Optimal Sensor Design for Estimation and Optimization of PDE Systems

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Abstract— In many situations one is interested in the knowledge of the state of a PDE system on a sub-domain or region in the spatial domain. This type of problem occurs naturally when one is interested in regional control. Also, in cases when full spatial control is the goal and the feedback law has compact spatial support so that the observer need only estimate the local spatial behavior of the system. We consider a class of sensor location problems where the goal is to provide optimal zonal estimation and control for a parabolic distributed parameter system. Numerical examples are provided to illustrate the approach. We observed that in many cases the optimization problem is mesh independent and lends itself to parallel and multi-grid optimization approaches.

I. INTRODUCTION AND PROBLEM FORMULATION

In the scenario described by the Figure 1 below, one is interested in estimating the state in the green zone rather than in the entire space. This problem falls into a class of so-called regional observation and estimation problems introduced by El Jai and his co-workers (see [1]–[4], [14]–[17], [25]–[32]). The concept of zonal “protector control” discussed in [21] has applications to a wide variety of problems. Focusing on regional observer design should allow for a reduction in the number of physical sensors and, as we illustrate below, offers the potential to reduce computational requirements in some cases. To illustrate these points and discuss research challenges we focus on the advection diffusion system.

Consider an advection-diffusion process in the region \( \Omega = [0, 1] \times [0, 1] \times [0, 1] \subset \mathbb{R}^3 \) with boundary \( \Gamma \). The system is described by the partial differential equation with disturbance \( \eta(t) \) given by

\[
\frac{\partial}{\partial t} T(t, \bar{x}) = c^2 \nabla^2 T(t, \bar{x}) + \sum_{k=1}^{m} g_k(\bar{x}) \eta_k(t),
\]

with boundary and initial conditions

\[
T(t, \bar{x}) \mid_{\Gamma} = 0, \quad T(0, \bar{x}) = T_0(\bar{x}).
\]

Here, we assume the advection coefficient \( \kappa(\bar{x}, \bar{\theta}) \) is parameterized by a known constant vector \( \bar{\theta} \). Although we do not treat the case where \( \bar{\theta} \) is unknown, one can reformulate the problems below to include a joint state estimation / parameter identification problem (see [22]). However, this problem lies outside the scope of this short paper. Assume one has two mobile sensors and one fixed sensor as illustrated in Figure 1. Also, we will be concerned with estimating the state or averaged value of the state \( T(t, x, y, z) = T(t, \bar{x}) \) on \( \Omega \) (or on the sub-domain \( \mathcal{R} \subset \Omega \)). Note that the region \( \mathcal{R} \) could include part (or all) of the boundary of \( \Omega \).

We assume that there are \( p = 3 \) heterogeneous sensor platforms and each produces a single local spatial average of the the state \( T(t, \bar{x}) \). If \( \bar{\gamma}_i(t) = [x_i(t), y_i(t), z_i(t)]^T \in \Omega \), \( i = 1, 2 \) is the position of the \( i^{th} \) mobile sensor, then we let

\[
y_i(t) = \int_{B_{\delta}(\bar{\gamma}_i(t)) \cap \Omega} h_i(\bar{x}) T(t, \bar{x}) d\bar{x} + v_i(t)
\]

denote the (noisy) measured output from these two sensors.

Fig. 1. Zonal Problem with Fixed and Mobile Sensors

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Observe that the weighting function \( h_i(\bar{x}) \) and sensor range

\[
B_h(\gamma_i(t)) \triangleq \{ \bar{x} \in \mathbb{R}^3 : \| \bar{x} - \gamma_i(t) \| < \delta_i \}
\]

can be different for each type of mobile platform. The fixed sensor is centered at \( \gamma_3 = \bar{x}_3 \) and produces sensed information in the region \( \Omega_3 = B_h(\gamma_3) \subset \bar{\Omega} \) of the form

\[
y_3(t) = \int_{B_h(\gamma_3)} h_3(\bar{x})T(t, \bar{x})d\bar{x} + v_3(t).
\]

We assume the mobile sensor platforms satisfy the differential equations

\[
\dot{\tilde{w}}_i(t) = f_i(\tilde{w}_i(t), \bar{u}_i(t)), \quad \tilde{w}_i(0) = \tilde{w}_{i,0}
\]

and the platform locations are given by

\[
\gamma_i(t) = H_i\bar{w}_i(t).
\]

For the network of mobile and fixed sensors defined by \( \gamma_i(t), \quad i = 1, 2 \) and \( \gamma_3(t) = \gamma_3 \), respectively, we define the output map \( C(t) : L_2(\Omega) \to \mathbb{R}^3 \) by

\[
C(t)\varphi(\cdot) = \begin{bmatrix} C_1(t) \\ C_2(t) \\ C_3(t) \end{bmatrix} \varphi(\cdot) \in \mathbb{R}^3
\]

where

\[
C_i(t)\varphi(\cdot) = \int_{B_h(\gamma_i(t))} h_i(\bar{x})T(t, \bar{x})d\bar{x}.
\]

Thus, the measured output has the form

\[
y(t) = C(t)T(t, \cdot) + E\bar{v}(t),
\]

where \( E : \mathbb{R}^3 \to \mathbb{R}^3 \).

One can formulate an abstract (infinite dimensional) model of the form

\[
\dot{z}(t) = A(\bar{\theta})z(t) + G\bar{q}(t) \in Z,
\]

\[
\tilde{w}_i(t) = f_i(\tilde{w}_i(t), \bar{u}_i(t)), \quad \tilde{w}_i(0) = \tilde{w}_{i,0}
\]

with outputs

\[
y(t) = C(t)z(t) + E\bar{v}(t),
\]

\[
\gamma_i(t) = H_i\bar{w}_i(t),
\]

where the state of the distributed parameter system is \( z(t) : \Omega \to \mathbb{R}^3 \).

The case with a stationary sensor fixed at \( \gamma_3 \) can be included in the above framework by defining \( \gamma_3(t) = \bar{w}_3(t) \), setting the third right hand side to zero

\[
f_3(\bar{w}_3, \bar{u}_3) = 0,
\]

and using the initial condition

\[
\bar{w}_3(0) = \gamma_3.
\]

Define the variables \( \bar{u}(\cdot) = (\bar{u}_1(\cdot), \bar{u}_2(\cdot), \bar{u}_3(\cdot)) \) and \( \bar{q} = (\bar{q}_1, \bar{q}_2, \bar{q}_3) \). In this setting, the problem is to find "optimal controls" \( \bar{u}(\cdot) \) and initial states \( \bar{q} \) to minimize a specified cost function of the form

\[
J(\bar{u}(\cdot), \bar{q}(\cdot), \bar{q})
\]

depending on \( \bar{u}(\cdot), \bar{q}(\cdot) \) and \( \bar{q} \).

To illustrate a "typical" estimation problem associated with this type of system we consider a modification of the filtering problem first formulated by Bensoussan, Curtain and Ichikawa in the early 1970’s. (see [6], [7], [12]). One approach to full state optimal estimation is to observe that the variance equation for the Kalman filter satisfies an infinite dimensional Riccati (partial) differential equation of the form

\[
\Sigma(t) = A(\bar{\theta})\Sigma(t) + \Sigma(t)A(\bar{\theta})^* + GG^* - \Sigma(t)C^*(t)C(t)\Sigma(t),
\]

with initial data

\[
\Sigma(t_0) = \Sigma_0,
\]

and to formulate an optimal sensor management problem as a control problem for (I.13). This formulation leads to the following (nonlinear) distributed parameter optimal control problem:

**Problem A**: Find \( \bar{u}(\cdot) \) and \( \bar{q}_{opt} \) so that when

\[
C_{opt}(\cdot) = C(\bar{u}_{opt}, \bar{q}_{opt})
\]

is defined by (I.5)-(I.6), then

\[
J(\bar{u}(\cdot)) = \text{Tr}(S_{i,1}\Sigma(t_f)) + \int_{t_0}^{t_f} \text{Tr}(Q(t)\Sigma(t))dt
\]

is minimized.

Here, \( \Sigma(t) = \Sigma(\bar{u}, \bar{q}) \) is the mild solution of (I.13)-(I.14), and for each \( t \in [t_0, t_f] \), the linear operators \( S_{ij} \) and \( Q(t) : L_2(\Omega) \to L_2(\Omega) \) are bounded and positive definite. The (time-varying) map \( Q(\cdot) \) allows one to weight significant parts of the state estimate and account for zonal control. The cost on the control \( N((\bar{u}, \bar{q})) \) could account for energy or fuel consumption and additional state and control constraints might be imposed.

Existence of the optimal solution for **Problem A** above were proven in [8] and will not be repeated here. The focus
of this paper is on the computational aspects of these types of problems.

A. Numerical Results for a 2D Problem

To illustrate the typical results we present numerical solutions to the 2D version of Problem A with 3 mobile sensors. In this case we assumed linear dynamics for the platforms of the form

\[ \dot{w}_i(t) = A_i w_i(t) + B_i u_i(t), \quad w_i(0) = w_{i,0}, \]

and the platform locations are given by

\[ \vec{\gamma}_i(t) = [x_i(t) \ y_i(t)]^T = H_i w_i(t) \]

for \( i = 1, 2, 3 \).

A standard linear finite element scheme was used to discretize the advection-diffusion equation. Figures 2 and 3 illustrate typical numerical results. Here \( \kappa(\vec{x}, \vec{\theta}) = [0 \ 0]^T \) and the initial controls were set to zero. With no control the sensor trajectories move in straight line paths. After five Gauss-Newton steps, the Gauss-Newton algorithm converged and the optimal trajectories are shown in Figure 2. Figure 3 shows the optimal controllers. Figure 4 show the optimal sensor paths when \( \kappa(\vec{x}, \vec{\theta}) = [25 \ 25]^T \). This example is typical of the numerical runs conducted on this class of problems. In particular, the Gauss-Newton algorithm converged in relatively few steps and in most cases the time-stepper could be very coarse. We also used reduced order models and observed essentially the same behavior.

II. ZONAL ESTIMATION

Consider the case where one is interested in estimating the state or function of the state only in the zone \( \mathcal{R} \). Here we assume that \( Y \) is a Hilbert space and \( D : L_2(\Omega) \rightarrow Y \) is a bounded linear operator. The case where \( Y = L_2(\mathcal{R}) \) and \( D : L_2(\Omega) \rightarrow L_2(\mathcal{R}) \) is the restriction operator

\[ [D \varphi](\vec{x}) \triangleq \varphi(\vec{x})|_{\mathcal{R}}. \quad (II.1) \]

includes the problem of full state estimation considered above when \( \mathcal{R} = \Omega \).

The problem of estimating a zonal average (e.g. average temperature in a zone) leads to the case where \( D : L_2(\Omega) \rightarrow \mathcal{R} \) is defined by

\[ r(t) = DT(t, \cdot) \triangleq \frac{1}{\text{mea}(\Omega)} \int_{\Omega} \int d(\vec{x}) T(t, \vec{x}) d\vec{x} \]

where the kernel has support over the region \( \mathcal{R} \).

In any case, the problem is to construct a zonal estimator

\[ \hat{T}(t, \vec{x}) \]

of \( T(t, \vec{x})|_{\mathcal{R}}, \vec{x} \in \mathcal{R} \) to minimize a measure of the error

\[ e(t) = r(t) - D\hat{T}(t, \cdot). \quad (II.2) \]
There are several possible measures of the error and each leads to distinct optimization problems with varying computational complexity. However, this zonal estimation problem often has a structure that can be exploited to build low order observers and reduce computational requirements. The following trivial example illustrates these points.

**Example 1:** Consider the 1D heat equation on $\Omega = (0, 2)$ defined by

$$\frac{\partial}{\partial t} T(t, x) = c^2 \frac{\partial^2}{\partial x^2} T(t, x), \quad t > 0, \ x \in (0, 2) \quad (\text{II.3})$$

with initial and boundary conditions

$$T(0, x) = \varphi(x), \quad T(t, 0) = 0 \quad \text{and} \quad T(t, 2) = 0. \quad (\text{II.4})$$

Let $R = [0, 1] \subset \Omega$ and define $\hat{T}(t, x) \triangleq DT(t, x) = T(t, x)|_R$. Clearly $\hat{T}(t, x)$ satisfies

$$\frac{\partial}{\partial t} \hat{T}(t, x) = c^2 \frac{\partial^2}{\partial x^2} \hat{T}(t, x), \quad t > 0, \ x \in (0, 1) \quad (\text{II.5})$$

with initial and boundary conditions

$$\hat{T}(0, x) = \varphi(x), \quad \hat{T}(t, 0) = 0 \quad \text{and} \quad \hat{T}(t, 1) = T(t, 1). \quad (\text{II.6})$$

Note that any solution $\zeta(t, x)$ of (II.5)-(II.6) must equal $\hat{T}(t, x)$ on $R = [0, 1]$. Therefore if one ignores noise in the system, then the issue of what and where to sense is obvious. If a “point sensor” is placed at $\gamma_e = 1$, then the output would be

$$y(t) = T(t, 1) \quad (\text{II.7})$$

and the zonal estimator is given by a PDE on $(0, 1)$ defined by

$$\frac{\partial}{\partial t} \zeta(t, x) = c^2 \frac{\partial^2}{\partial x^2} \zeta(t, x), \quad t > 0, \ x \in (0, 1) \quad (\text{II.8})$$

with initial and boundary conditions

$$\zeta(0, x) = \varphi(x), \quad \zeta(t, 0) = 0 \quad \text{and} \quad \zeta(t, 1) = y(t). \quad (\text{II.9})$$

This simple example also illustrates some obvious theoretical, practical and computational challenges. First, locating an idealized point sensor exactly at $\gamma_e = 1$ is not an option in most realistic problems. In fact, the “point sensors” idealized by $y(t) = T(t, 1)$ actually produce local spatial averages of the form

$$y_{\text{real}}(t) = \frac{1}{2\delta} \int_{\gamma_e - \delta}^{\gamma_e + \delta} h(x)T(t, x)dx.$$ 

Also, because of physical limitations one may only be able to place a sensor near the wall, say at $\gamma = 2$. Of course, what to sense and how to use this sensed information to build a lower order observer depends on the exact problem formulation and is the subject of several papers cited above.

However, we note that because the system (II.8)-(II.9) defines a good zonal estimator if one has a good estimate of the value $\xi(t) = T(t, 1)$, it makes sense to focus on an observer that uses available sensed information to construct a 1D estimate $\hat{\xi}(t)$ for $\xi(t)$ and then feed this estimate into the system (or finite dimensional approximation) to yield an approximate zonal state estimator.

The following example illustrates how one might approach this problem for the case of a static sensor.

**A. A 1D Static Numerical Example**

To illustrate this idea, consider the case for the 1D advection-diffusion equation on the interval $\Omega = (0, 1)$ defined by

$$\frac{\partial}{\partial t} T(t, x) = c^2 \frac{\partial^2}{\partial x^2} T(t, x) + \kappa \frac{\partial}{\partial x} T(t, x) + g(x)\eta(t), \quad (\text{II.10})$$

with boundary and initial conditions

$$T(t, 0) = 0, \quad T(t, 1) = 0, \quad T(0, x) = T_0(x). \quad (\text{II.11})$$

Assume the average state is given by

$$z(t) = DT(t, \cdot) \triangleq \int_0^1 d(x) T(t, x)dx \quad (\text{II.12})$$

and a sensed output of the form

$$y(t) = CT(t, \cdot) \triangleq \int_0^1 c(x) T(t, x)dx + Ev(t), \quad (\text{II.13})$$

where $c(\cdot)$, $d(\cdot)$ and $g(\cdot)$ belong to $L_2(0, 1)$ and $\eta(t)$, $v(t)$ are normally distributed random disturbances.

In order to capture the zonal nature of the problem we assume that $c(\cdot)$ and $d(\cdot)$ have “small” compact support around the points $\gamma$ and $\bar{d} = .5$, respectively. In particular,

$$d(x) = \begin{cases} 
1 & t \in (.5 - \delta_1, .5 + \delta_1) \\
0 & \text{elsewhere}
\end{cases}$$
and
\[ c(x) = \begin{cases} 1 & t \in (\gamma - \delta_2, \gamma + \delta_2) \\ 0 & \text{elsewhere} \end{cases}, \]
where \(0 < \delta_1 = .2\) and \(0 < \delta_1 = .01\). Thus, we are interested in estimating the signal \(z(t)\) by a sensor centered at \(\gamma_c\). In the abstract formulation of the problem we define \(G : R^1 \rightarrow L_2(0,1)\) by \([G\eta](x) = g(x)\eta\) and \(D : L_2(0,1) \rightarrow R^1\) and \(C = C(\gamma) : L_2(0,1) \rightarrow R^1\) by (II.12) and (II.13), respectively.

Let
\[ M = GG^*, \quad Q = D^*D \quad \text{and} \quad N = EE^* = 1 \]
and consider the operator Riccati equation corresponding the Min-Max optimal filter given by
\[ A\Sigma + \Sigma A^* - \Sigma[C(\gamma)^*N^{-1}C(\gamma) - \theta^2Q]\Sigma + M = 0, \quad (II.14) \]
where \(\theta \geq 0\). The static optimal sensor placement problem is to find \(\gamma^{opt} \in [0,1]\) so that \(\gamma^{opt}\) minimizes
\[ J(\gamma) = trace(\Sigma_\gamma) \quad (II.15) \]
where \(\Sigma_\gamma\) is the solution to the Riccati equation (II.14).

For this example we set \(\theta = .25\) and \(g(x) = 5\) for all \(x\) and used linear finite elements to solve this problem. Observe that all the operators \(M = GG^*, \quad Q = D^*D, \quad N = EE^* = 1\) and \(C(\gamma)^*N^{-1}C(\gamma)\) are nuclear and hence \(\Sigma_\gamma\) is nuclear for all \(\gamma \in [0,1]\).

The numerical runs shown below illustrate two very important points: (i) the optimal sensor location problem shows a certain amount of mesh-independence and (ii) by carefully constructing low order reduced models one can compute sub-optimal sensor network controllers. For example in this 1D problem, optimal locations on coarse grids proved to be good initial estimates for fine grid computations which suggests that one can and should use some type of multi-grid optimization algorithm.

In Figure 5 we see that the optimal location for \(N = 16\) elements is already close to the optimal location for the \(N = 32\) and \(N = 64\) models even though the cost function \(trace(\Sigma_{N})\) does not converge until \(N = 128\) elements are used (see Figure 6). Therefore, this indicates that a multi-grid based optimization algorithm should perform well. Indeed, when we used the \(N = 16\) coarse grid solution as a starting location in a Gauss-Newton algorithm in the \(N = 32\) “fine” grid model, the algorithm converged in three steps. Similar results were observed in the more complex optimal control problem defined by the mobile sensor network problem.

![Fig. 5. Convergence of Sub-optimal Sensor Locations](image1)

![Fig. 6. Convergence of Sub-optimal Cost Functions](image2)

### III. Conclusions

There are advantages of focusing on the spatial differential and integral operators that define the hybrid-PDE systems since the concepts of zonal control and estimation are naturally formulated in physically meaningful terms. In addition, if the goal is to estimate an averaged value such as (II.12) above, then one would expect that a low order observer would suffice. Indeed, one can show that the basic ideas set forth by Lunenberger in [19] and [20] can be extended to PDE systems when one has additional information about the nature of the PDE (parabolic, hyperbolic, etc.).

Mesh independence for the finite element scheme offers the potential for practical algorithms based on model reduction and/or multi-grid like methods. For example, when we used a low order “modal model” we observed that the optimal sensor location was very well approximated even for
a reduced order model of degree four. This behavior was also observed in [24] where POD based models were employed and suggest that reduced order modeling techniques may also provide an approach to real-time mobile sensor control problems.

Finally, given the framework defined by a hybrid system of the form (I.8)-(I.11) with static and dynamic mobile sensor network it is now possible to formulate a wide range of “optimal sensor problems” depending on the particular problem to be addressed. Problems such as the one above were considered for finite dimensional linear systems by Speyer in 1979 [23] where issues of decentralized control were addressed. In this paper Speyer noted that optimal decentralization could be achieved under certain assumptions with the increase of the computational requirements on individual mobile sensors. Infinite dimensional versions have been considered by [5], [8]–[11]. The recent papers by Demetriou and Hussein (see [13], [18]) provide the best description of the infinite dimensional problems. Problems of parameter identification (estimating \( \hat{\theta} \) in the above model) are of a similar nature but with different cost functionals.

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