Extraction of Relative Proximity from Electrostatic Images Using Wide-Field Integration Methods

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Abstract—Weakly electric fish use a self-generated electric field to interrogate the environment around them for obstacles, predators, and prey. These objects appear as spatial perturbations to an electric image formed by electro-receptors which are distributed over their bodies. The influence on an electric image, formed on a simulation model, due to the proximity of wide-field obstacles is derived here from standard principles of electrostatics. Wide-Field Integration (WFI) methods are then applied to perturbed electric images to extract relative position and orientation information, and a static output feedback methodology is presented to achieve local unmapped reflexive obstacle avoidance.

I. INTRODUCTION

Electrolocation, which is observed in weakly electric fish, has been established as a technique with a rich source of motion cues for underwater navigation. In prior work, vision [1] and acoustic [2] sensing have been employed for performing object identification and object avoidance in underwater environments. Camera images in vision-based systems are affected by low levels of illumination in aquatic environments due to chlorophyll in aquatic fauna, suspended materials, and the medium itself. Sonar, another sensing modality, possesses a highly directional nature and puts limitation on the extent of the sensed region in the environment. Electric fields are not dependent on the illumination in the surroundings, and due to omni-directional propagation from their source, provide with a wide-field sensing of the environment. Electrical sensing and signal processing are much simpler and computationally cheaper processes, making electrolocation a viable candidate for proximity analysis in underwater navigation.

Early experiments with electric fishes provided knowledge of the generation and measurement of the electric field of the fish [3], [4]. In [5], the theory for quantifying the perturbation caused by single spherical and ellipsoidal objects to the self-generated electric field was developed. It has been postulated that the relative proximity of an object can be given either by the relative width [6] or the slope and amplitude [7] of its electric image. Electrolocation has been observed to be most active within a limited distance from the fish’s body [8].

In recent work, researchers have made progress in the interpretation of electric images for locating objects in underwater environments. In a robotic electrolocator, the differential voltage at two sensors has been used to measure the perturbation of an electric field due to the presence of an object [9]. Due to the presence of sensor noise, the location estimate of the object often has a probabilistic nature. The initial estimate of the location of a moving object is improved iteratively by applying a particle filter, which weights the particles in the simulation according to the motion model, eventually converging to the accurate location of the object, as demonstrated in [10].

In this effort, wide-field processing of electric images is utilized to extract signals encoding relative proximity of objects in the surrounding environment, which are applied as feedback to generate reflexive avoidance behaviors. The technique, termed Wide-Field Integration (WFI) [11], [12], [13], [14] was originally formulated to model spatial decompositions of patterns of optic flow by specialized neurons in the insect visuomotor system [15], [16], [17]. Here it is extended to develop a method to rapidly decompose electric images into compensatory commands that maneuver an underwater vehicle safely between obstacles.

The paper is outlined as follows. In Section II, the electrostatic equations that describe the image formed on a circular sensor for an arbitrary pose within corridor and arena-like environments is derived. In Section III analysis of spatial decompositions of these electric images using WFI methods is presented, and an output feedback methodology which generates local unmapped obstacle avoidance is demonstrated in Section IV.

II. MATHEMATICAL ANALYSIS OF ELECTROLOCATION WITH A SENSING MODEL

In this section the method of images from classical electrostatic analysis [18] is used to derive the equations that describe the image formed on a circular sensor for an arbitrary pose within corridor and arena-like environments.

An electric image is the pattern of electric energy at the level of electoreceptors [19], implying that for a robotic vehicle, it is the voltage that is measured at the sensors. This image depends on various factors including shape, size, distance, and impedance of surrounding objects.

Consider a dipole with magnitude of individual pole current $I$ and inclined at an angle $\theta$ to the real axis with the poles, $R$ distance apart, inserted in a medium with electrical conductivity $\sigma$. For the entire treatment in this paper, the coordinate axes are assumed to be rotated through $90^\circ$ to be consistent with the convention set in [11] for representing the coordinate frame of a straight tunnel. The electric potential and the flux of a dipole field can together be represented as a holomorphic function. This function is called as the complex potential of the dipole field and is given as:

\[ V(z) = \Phi + i\Psi \]  

(1)
Here, $\Phi$ is the electric potential and $\Psi$ is the electric flux due to the dipole field. Any point in the field is represented by $z = x + iy$ in the complex plane, with the origin at the centre of the dipole. The equipotential lines are perpendicular to the flux lines at any given point in the field and they are related using the Cauchy-Reimann equations. For a far-field approximation, the dipole potential equation from [20] and the electric flux equation using the Cauchy-Reimann equations are given by (2a) and (2b) respectively.

$$\Phi(r, \gamma) = \frac{IR \cos(\gamma)}{2 \pi \sigma r^2} \quad (2a)$$

$$\Psi(r, \gamma) = \frac{IR \sin(\gamma)}{2 \pi \sigma r^2} \quad (2b)$$

Here, $r$ is the distance of a point of interest from the dipole centre which also happens to be the origin of the coordinate system. The above equations, which are in the polar coordinate system can be conveniently represented in the Cartesian coordinates as:

$$\Phi(z) = \frac{IRx}{2 \pi \sigma |z|^3} \quad (3a)$$

$$\Psi(z) = \frac{IRy}{2 \pi \sigma |z|^3} \quad (3b)$$

The complex potential in (1) is then given as:

$$V(z) = \frac{IRz}{2 \pi \sigma |z|^3} \quad (4)$$

### A. Epidermal Electric Image

Consider a planar model, circular in shape, having electric sensors around its periphery and radiating a dipole field. The body frame is assumed to be inclined to the real axis at angle $\theta$. Under the influence of this dipole field, a secondary dipole field is created at an object present to the right of the model, as shown in Fig. 1A. The equipotential lines due to this secondary field when superimposed on the model give the corresponding electric image. It can be seen that the zero-potential line intersects the model at two points close to $\gamma = 0$. Fig. 1B shows the electric image for this setup, with the potentials shown being non-dimensional normalized values.

### B. Straight Tunnel

A straight tunnel consists of two parallel walls, some fixed distance apart. For a single conducting wall shown in Fig. 2A, the original dipole can be considered to cast its image on the other side of the wall, so as to satisfy the boundary conditions imposed by the dipole. For a conducting wall, the polarity of the image is reversed as compared to that of the original dipole.

The combined equation for complex potential of the dipole and its image can be derived as (5), by using (4) and the theorem of superposition of electric fields.

$$V_{wall}(z) = \frac{IR z e^{-i \theta}}{2 \pi \sigma |z|^3} - \frac{IR (z - 2i a) e^{i \theta}}{2 \pi \sigma |(z - 2i a)|^3} \quad (5)$$

In the case of a tunnel, two primary images are formed corresponding to the two walls. But each of these images, in turn, imposes boundary conditions at the wall farther from it to give one more image. These images continue the same behavior to give an infinite number of images as shown in figure 2B. For analyzing this, the images can be divided into two groups, so that all the images in a group have the same orientation. One of the groups containing and centred at the original dipole can be subscripted as $a$, while the other group, centred at the first image on the left of the dipole, as $b$. At any point represented as $z$ (with respect to the centre of the tunnel), the complex potentials for these groups due to all the images contained in them can be represented using previous formulation:

$$V_a(z) = \frac{IR e^{-i \theta}}{2 \pi \sigma} \sum_{n=0}^{\infty} \frac{z - i(4an + y)}{|z - i(4an + y)|^3}$$

$$+ \frac{z + i(4an + y)}{|z + i(4an + y)|^3} \quad (6a)$$

$$V_b(z) = -\frac{IR e^{i \theta}}{2 \pi \sigma} \sum_{n=0}^{\infty} \frac{z - i(4an + 2a - y)}{|z - i(4an + 2a - y)|^3}$$

$$+ \frac{z + i(4an + 2a - y)}{|z + i(4an + 2a - y)|^3} \quad (6b)$$

Clearly, according to the Law of Superposition of electric fields, the total complex potential at any point represented as $z$ in the reference frame of the tunnel, is simply the sum of (6a) and (6b). This further implies that the complex perturbation potential at any point in the tunnel frame of

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**Fig. 1.** Electric Image, (A) Sensory model and perturbing dipole, (B) Electric image formed
reference, $V_{p}(z)$ is difference between the total complex potential and the complex potential due to the original dipole:

$$V_{p_{\text{tunnel}}}(z) = V_{a}(z) + V_{b}(z) - \frac{I}{2\pi \sigma} \frac{(z - i y) e^{-i \theta}}{|z - i y|^3} \quad (7)$$

### C. Circular Arena

If a single source with current $I$ is placed inside a circular arena, its image is formed on the outside of the arena on the radius vector passing through the pole. Electrostatics theory [18] provides us with the formulae for pole current and radial distance (superscripted with primes below) of the image:

$$I' = \frac{I}{y'} \quad \text{and} \quad y' = \frac{a^2}{y}$$

![Circular Arena Diagram](image)

This theory can be further adapted to a dipole with slight modifications, considering the fact that the dipole is a pair of oppositely charged single poles. For this purpose, we consider a radiating dipole at a distance $y$ from the centre of the arena of radius $a$ ($y < a$), inclined at an angle $\theta$ with the radius vector passing through its centre, as shown in figure 3. The dipole has a pole current magnitude of $I$ and pole separation distance of $2R$. An image dipole is formed at a distance $y'$ from the centre and is inclined at an angle $\theta'$ to the same radius vector described above. In the following discussion, all the parameters related to the image dipole are denoted by primes of the respective parameters for the original dipole. For all practical purposes, the image dipole can be assumed to have:

$$I' = \frac{I}{y' \sqrt{y' + R^2}} \quad \text{and} \quad \theta' \approx -\theta$$

The complex perturbation potential due to a dipole inside a circular arena equals the complex potential due to the image of the dipole and is given as:

$$V_{p_{\text{arena}}} = \frac{I' R'(z - i(y' - y)) e^{-i\theta'}}{2\pi \sigma |z - i(y' - y)|^3} \quad (8)$$

The real parts of (7) and (8) are the perturbation potentials. The task of our control system is to characterize the WFI sensory outputs for these potentials, interpret them and to close the loop using a WFI-based static output feedback.

### III. WIDE-FIELD INTEGRATION PROCESSING OF ELECTRIC IMAGES

In this section, the methodology for extracting control-relevant motion cues from spatial electric images, using the wide-field integration methods has been developed.

For this treatment, the electric images are weighted with general weights $F_i(\gamma) \in L_2[0, 2\pi]$. Analogous to what was proposed originally in [11], these weights, applied at different peripheral locations, $\gamma$, define the sensitivity of the electoreceptors to the electric images formed. The orientation $\gamma$, is measured in the body frame of the dipole with the X-axis along the two poles. The spatial inner product of the measured electric potentials, $\Phi$, at these locations, and the basis functions, $F_i(\gamma)$, for a vehicle with the pose $q = (x, y, \theta)$, are given as,

$$z_i(q) = \langle \Phi, F_i \rangle = \frac{1}{\pi} \int_0^{2\pi} \Phi(\gamma, q) \cdot F_i(\gamma) \, d\gamma \quad (9)$$

### A. Application of WFI Processing to Electrolocation

We select the Fourier basis to decompose the electric image into multiple Fourier harmonics:

$$\Phi = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\gamma) + \sum_{n=1}^{\infty} b_n \sin(n\gamma)$$

where the Fourier harmonics are defined as,

$$a_0(q) = \langle \Phi, 1/\sqrt{2} \rangle = \frac{1}{\sqrt{2}} \sum_{\gamma=0}^{2\pi} \Phi(\gamma, q) / \sqrt{2} \delta \gamma$$

$$a_n(q) = \langle \Phi, \cos(n\gamma) \rangle = \frac{1}{\pi} \sum_{\gamma=0}^{2\pi} \Phi(\gamma, q) \cdot \cos(n\gamma) \delta \gamma$$

$$b_n(q) = \langle \Phi, \sin(n\gamma) \rangle = \frac{1}{\pi} \sum_{\gamma=0}^{2\pi} \Phi(\gamma, q) \cdot \sin(n\gamma) \delta \gamma$$

![Circular Arena Diagram](image)
The above equations for Fourier harmonics are the discrete counterparts of the usual integral equations. These discretized equations are helpful in processing the measurements at a fixed number of sensors which form a non-continuous set of electric potential values.

For all the possible values of $q$ and $\gamma$ substituted in the above equations, $size(y) \times size(\theta)$ matrices are obtained, each corresponding to a different harmonic. Here, $y$ and $\theta$ are the local lateral offset and orientation of the vehicle in the global frame of reference respectively. According to the WFI control procedure [11], [12], [21], the linearized form of each harmonic decomposes the electric image into the corresponding states of the vehicle. The linearization that has been used is $z(x) = z(x_0) + \sum_i \frac{\partial z}{\partial x_i}(x_0)(x_i - x_0)$ with respect to the desired equilibrium point $x_0 = (0,0)$ for a straight tunnel for instance. The terms in the linearization can be found out from the above matrices; $z(x_0)$ is the value in the matrices at the equilibrium point and $\frac{\partial z}{\partial x_i}(x_0)$ is computed by performing a numerical differentiation of the above matrices, with respect to the variables in $x_0$. This is performed for various values of characteristic dimension of an environment, $a$, which can be the width for the tunnel or the radius of the circular arena. Finally, a curve is fit to these linearized Fourier harmonics at different values of $a$ to incorporate the their dependence on it.

The electric images become weaker as $a$ increases, since this places the perturbations outside the sensory volume [8]. Second degree exponentials are fit to the linearized Fourier harmonics to capture their variation with $a$. The final expressions for the linearized Fourier coefficients are of the form $f_i = d_i + k_{1i}y + k_{2i}\theta$. Here $d_i$, $k_{1i}$ and $k_{2i}$ are constant $2^{nd}$ order exponentials in terms of $a$ and can be represented as $p_1e^{q_1a} + p_2e^{q_2a}$, where $p_1$, $p_2$, $q_1$ and $q_2$ are integers, $q_1$ and $q_2$ being strictly negative. Thus, all the linearized Fourier harmonics are fully coupled and linear in $y$ and $\theta$ and have a DC term $d$. The Curve-fitting Toolbox in MATLAB was used to perform these approximations which carried out the curve-fitting with very low root mean-square error.

B. Processing of WFI Outputs

The WFI outputs, contain the information of the current states of the vehicle. We employ a Linear Least Square Estimator (LLSE) for estimating the states $\hat{y}$ and $\hat{\theta}$ from the harmonics. If $x = [y, \theta]'$ is the state vector and $z(x)$ are the linearized Fourier harmonics, then the state equation is:

$$z = Cx + d \Rightarrow z' = z - d = Cx$$  
(10)

The matrix $C$ contains all the coefficients of $y$ and $\theta$ terms in the linearized Fourier harmonics. The LLSE, $L$, performs a pseudo-inverse of (10) to output the estimates $\hat{x}$ [22]:

$$L = (C^T C)^{-1} C^T$$  
(11)

Finally, the estimated state vector is:

$$\hat{x} = Lz' = (C^T C)^{-1} C^T z'$$  
(12)

Ideally, for the estimation of two states, a fully constrained system of two Fourier harmonics is sufficient. But having an over-constrained system with more harmonics helps in compensating the effects due to the noise present in the measurements [23]. In these simulations, four harmonics, $a_1$, $a_2$, $b_1$ and $b_2$ have been used.

IV. WFI-BASED OUTPUT FEEDBACK CONTROL

In this section, the methodology for generating local unmapped obstacle avoidance for a planar electric fish model via static output feedback (Fig. 4) has been demonstrated. For the analysis, the vehicle can be assumed to have unicycle type kinematics and a non-holonomic constraint,

$$\dot{x} \sin \theta - \dot{y} \cos \theta = 0$$  
(13)

which enforces $\dot{y}_b = 0$ and $\dot{x}_b = v$. The kinematics of this kind of a model in the inertial frame are defined as,

$$\dot{x} = v \cos \theta$$  
$$\dot{y} = v \sin \theta$$  
$$\dot{\theta} = u_\theta$$  
(14)

For a vehicle having a constant velocity $v$, implies that $\dot{v} = u_y = 0$. The linearized $y$, $\theta$ dynamics derived from (14) are,

$$\left( \begin{array}{c} \dot{y} \\ \dot{\theta} \end{array} \right) = \left( \begin{array}{cc} 0 & v_0 \\ 0 & 0 \end{array} \right) \left( \begin{array}{c} y \\ \theta \end{array} \right) + \left( \begin{array}{c} 0 \\ 1 \end{array} \right) u_\theta$$  
(15)

For stabilizing the system, the control $u_\theta$ should consist of lateral $y$ and rotational $\theta$ stiffness terms.

$$u_\theta = K_y \dot{y} + K_{\theta} \dot{\theta}$$  
(16)

This is the control input which can be fed in as the $\dot{\theta}$ input to the vehicle kinematics. The resulting linearized closed loop dynamics are,

$$\left( \begin{array}{c} \dot{y} \\ \dot{\theta} \end{array} \right) = \left( \begin{array}{cc} 0 & v_0 \\ K_y & K_{\theta} \end{array} \right) \left( \begin{array}{c} y \\ \theta \end{array} \right)$$  
(17)

which has a characteristic equation,

$$s^2 - sK_{\theta} - v_0K_y = 0$$  
(18)

and eigenvalues,

$$s_{1,2} = \frac{K_{\theta} \pm \sqrt{K_{\theta}^2 + 4v_0K_y}}{2}$$  
(19)
We select $K_\theta < 0$ for the roots, $s$, to lie in the left half plane, and with the proper choice of $K_y$, stability as well as the desired performance can be achieved.

V. Simulation Results

In this section, the results for some obstacle avoidance simulations in a corridor and an arena, have been reproduced. For simulating the path of the autonomous vehicle in two different environments, bitmap images of the environment geometry were used. The code measured the depth at the current position and orientation at equally spaced peripheral points on the vehicle. The instantaneous perturbation potential, $\Phi$, was computed at these peripheral points by using either (7) or (8), depending on the type of the environment. Fourier harmonics of $\Phi$ at these points are used to generate the WFI outputs.

A. Infinite Straight Tunnel

The objective of this simulation is to make the vehicle center itself in the tunnel and maintain a constant zero heading afterwards. Fig. 5A shows the traversed path in 2.1 meters wide tunnel, first two sine and cosine Fourier harmonics and the variation of orientation, $\theta$, of the vehicle. The output gains are selected to be $K_y = 5$ and $K_\theta = -5$ based on the desired performance. It can be seen that the vehicle centers itself rapidly and maintains a constant zero heading afterwards as expected.

B. Circular Arena: Wall Following

In this simulation, the vehicle is expected to start at any initial condition in a circular arena of 6 meters diameter and align itself to the circular wall at a distance of 0.5 meter from the wall and hold that distance constant throughout.
by maintaining a constant turn rate. It can be seen in Fig. 5B, that the vehicle converges to the final path rapidly and maintains it throughout. Some rapid bumps, seen in the orientation and the response plots, result due to the fact that low DPI, non-anti-aliased bitmap images have been used as the environment, having a jagged periphery of the arena wall. When the vehicle is navigating closer to the wall, it is not seeing a smooth wall. So the vehicle does instantaneous adjustments depending upon the nature of the image at a particular point. Also, it can be noted that a constant mean non-zero turn rate $\theta$ is maintained by the vehicle after it converges to its desired path.

C. Circular Arena: Saccade

Simulation for this case involves a saccadic motion of the vehicle away from the wall when it senses the wall within its sensory volume. Same set of linearized Fourier harmonics as in the wall following case can be used for the saccade case as well. Whenever the vehicle senses the arena wall within a fixed distance, it initiates a rapid turn away from it at a random turn rate between $45 - 60 \, ^\circ/\sec$. This saccadic motion continues until the wall is sensed within the sensory volume. Fig. 5C shows the traversed path, responses of first two sine and cosine harmonics and the orientation plots for the saccade case.

VI. Conclusion

In this paper, the technique to achieve autonomous underwater navigation by extracting relative proximity from electric images was demonstrated. The basis of this exercise is the spatial decomposition of the wide-field patterns of electric images with sets of weighting functions. The major contribution of this paper is the representation of electric images in terms of analytical formulas and the development of a control-theoretic framework, used to directly derive the distance and orientation of a robotic vehicle relative to an obstacle from these images.

The linearized Fourier harmonics of an electric image are linearly coupled in $y$ and $\theta$ and the positional information cannot be directly inferred from them. So, a Linear Least Square Estimator was used to estimate the $[\hat{y}, \hat{\theta}]$ states from the harmonics. It was shown through simulations that the proposed robotic vehicle model can be made capable of performing centring, wall following and saccade behaviors by using these state estimates as feedback.

The WFI processing method was effectively used to decompose the electric images to extract the states of the system. For this purpose, we employ a control-theoretic approach which makes the use of standard control tools, to design gains, analyse control loop performance and examine the stability of the system, possible. Simplest type of estimator is implemented in the form of a static linear estimator, which is computed offline and occupies minimal memory storage space. The behaviors discussed in this paper were achieved without the use of any computationally expensive algorithms like particle filters [10], Kalman filters or other dynamic estimators. Hence, the whole process of extraction of the motion cues from the electric images via the WFI processing can be considered as a computationally efficient method for autonomous underwater navigation.

REFERENCES