A Multivariable MRAC Design Using State Feedback for Linearized Aircraft Models with Damage

Jiaxing Guo, Yu Liu and Gang Tao
Department of Electrical and Computer Engineering
University of Virginia
Charlottesville, VA 22904

Abstract—Aircraft damage causes large uncertain system parametric and structural changes which lead to unknown deviations of the aircraft equilibrium points and unknown coupling of the aircraft dynamics. To deal with such unknown changes, we use a sequential linear model with a dynamics offset to represent a linearized aircraft system before and after damage. A state feedback multivariable model reference adaptive control (MRAC) scheme is developed for such a linear model, with adaptive compensation of the uncertain dynamics offset as well as system parametric uncertainties. Desired closed-loop signal boundedness and asymptotic output tracking are established. Numerical values of linearized models from the NASA GTM were used in the simulation study and its results verified the desired system performance despite uncertain damage.

Keywords: Adaptive control, state feedback, output tracking, aircraft damage, flight control, linearization, GTM.

I. INTRODUCTION

Aerodynamic and structural damage, such as locked flaps or loss of wing tip, which can cause uncertain structural and parametric changes for aircraft systems, may lead to severe accidents. A controller is needed to tolerate the damage, keep desired aircraft performance before and after damage, and guarantee aircraft safety. Considerable effort has been devoted on it (e.g., [1], [3], [4], [5], [6], [7], [9], [10]).

A typical procedure of linearization-based design for the aircraft flight system is to linearize it at a given operating point, to design a control law based on the linearized system model, and to apply an augmented control law to the aircraft system around a neighborhood of the operating point. In this paper, we will focus on the first two steps to design a control scheme for the linearized aircraft system with damage. Since there are uncertainties in the linearized system due to the unknown changes of the nonlinear aircraft system caused by damage, we will employ a multivariable MRAC design to compensate the uncertainties and make the output signals of the linearized aircraft system with damage track some given reference signals.

For a multivariable MRAC design, there are two essential conditions: knowledge of the infinity zero structure of the system and that of signs of principal minors of the high frequency gain matrix. Whether these characteristics of the linearized aircraft system change due to damage is an important problem to study for MRAC. In [6], it gives a thorough analysis of these characteristics based on a generic aircraft model structure, and it concludes that the infinity zero structure is invariant, and so are the matrix signs, under certain damage conditions. Based on such system invariance, an output feedback for output tracking MRAC scheme is developed for the linearized system with damage in [6]. However, such a linearization-based design cannot be directly applied to the nonlinear aircraft system, since the system is linearized at an unknown equilibrium point due to the uncertain damage. The state or output signals of the linearized system are actually obtained by subtracting the operating point from the nonlinear signals, so that when the operating point is unknown, the state or output signals of the linearized system are unavailable. That is, to apply the linearization-based design, we cannot linearize a nonlinear system at an unknown operating point. For the damaged aircraft system, the equilibrium points are unknown due to the uncertain damage, we need to linearize the damaged aircraft system at a known operating point rather than the unknown equilibrium point.

In this paper, we will linearize the aircraft system at the same given operating point before and after damage to obtain a sequence of linear systems. Since the chosen operating point is not an equilibrium point of the damaged system, the sequential linear system has a dynamics offset. Such a dynamics offset is unknown because of the uncertain damage. Moreover, the linearized system has parametric uncertainties due to the damage. To compensate the unknown dynamics offset and the parametric uncertainties, we will employ a multivariable MRAC scheme. By applying the analysis method in [6], we can verify that the two important characteristics (invariance of infinity zero structures and matrix signs) of the linearized system model (the sequential linear system with the dynamics offset) do not change before and after damage, by selecting the operating point in the same form as the unknown equilibrium in [6]. That is the multivariable MRAC scheme can be used to compensate the uncertainties and offset for the sequential linear system.

We will apply an adaptive state feedback controller since a state feedback control law has a simpler structure than an output feedback one and the aircraft state signals are available for measurement. For a state feedback state tracking adaptive scheme, it will require different reference models.
under different matching conditions, which may not be satisfied due to the damage uncertainty for the nominal and damage cases. On the other hand, the state feedback for output tracking adaptive scheme has less restrictive matching conditions [2], so that we can design a uniform state feedback adaptive control law to make the output signals of the aircraft system track reference signals from a common reference model system, before and after damage. Moreover, a state feedback for output tracking MRAC is less complex than an output feedback for output tracking MRAC scheme [2]. Therefore we will employ a state feedback adaptive control law for the linearized system to achieve closed-loop signal boundedness and output signal tracking. A stability analysis and a simulation study of the linearized system with damage obtained from the NASA GTM model will demonstrate that the proposed controller can guarantee all the closed-loop signals of the linearized system are bounded and the output signals asymptotically track the reference signals.

The paper is organized as follows. In Section II, we will give a sequential linear system with an uncertain dynamics offset which can represent the linearized aircraft system before and after damage. An adaptive state feedback controller will be designed in Section III, where a nominal controller will be developed as a priori knowledge for the design of the adaptive scheme. In Section IV, the proposed scheme will be applied to a linearized GTM model with damage to demonstrate the desired output tracking performance.

II. PROBLEM STATEMENT

In this paper, we will consider a sequence of linear systems with unknown dynamics offsets, which can represent the linearized aircraft system models before and after damage.

A. Control Problem

Consider a sequential linear model with an unknown dynamics offset described as

\[ \dot{x}(t) = Ax(t) + Bu(t) + f_0, \quad y(t) = Cx(t), \]  

(1)

where \( A \in \mathbb{R}^{n \times n} \), \( B \in \mathbb{R}^{n \times m} \), and \( C \in \mathbb{R}^{m \times n} \) are unknown piecewise constant matrices with a finite number of unknown jumps \((A_i, B_i, C_i)\), \( i = 1, 2, \ldots, N \), and \( f_0 \in \mathbb{R}^{n} \) is an unknown piecewise constant offset with a finite number of unknown jumps \( f_{0i}, \quad i = 1, 2, \ldots, N \), such that \( A = A_i, \quad B = B_i, \quad C = C_i, \quad f_0 = f_{0i} \), for \( t \in [t_i-1, t_i) \), \( i = 1, 2, \ldots, N \), with \( t_0 = 0 \) and \( t_N = \infty \).

Control objective. Given that the state signals are available in an aircraft control system, and the state feedback for output tracking MRAC is less complex than the output feedback for output tracking MRAC scheme [2], our control objective is to construct a state feedback control law \( u(t) \) in (1) to make all the closed-loop signals bounded and the plant output signal \( y(t) \) track a given reference signal \( y_m(t) \in \mathbb{R}^{M} \) generated from the reference model system

\[ y_m(t) = W_m(s)[r](t), \]  

(2)

where \( W_m(s) \in \mathbb{R}^{M \times M} \) is a stable transfer matrix, and \( r(t) \in \mathbb{R}^{M} \) is a bounded reference input signal.

Assumptions. To proceed the control design, we assume:

(A1) All zeros of \( G_i(s) = C_i(sI - A_i)^{-1}B_i, \quad i = 1, 2, \ldots, N \), have negative real parts;

(A2) \( G_i(s), i = 1, 2, \ldots, N \), have full rank, there is an unknown modified left interactor matrix \( \xi_m(s) \) for all \( G_i(s), i = 1, 2, \ldots, N \), and the reference system transfer matrix \( W_m(s) = \xi_m^{-1}(s) \);

(A3) All leading principal minors \( \Delta_{ij}, i = 1, 2, \ldots, N, j = 1, 2, \ldots, M \), of each high frequency gain matrices \( K_{pi} = \lim_{s \to \infty} \xi_m(s)G_i(s) \) are nonzero and their signs are known, and \( \text{sign} [\Delta_{ij}] = \text{sign}[\Delta_{ij}], p, q = 1, 2, \ldots, N, j = 1, 2, \ldots, M \);

(A4) \((A_i, B_i)\) is controllable and \((A_i, C_i)\) is observable.

Next, we will show that the sequential linear system (1) can represent the linearized aircraft systems before and after damage. That is, based on such a linear system (1), we can apply a linearization-based design for the control problem of nonlinear aircraft systems with damage.

B. Control of Nonlinear Aircraft Systems with Damage

A nonlinear aircraft system without damage can be denoted as

\[ \dot{x}(t) = f(x(t), u(t)), \quad y(t) = Cx(t) = [\theta, \psi]^T, \]  

(3)

with state variables and input variables:

\[ x(t) = [u_b, v_b, \theta, v_r, p_b, \phi, \psi]^T, \quad u(t) = [d_e, d_r]^T, \]  

(4)

where \( u_b, v_b \) and \( \theta \) are the body-axis velocity components of the origin of the body-axis frame whose units are \( \text{ft/sec} \), \( p_b \), \( q_b \) and \( r_b \) are the body-axis components of the angular velocity whose units are \( \text{rad/sec} \), \( \phi \), \( \theta \) and \( \psi \) are the Euler roll, pitch and yaw angles of the aircraft body axes with respect to the reference axes whose units are radian, and \( d_e \) and \( d_r \) are the control surfaces—the elevator and rudder angular positions whose units are degree.

When damage occurs, it will cause uncertain parametric and structural variations. We denote the nonlinear aircraft dynamic model in the presence of damage as

\[ \dot{x}(t) = f_d(x(t), u(t)), \quad y(t) = Cx(t) = [\theta, \psi]^T, \]  

(5)

where \( f_d(t) \) is different from \( f(t) \) in (3) due to the damage.

Linearization. We linearize the nonlinear aircraft system at an operating point \((x_0, u_0)\) to obtain a linearized model. The operating point \((x_0, u_0)\) is important for the linearization-based design, since it connects the state, output, and control input signals between the nonlinear system and its linearized system in the following way:

\[ \Delta x(t) = x(t) - x_0, \quad \Delta y(t) = y(t) - Cx_0, \quad \Delta u(t) = u(t) - u_0, \]  

where \( \Delta x(t), \Delta y(t), \) and \( \Delta u(t) \) are the linearized system’s state, output, and input signals, and \( x(t), y(t), \) and \( u(t) \) are the nonlinear aircraft system’s signals.
If \((x_0, u_0)\) is unknown, we cannot obtain \(\Delta x(t)\) and \(\Delta y(t)\) from \(x(t)\) and \(y(t)\). Moreover, the linearization-based control law \(\Delta u(t)\) cannot be applied to the nonlinear system, since \(u_0\) is unknown in the nonlinear controller signal \(u(t) = \Delta u(t) + u_0\). That is, when operating point \((x_0, u_0)\) is unknown, we cannot employ a linearization-based design for the nonlinear system. So we need to linearize the system at a known operating point.

For the damaged system (5), the equilibrium point is unknown due to the uncertain damage, so that it cannot be chosen as linearization operating point. We need to choose a known operating point for the damaged system (5). Since the equilibrium point of the nominal system (3) can be obtained, in this paper, we will use an equilibrium point \((x_0, u_0)\) of the nominal system (3) as a linearization operating point for both the nominal system (3) and the damaged system (5) to ensure that the linearization-based design is meaningful.

The linearized nominal system at \((x_0, u_0)\) is expressed as
\[
\dot{\Delta}x = A_n \Delta x + B_n \Delta u + f(x_0, u_0), \quad \dot{\Delta}y = C \Delta x,
\]
(6)
since \((x_0, u_0)\) is the equilibrium point, \(f(x_0, u_0) = 0\). The linearized damaged system at \((x_0, u_0)\) is expressed as
\[
\dot{\Delta}x = A_d \Delta x + B_d \Delta u + f_d(x_0, u_0), \quad \dot{\Delta}y = C \Delta x,
\]
(7)
since \((x_0, u_0)\) is not the equilibrium of the damaged system (3), \(f_d(x_0, u_0) \neq 0\). Since the aircraft system has parametric uncertainties, the parameters in the linearized systems (6) and (7) are unknown, and the offset \(f_d(x_0, u_0)\) is unknown as well. The damaged system (7) is coupled due to damage, so we consider the lateral and longitudinal dynamics together for the aircraft systems with damage.

**Linearized aircraft system with damage.** From (6) and (7), the linearized aircraft systems before and after damage can be expressed as the sequential linear system (1):
\[
\dot{x}(t) = Ax(t) + Bu(t) + f_0, \quad y(t) = Cx(t),
\]
where \(A = A_n, B = B_n, f_0 = f(x_0, u_0)\), before damage occurs, and \(A = A_d, B = B_d, f_0 = f_d(x_0, u_0)\), after damage occurs. Thus, in this paper, for the linearization-based design of the nonlinear aircraft system with damage, we consider the control problem of the sequential linear system (1) presented in subsection II-A.

**Invariance of infinity zero structures.** Based on the analysis in [6], under certain damage conditions, the infinity zero structure and the signs of the leading principal minors of the high frequency gain matrix do not change before and after damage, which implies that the above assumptions (A2) and (A3) hold, for the linearized aircraft system (1). Thus, for the linearized aircraft system (1), we can apply a multivariable MRAC scheme, which will be derived next.

**State feedback controller design.** To compensate the constant offset term \(f_0\) in (1), we choose the state feedback controller structure as
\[
u(t) = K_1^T(t)x(t) + K_2(t)r(t) + k_3(t),
\]
(8)
where \(k_3(t) \in R^M\) is the adaptive estimate of an unknown constant compensation term \(k^*_3\), which will be derived next, for canceling the effect of the constant offset \(f_0\) and \(K_1(t)\) and \(K_2(t)\) are the estimates of the nominal \(K_1^*\) and \(K_2^*\) which satisfy the matching conditions
\[
C(sI - A - BK_1^*T)^{-1}BK_2^* = W_m(s), K_2^*-1 = K_p,
\]
(9)
where \(K_p\) is the piecewise constant high frequency gain matrix, for each jump, \(K_p = K_{pi} = \lim_{s \to \infty} \xi_m(s)G_i(s), i = 1, 2, \ldots, N\).

**Remark 1:** From Assumption (A2), all \((A_i, B_i, C_i), i = 1, 2, \ldots, N\), have the same interactor matrix \(\xi_m(s)\), and \(W_m(s) = \xi_m^{-1}(s)\). Based on [8], for each \((A_i, B_i, C_i), i = 1, 2, \ldots, N\), we can obtain a set of constant parameters \(K_1^*\) and \(K_2^*\) to satisfy the plant-model matching equations (9), so that the nominal \(K_1^*\) and \(K_2^*\) are piecewise constants. □

To derive \(k_3\), we apply a nominal controller
\[
u(t) = K_1^T(t)x(t) + K_2^*r(t) + k_3^*
\]
(10)
to the system (1) to achieve exact plant-model matching.

To define the matching parameter vector \(k_3^*\), we consider a particular set of constant values of the system parameters \((A, B, f_0)\). Then, substituting (10) in the plant (1), we have the closed-loop system in the frequency \(s\)-domain as
\[
y(s) = C(sI - A - BK_1^*T)^{-1}BK_2^*r(s) + \Delta(s),
\]
(11)
where
\[
\Delta(s) = C(sI - A - BK_1^*T)^{-1}(Bk_3^* + f_0)\]
(12)
From the reference system (2) and the matching conditions (9), we can have the output tracking error in the frequency \(s\)-domain as
\[
e(s) = y(s) - y_m(s) = \Delta(s).
\]
(13)
Applying the Laplace final value theorem, we obtain
\[
\lim_{t \to \infty} e(t) = \lim_{s \to 0} s\Delta(s) = Dh_3^* + d
\]
(14)
for some constant invertible matrix \(D\) and vector \(d\). For offset rejection, we set
\[
k_3^* = -D^{-1}d,
\]
(15)
and then from (14)–(15), we have
\[
\lim_{t \to \infty} (y(t) - y_m(t)) = \lim_{t \to \infty} \delta(t) = 0
\]
(16)
exponentially fast, where \(\delta(t) = L^{-1}[\Delta(s)]\).

Since the system parameters \((A, B, f_0)\) are piecewise constant, the nominal matching parameter \(k_3^*\) is also piecewise constant, as defined above for each set of \((A, B, f_0)\).

From the matching conditions (9) and (15), we can conclude that there exists a nominal controller (10) to achieve
the asymptotic output tracking. However, we will use the adaptively updated control law (8), since the parameters $K_1^*, K_2^*$, and $k_3^*$ are unknown. In the following, we will first develop the tracking error equation by applying the control law (8) to the system (1).

**Tracking error equation.** Substituting the control law (8) in the plant (1), we have
\[
\dot{x}(t) = AX(t) + B(K_1^T(t)x(t) + K_2^T(t)r(t) + k_3(t)) + f_0 \\
= (A + BK_1^T(t)x(t) + BK_2^T(t)r(t) + Bk_3^* + f_0 \\
+ B(K_1^T(t)x(t) + K_2^T(t)r(t) + k_3(t)) \\
y(t) = Cx(t),
\]
(17)
where $K_1^T(t) = K_1^T(t) - K_1^*$, $K_2^T(t) = K_2(t) - K_2^*$, and $k_3(t) = k_3(t) - k_3^*$.

In view of the reference model (2), matching conditions (9), (15), and (17), the output tracking error is
\[
e(t) = y(t) - y_m(t) = W_m(s)K_p[\tilde{\Theta}^T(t)\omega(t) + \delta(t),
\]
(18)
where
\[
\tilde{\Theta}(t) = \Theta(t) - \Theta^*, \quad \Theta(t) = [K_1^T(t), K_2^T(t), k_3(t)]^T, \\
\Theta^* = [K_1^*, K_2^*, k_3^*]^T, \quad \omega(t) = [x(t), r^T(t), 1]^T.
\]
(19)
To deal with the uncertainty of the high frequency gain matrix $K_p$, we use its LDS decomposition
\[
K_p = L_sD_sS
\]
(20)
where $S \in R^{M \times M}$ with $S = S^T > 0$, $L_s$ is an $M \times M$ unit lower triangular matrix, and
\[
D_s = \text{diag}\{s_1^*, s_2^*, \ldots, s_M^*\} \\
= \text{diag}\{\text{sign}[\Delta_1^\gamma_1], \ldots, \text{sign}[\frac{\Delta_M}{\Delta_{M-1}^\gamma_M}]\gamma_M\}
\]
(21)
such that $\gamma_i > 0$, $i = 1, \ldots, M$, may be arbitrary [11].

**Remark 2:** In the adaptive laws design, we will use $D_s$ matrix as a gain matrix. Although $K_p$ is a piecewise constant, we can choose a uniform $D_s$ for all the high frequency gain matrices from the Assumption (A3) for the adaptive laws. \(\square\)

Substituting the LDS decomposition of $K_p$ (with a uniform $D_s$) (20) in (18), and ignoring the exponentially decaying term $\delta(t)$, we have
\[
L_s^{-1}\xi_m(s)[e(t)] = D_sS\tilde{\Theta}^T(t)\omega(t).
\]
(22)
We introduce a filter $h(s) = 1/f_h(s)$, where $f_h(s)$ is a stable and monic polynomial of degree equals to the degree of $\xi_m(s)$. Operating both sides of (22) by $h(s)I_M$ leads to
\[
L_s^{-1}\xi_m(s)h(s)[e(t)] = D_sS h(s)[\tilde{\Theta}^T(t)\omega(t)].
\]
(23)
To parameterize the unknown matrix $L_s$, we introduce
\[
\Theta_0^* = L_s^{-1} - I = \begin{bmatrix}
0 & 0 & \cdots & 0 \\
0 & \theta_{21}^* & \cdots & 0 \\
\theta_{31}^* & \theta_{32}^* & \cdots & 0 \\
\theta_{M-1}^* & \cdots & 0 & 0 \\
\theta_{M1}^* & \cdots & \theta_{MM-1}^*
\end{bmatrix}
\]
(24)
Then we have
\[
\bar{e}(t) + [0, \theta_{21}^T(t)\eta_2(t), \theta_{31}^T(t)\eta_3(t), \ldots, \theta_{MM}^T(t)\eta_M(t)]^T \\
= D_sS h(s)[\tilde{\Theta}^T(t)\omega(t)],
\]
(25)
where
\[
\bar{e}(t) = \xi_m(s)h(s)[e(t)] = [\bar{e}_1(t), \ldots, \bar{e}_M(t)]^T,
\]
(26)
\[
\eta_i(t) = [\bar{e}_1(t), \ldots, \bar{e}_{i-1}(t)]^T \in R^i, i = 2, \ldots, M,
\]
(27)
\[
\theta_{ii}^* = [\theta_{11}^*, \ldots, \theta_{ii-1}^*]^T, \quad i = 2, \ldots, M.
\]
(28)

**Estimation error.** From (25), we introduce the estimation error signal
\[
e(t) = \begin{bmatrix}
0, \theta_{21}^T(t)\eta_2(t), \theta_{31}^T(t)\eta_3(t), \ldots, \theta_{MM}^T(t)\eta_M(t)
\end{bmatrix}^T \\
+ \Psi(t)\xi(t) + \bar{e}(t),
\]
(29)
where $\theta_{ii}(t), i = 2, 3, \ldots, M$ are the estimates of $\theta_{ii}^*$, and $\Psi(t)$ is the estimate of $\Psi^* = D_s S$.

\[
\xi(t) = \Theta(t)\zeta(t) - h(s)[\Theta^T(t)\omega(t)]
\]
(30)

From (25)–(30), we can derive that
\[
e(t) = \begin{bmatrix}
0, \theta_{21}^T(t)\eta_2(t), \theta_{31}^T(t)\eta_3(t), \ldots, \theta_{MM}^T(t)\eta_M(t)
\end{bmatrix}^T \\
+ D_s S\tilde{\Theta}^T(t)\xi(t) + \Psi(t)\xi(t),
\]
(31)
where $\hat{\theta}_{ii}(t) = \theta_{ii}(t) - \theta_{ii}^*$, $i = 2, 3, \ldots, M$, and $\hat{\Psi}(t) = \Psi(t) - \Psi^*$ are the related parameter errors.

**Adaptive laws.** With the estimation error model (31), we choose the adaptive laws
\[
\dot{\theta}_{ii}(t) = -\frac{\Gamma_{ii}e_i(t)\eta_i(t)}{m^2(t)}, \quad i = 2, 3, \ldots, M
\]
(32)
\[
\dot{\Theta}^T(t) = -\frac{D_s e(t)\zeta(t)}{m^2(t)}
\]
(33)
\[
\dot{\Psi}(t) = -\frac{\Gamma(m(t))\zeta(t)}{m^2(t) + \zeta^T(t)\zeta(t) + \sum_{i=2}^M \eta_i^T(t)\eta_i(t)}
\]
(34)
where the signal $e(t) = [e_1(t), e_2(t), \ldots, e_M(t)]^T$ is computed from (29), $\Gamma_{ii}(t) = \Gamma_{ii}^* > 0$, $i = 2, 3, \ldots, M$, and $\Gamma = \Gamma^T > 0$ are adaptation gain matrices, and
\[
m(t) = (1 + \zeta^T(t)\zeta(t) + \zeta^T(t)\xi(t) + \sum_{i=2}^M \eta_i^T(t)\eta_i(t))^{1/2}
\]
(35)
is a standard normalization signal.

**Stability analysis.** From the adaptive laws (32)–(34), we have the following desired stability properties.

**Lemma 1:** The adaptive laws (32)–(34) ensure that
(i) $\theta_{ii}(t) \in L^\infty$, $i = 2, 3, \ldots, M$, $\Theta(t) \in L^\infty$, $\Psi(t) \in L^\infty$, and $\frac{\epsilon_i(t)}{m^2(t)} \in L^2 \cap L^\infty$;
(ii) $\dot{\theta}_{ii}(t) \in L^2 \cap L^\infty$, $i = 2, 3, \ldots, M$, $\dot{\Theta}(t) \in L^2 \cap L^\infty$, and $\dot{\Psi}(t) \in L^2 \cap L^\infty$.

**Proof:** Consider a positive definite function
\[
V = \frac{1}{2} \sum_{i=2}^M \bar{\theta}_{ii}^T \Gamma_{ii} \bar{\theta}_{ii} + \text{tr}[\hat{\Psi}^T \Gamma^{-1} \hat{\Psi}] + \text{tr}[\hat{\Theta} S \hat{\Theta}^T]
\]
(36)
which is continuous at each interval \((t_{i-1}, t_i), i = 1, 2, \ldots, N\), with \(t_0 = 0\) and \(t_N = \infty\), and has a finite jump at \(t_i, i = 1, 2, \ldots, N - 1\), i.e.,

\[
V(t_i^-) - V(t_i^+) < \infty, i = 1, 2, \ldots, N - 1. \tag{37}
\]

From the adaptive laws (32)-(34), we obtain the time-derivative of \(V\) in each \((t_{i-1}, t_i), i = 1, 2, \ldots, N\) as

\[
\dot{V} = \frac{e^T(t)ce(t)}{m^2(t)} \leq 0. \tag{38}
\]

That is \(V(t_i^-) \leq V(t_{i-1}^+)\), and from (37), we can conclude that \(V(t)\) is bounded for \([0, \infty)\). So that \(\dot{\theta}_i(t) \in L^\infty, i = 2, 3, \ldots, M, \Theta(t) \in L^\infty, \Psi(t) \in L^\infty, \frac{d\Theta}{dt}(t) \in L^2 \cap L^\infty, \quad \hat{\theta}_i(t) \in L^2 \cap L^\infty, i = 2, 3, \ldots, M, \Theta(t) \in L^2 \cap L^\infty, \quad \nabla \Psi(t) \in L^2 \cap L^\infty\).

Based on these properties, we have the desired closed-loop system properties as summarized in the following theorem.

**Theorem 1:** The multivariable MRAC scheme with the state feedback control law (8) updated by the adaptive laws (32)-(34), when applied to the plant (1), guarantees the closed-loop signal boundedness and asymptotic output tracking: \(\lim_{t \to \infty}(y(t) - y_m(t)) = 0\), for any initial conditions.

### IV. AIRCRAFT FLIGHT CONTROL APPLICATION

In this section, we will apply the above MRAC scheme to a linearized aircraft model with damage, which is obtained by linearizing the NASA generic transport model (GTM) under nominal and damage conditions. The GTM model contains several damage scenarios, in this study, we choose the damage case with loss of outboard left wing-tip.

**System description.** Based on the analysis in [6], we linearize the nominal and the damaged GTM models (3) and (5) at an operating point \((x_o, t_o)\), where \(x_o = [u_o, \omega_o, 0, \theta_o, 0, 0, 0, 0, \psi_o]^T\), and \(u_o = [d_o, d_v]^T\) to ensure the invariance of the infinity zero structure and the signs of principal minors of the high frequency gain matrices. To obtain such an operating point, we trim the nominal GTM at a wings-level steady-state flight condition with the equivalent airspeed as 90 knots and roll angle as 0 radian. The linearized aircraft system is described as

\[
\dot{x}(t) = Ax(t) + Bu(t) + f_o, \quad y(t) = Cx(t), \tag{39}
\]

where \(A = [A_{11}, A_{12}], B = B_1, f_o = 0\), during the nominal flight condition, \(A = [A_{21}, A_{22}], B = B_2\), and \(f_o = [0.04, -0.93, -0.09, 0.09, 0.21, 0.48, 0.02, 0]^{T}\), after the loss of outboard left wing-tip, and

\[
A_{11} = \begin{bmatrix}
-0.0380 & 0.2807 & -7.4780 & -32.10 \\
-0.2460 & -3.4110 & 146.2 & -2.1750 \\
0.0130 & -0.3579 & -4.4020 & 0 \\
0 & 0 & 1 & 0 \\
0 & -0.0004 & 0.0017 & 0 & -0.0005 \\
0 & -0.0009 & -0.0293 & 0 \\
0 & 0 & 0.0020 & 0 \\
0 & 0 & 0 & 0 & -0.0002 \\
0 & 0 & 0.0002 & 0
\end{bmatrix},
\]

\[
A_{12} = \begin{bmatrix}
-0.0037 & -0.0005 & 0 & 0 & 0 \\
0.0004 & 0 & 0.0005 & -0.0069 & 0 \\
0.0085 & 0.0377 & -0.0016 & 0 & 0 \\
0 & -0.0002 & 0 & 0.0002 & 0 \\
-0.6868 & -151.5 & 7.9150 & 32.10 & 0 \\
0.2735 & -1.6850 & -0.2679 & 0 & 0 \\
-0.7289 & 1.8910 & -6.7690 & 0 & 0 \\
0 & 0.0678 & 1 & 0 & 0 \\
0 & 1.0020 & 0 & 0.0002 & 0
\end{bmatrix},
\]

\[
A_{21} = \begin{bmatrix}
0.0099 & -0.2118 & 0 & 0 & 0 \\
0.5486 & 0 & 0.2118 & 0.7877 & 0 \\
-0.0046 & -0.4568 & 0.1079 & 0 & 0 \\
0 & 0.0245 & 0 & 0.0082 & 0 \\
0.6777 & -151.4 & 8.86 & 32.08 & 0 \\
0.2803 & -1.7270 & -0.3438 & 0 & 0 \\
-0.8383 & 2.3290 & -6.1570 & 0 & 0 \\
0 & 0.0728 & 1 & 0.0070 & 0 \\
0 & 1.0020 & 0 & 0.0962 & 0
\end{bmatrix},
\]

\[
A_{22} = \begin{bmatrix}
0.0011 & -0.0098 & 0 & 0 & 0 \\
-0.8699 & -0.1217 & 0 & -0.8875 & -0.1349 \\
-1.0860 & 0 & -1.0690 & -0.0015 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0.6157 & 0 & 0.6269 & 0 \\
0 & -0.5951 & 0.0056 & -0.6114 & 0 \\
0 & 0.3826 & 0.1264 & 0.4697 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix},
\]

\[
B_1 = \begin{bmatrix}
0 & 0 \\
0 & 0.6157 \\
0 & -0.5951 \\
0 & 0.3826 \\
0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}, \quad B_2 = \begin{bmatrix}
0 & 0 \\
0 & 0.6269 \\
0 & 0.6114 \\
0 & 0.4697 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix},
\]

\[
C = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}.
\]

**Verification of design conditions.** It can be verified that all zeros of \(G_1(s) = C(sI - A_1)^{-1}B_1\) with \(A_1 = [A_{11}, A_{12}]\) and \(G_2(s) = C(sI - A_2)^{-1}B_2\) with \(A_2 = [A_{21}, A_{22}]\) have negative real parts, and \(G_1(s)\) and \(G_2(s)\) have full rank.

We can choose a common interacter matrix for both \(G_1(s)\) and \(G_2(s)\) as \(\xi_m(s) = \text{diag}\{(s + 1)^2, (s + 1)^2\}\), such that the high frequency gain matrix for the nominal case is

\[
K_{p1} = \lim_{s \to \infty} \xi_m(s)G_1(s) = \begin{bmatrix}
-1.086 & 0 & 0 \\
0 & 0 & -0.596
\end{bmatrix}, \tag{40}
\]

and the high frequency gain matrix for the damage case is

\[
K_{p2} = \lim_{s \to \infty} \xi_m(s)G_2(s) = \begin{bmatrix}
-1.069 & 0.032 \\
0 & -0.617 & -0.612
\end{bmatrix}. \tag{41}
\]

Signs of first leading principal minor of \(K_{p1}\) and \(K_{p2}\) are

\[
\text{sign} (\Delta_{11}) = \text{sign} (\Delta_{21}) = -1, \tag{42}
\]
and signs of second leading principal minor are
\[ \text{sign}(\Delta_{21}) = \text{sign}(\Delta_{22}) = 1, \quad (43) \]
which verifies there is no sign change of the principal minors.

**Reference model.** From the common interactor matrix \( \xi_m(t) \) for both nominal and damage cases, we choose the transfer matrix of the reference model (2) as \( W_m(s) = \xi_m^{-1}(s) = \text{diag}\{1/(s+1)^2, 1/(s+1)^2\} \).

**Design parameters.** Since the degree of \( \xi_m(t) \) is 2, we choose the filter \( h(s) = 1/(s+8)^2 \). For the adaptive laws (32)–(34), we choose \( \Gamma \theta_2 = 10, \Gamma = \text{diag}\{10, 10\} \), and \( D_s = \text{diag}\{-30, -30\} \) because of the no sign change property of the principal minors.

**Simulation results.** To make a reasonable aircraft flying trajectory, we select a reference input as \( r(t) = [8\pi/180, 15\pi/180]^T \). By applying the control law (8) with the adaptive laws (32)–(34), we can see that, from Fig. 1, the output signal \( y(t) = [\theta(t), \psi(t)]^T \) (solid) tracks the reference signal \( y_m(t) = [\theta_m(t), \psi_m(t)]^T \) (dotted) after damage occurs at 200 seconds.

Moreover, when the reference inputs are varying, we can also have the similar simulation results. We choose a reference input as \( r(t) = [8\pi/180 \sin(0.015t), 8\pi/180 \sin(0.015t)]^T \). From Fig. 2, we can see that the output signals track the reference signals after damage happens at 500 seconds.

**V. CONCLUSIONS**

In this paper, a multivariable state feedback for output tracking MRAC scheme has been developed for the linearized aircraft model with damage. We linearized both nominal and damaged aircraft systems at a given operating point to obtain a sequential linear system with a dynamics offset. We designed an adaptive state feedback controller to compensate the parametric uncertainties and the offset, and make the signals of closed-loop system bounded and the output signals track some reference signals. A simulation study of the linearized system obtained from the GTM model with damage verified desired performance of the scheme. Our future work is to apply this linearization-based design to the nonlinear GTM model to assess the performance.

**ACKNOWLEDGEMENTS**

This research was supported in part by NASA under the grant NNX08AB99A and NSF under the grant ECS0601475.

**REFERENCES**