jx(t)d\tau + \int_{t_e}^{t_e + s} x(t)'C_jx(t)d\tau \geq \int_{t_e}^{t_e + s} x(t)'C_jx(t)d\tau,$$

leading to

$$W^{s+t_e}(x(T), \theta(T)) \geq E\{W^{s+t_e}(x(T), \theta(T))|\mathcal{F}_T\} \geq E\{W^x(x(t_e), \theta(t_e))|\mathcal{F}_T\} > 0$$

(a.s.).

Corollary 2: If $W^x(x, i) = 0$ for all $t \geq 0$ then $C_{im}x(t) = 0$ (a.s.), $\forall t \geq 0$ whenever $i_m$ is accessible from $i$.

Proof: Since $W^x(x, i) = 0$ for any $t \geq 0$, in particular we have for some fixed $s \geq 0$ and $t \geq 0$ that $W^{s+t}(x, i) = 0, \forall t \geq 0$, and using Lemma 1 we obtain that

$$W^{s+t}(x(t), \theta(t)) = 0$$

(a.s.), $\forall t \geq 0$.

Thus, Lemma 3 yields

$$x(t) \in \mathcal{N}(C_{im})$$

(a.s.) for all $t \geq 0$.

or, equivalently,

$$C_{im}x(t) = 0$$

(a.s.), $\forall t \geq 0$.

The following basic fact shall be employed in the main result.

Lemma 4: For $Q \in \mathbb{R}^{m \times n}$ we have $Q\mathcal{A}_{\hat{h}(\cdot)x}(t) = 0$ (a.s.) $t \geq 0$, $n > 0$, whenever $Qx(t) = 0$ (a.s.) $t \geq 0$.

Proof: For any $v \in \mathcal{N}(Q)^\perp$ we have $\nabla v(x(t)) = 0$ (a.s.), $\forall t \geq 0$. Now, for all $t \geq 0$ and $n > 0$ we evaluate:

$$\frac{d^n(v'x(t))}{dt^n} = v'\left(\frac{d^n(x(t))}{dt^n}\right) = 0$$

(a.s.)
or, equivalently, \( d^n x(t)/dt^n \in \mathcal{N}(Q) \) \((a.s.)\), \( t \geq 0 \), yielding
\[
Q A_{\theta(t)}^n x(t) = 0 \quad (a.s.), \quad \forall t \geq 0.
\]

The preceding results allow one to show that, provided \( x_0 \) is accessible from \( i \), then \( C_i A_{\theta(t)}^n x(t) \equiv 0 \) \((a.s.)\) \( t \geq 0 \). However, we would like to generalize this result to any Markov state \( i_m \) accessible from \( \theta(t) \), as follows.

**Lemma 5:** Assume that the Markov states sequence \( \theta(t), i_1, i_2, \ldots, i_{m-1}, i_m \) is such that \( \lambda_{i(i_i)} \lambda_{i_j} \cdots \lambda_{i_{m-1} i} \neq 0 \). If \( Q A_{\theta(t)}^n x(t) = 0 \) \((a.s.)\), \( t \geq 0 \), then \( Q A_{i_m}^n x(t) = 0 \) \((a.s.)\), \( t \geq 0 \), \( p \geq 0 \).

**Proof:** We shall assume the contrary, i.e., that there exists \( T \), such that \( Q A_{\theta(t)}^n x(T) \neq 0 \) with positive probability \( \alpha \), that is, \( P(x(T) \notin \mathcal{N}) = \alpha > 0 \), where we define \( \mathcal{N} = \mathcal{N}(Q A_{i_m}^n) \). Then, \( \forall e > 0, \exists \bar{e} > 0 \) (possibly dependent on \( x(T) \)) satisfying:
\[
Q A_{i_m}^n x(l) \neq 0, \quad (a.s.) \quad T < l \leq T + \bar{e}.
\]

Now, assume \( T_i \) is the time of the first visit to the state \( i_m \) starting from \( T \), i.e., \( T_i = \inf_{t \geq 0} \{ t \geq T : \theta(t) = i_m \} - T \), then we have \( P(T_i < \bar{e}) > 0 \). Thus,
\[
P(Q A_{\theta(t)}^n x(l) \neq 0) = P(Q A_{\theta(t)}^n x(T) \neq 0) (1 - \alpha) + P(Q A_{\theta(t)}^n x(T) \neq 0) \alpha
\]
\[
\geq P(Q A_{\theta(t)}^n x(l) \neq 0) (1 - \alpha) + P(l < \bar{e}) \alpha P(l < \bar{e})
\]
\[
= \alpha P(l < \bar{e}) > 0,
\]
which contradicts the hypothesis that \( Q A_{\theta(t)}^n x(t) = 0 \) \((a.s.)\), \( t \geq 0 \). \( \blacksquare \)

Lemma 4 can be employed to obtain the key evaluation for the necessity of \( P \), as follows.

**Corollary 3:** For each \( x \in \mathbb{R}^n \) and sequence of Markov states \( i, i_1, \ldots, i_n \) such that \( xO_1(k)x = 0 \), \( k = 0, \ldots, n^2N - 1 \), and \( \lambda_{i i_1} \lambda_{i_1 i_2} \cdots \lambda_{i_{n-1} i_n} \neq 0 \), we have
\[
C_{i_1} A_{i_{m-1}}^{p_1} A_{i_{m-2}}^{p_2} \cdots A_{i_1}^{p_n} x = 0,
\]
(11)
for each \( p_1 \geq 0, \ell = 1, \ldots, m \).

**Proof:** If each \( k = 0, \ldots, n^2N - 1 \) we have \( xO_1(k)x = 0 \), then, according to Corollaries 1 and 2 we obtain
\[
C_{i_m} x(t) = 0 \quad (a.s.), \quad t \geq 0
\]
and (see Lemma 4)
\[
C_{i_m} A_{\theta(t)}^n x(t) = 0 \quad (a.s.), \quad \forall t \geq 0.
\]
Assume that \( \theta(t) = i \) (since \( \theta(0) \neq i \)). We have that \( \theta(t) = i \) with positive probability. By hypothesis, \( \theta(t) \) reaches \( i_{m-1} \) and, thus, applying Lemma 5 we get,
\[
C_{i_m} A_{i_{m-1}}^{p_1} x(t) = 0 \quad (a.s.), \quad \forall t \geq 0.
\]

If we set \( Q = C_{i_m} A_{i_{m-1}}^{p_1} \), then proceeding similarly as above (replacing \( C_{i_m} \) by \( Q \) in (12)) we obtain
\[
Q A_{\theta(t)}^n x(t) = 0 \quad (a.s.), \quad \forall t \geq 0
\]
and, recalling we assume \( \theta(t) = i \), we have again by hypothesis that \( \theta(t) \) reaches \( i_{m-2} \), leading to
\[
C_{i_m} A_{i_{m-1}}^{p_1} A_{i_{m-2}}^{p_2} x(t) = 0 \quad (a.s.), \quad \forall t \geq 0.
\]

Proceeding recursively,
\[
C_{i_m} A_{i_{m-1}}^{p_1} A_{i_{m-2}}^{p_2} \cdots A_{i_1}^{p_n} x(t) = 0 \quad (a.s.), \quad \forall t \geq 0,
\]
and, in particular for \( t = 0 \), we can see that (11) holds. \( \blacksquare \)

The necessity of \( P \) follows by direct evaluation of \( \theta(t) \) with the aid of Corollary 3. In fact, from (2), (3) and (8) one can check that \( O_1(k) \) is a sum involving terms of the form
\[
\pi A_{i_1}^{p_1} A_{i_2}^{p_2} \cdots A_{i_{m-1}}^{p_{m-1}} x, \quad (13)
\]
where \( i, i_1, \ldots, i_k \) and \( i, j_1, \ldots, j_{k-1}, i_k \) are sequences of Markov states, \( p_1 + \cdots + p_k \leq k, j_1 + \cdots + j_k \leq k \), and \( \pi \) stands for a productory of transition rates related to the transitions \( i, i_1, \ldots, i_k \) and \( j_1, j_2, \ldots, j_{k-1}, i_k \). One can also check that \( \pi \) is different from zero if and only if \( \lambda_{i i_1} \lambda_{i_1 i_2} \cdots \lambda_{i_{k-1} i_k} \neq 0 \) and \( \lambda_{j_1} \lambda_{j_1 j_2} \cdots \lambda_{j_{k-1} j_k} \neq 0 \). Then, assuming \( xO_1(k)x = 0 \) for each \( k = 0, \ldots, n^2N - 1 \), and \( \pi \neq 0 \), Corollary 3 yields that \( C_{i_1} A_{i_1}^{p_1} A_{i_2}^{p_2} \cdots A_{i_{m-1}}^{p_{m-1}} x \), in such a manner that the term (13) is zero. As an illustration, let us assume \( xO_1(k)x = 0, k = 0, \ldots, n^2N - 1 \), and carry out the evaluations for \( O_1(k), k = 0, 1, 2 \). We have that
\[
O_i(0) = C_i C_i
\]
and it follows immediately from Corollary 3 that \( C_i x = 0 \), hence \( O_i(0)x = C_i^2 C_i x = 0 \). From (2), (3) and (8),
\[
O_i(1) = A_i^2 C_i C_i + C_i^3 C_i A_i + \sum_{j=1}^{N} \lambda_{ij} C_j C_i + C_i^2 C_i
\]
and post multiplying each term of \( O_i(1) \) by \( x \) and using again Corollary 3 we obtain \( O_i(1)x = 0 \). Now,
\[
O_i(2) = A_i^2 C_i C_i + 2A_i^2 C_i C_i A_i + \sum_{j=1}^{N} \lambda_{ij} C_j C_i A_i + C_i^3 C_i A_i + \sum_{j=1}^{N} \lambda_{ij} C_j C_i A_i + \sum_{j=1}^{N} \lambda_{ij} C_j C_i A_i + \sum_{j=1}^{N} \lambda_{ij} C_j C_i A_i + C_i^2 C_i
\]

Similarly as above, Corollary 3 yields \( O_i(2)x = 0 \).

**Theorem 1:** For each \( k = 0, \ldots, n^2N - 1 \) we have that \( xO_1(k)x = 0 \) if and only if \( O_i(k)x \equiv 0 \).
IV. EXAMPLES

This section contains numerical examples illustrating the main results of the paper. Example 1 addresses the result in Corollary 3. In Example 2 we evaluate the matrix $\Theta$ and check that $\Theta x = 0$ provided $x' O(k) x = 0$, in accordance with Theorem 1.

Example 1: Consider the continuous-time MJLS $\Phi$ with

\[
A_1 = \begin{bmatrix} 2 & 0 & 0 \\ 0.01 & 1 & 0 \\ 0.9 & 1 & 0.1 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},
\]

\[
A_2 = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.99 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},
\]

\[
A_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0.1 & 0.25 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad C_3 = 0, \quad \Lambda = \begin{bmatrix} -3 & 1 & 2 \\ 2 & -5 & 3 \\ 0.5 & 0.5 & -1 \end{bmatrix}.
\]

Note that $\mathcal{N} = \{1, 2, 3\}$, $n = 3$, $N = 3$. We consider initial condition $x(0), \theta(0)$ with $x(0) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}'$ and $\theta(0) = 1$ (compatible with initial distribution $\mu(0) = [1 \ 0 \ 0]$). Consider the sequence $1, 3, 1, 2, 3, 2$, for which $\lambda_i(x) \neq 0$. We have checked that $x' O(k) x = 0$, $k = 0, \ldots, 26$, thus satisfying the hypotheses of Corollary 3, and

\[
C_2 A_3^{p_1} A_2^{p_2} A_1^{p_3} A_3^{p_4} A_1^{p_5} x(0) = 0,
\]

for each $p_\ell = 0, \ldots, 20$, $\ell = 1, \ldots, 5$. This confirms (11).

Example 2: Consider system $\Phi$ as in Example 1, with $C_2 = 0$. The initial condition remains unaltered, $\theta(0) = 1$ and $x(0) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}'$. We have carried out the calculation of $O_i(k)$, $k = 0, \ldots, 26$, and $\Theta_i$ via (2) and (3). The null space of $\Theta$ coincides with the space spanned by the vectors $v_1 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}'$ and $v_2 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}'$; all elements of $O_i(k)$ are null except from the element $(1, 1)$, in such a manner that $x' O(k) x = 0$ if and only if $x$ belongs to the space spanned by vectors $v_1, v_2$, confirming Theorem 1. Note that $x(t) \notin \mathcal{N}(\Theta_i)$, in such a manner that $W'(x(0), 1) = 0$ (see Corollary 1), and the hypotheses of Corollary 2 are satisfied for $i_0 = 1, 2, 3$; Figure 2 confirms that $x(t)$, $t \geq 0$, is in the null space of $C_1$ (obviously, the same goes for $C_2 = C_3 = 0$).

V. CONCLUSIONS

In this paper we established some invariance results for the trajectory $x(t)$, aiming at the evaluation presented in Corollary 3. This evaluation is then employed to show the equivalence (P) (see Section I), which is essential for the weak observability and weak detectability notions for continuous-time MJLS and related properties.

REFERENCES
