The Measurement Selection of Inventory Control

Juan Du, Christy M. Laird, B. Erik Ydstie
Department of Chemical Engineering
Carnegie Mellon University
5000 Forbes Ave, Pittsburgh, PA 15213

Abstract—This work addresses how input-output pairing is related to stability of inventory control systems. Input-output pairing divides the process dynamics into a measured subset and an unmeasured subset. The unmeasured dynamics are called the zero dynamics. We demonstrate that if the input-output pairing is chosen so that the zero dynamics are stable, then the overall stability of the system is guaranteed. Otherwise, the unmeasured inventories do not converge to setpoints due to the instability of internal dynamics. A system with stable zero dynamics is called a minimum phase system. The impact of different input-output pairings on the stability of the overall system is illustrated with two typical examples characterized by nonlinearities. The capability of inventory control is illustrated with an industrial solar grade silicon production process.

I. INTRODUCTION

Passivity plays a significant role in the stability analysis of a nonlinear system. Passivity is an input-output property, and it serves as a counterpart to the Lyapunov stability analysis which is based on internal dynamics. The Kalman-Yacubovitch-Popov lemma makes a direct connection between Lyapunov stability and passivity[1]. Passivity is related to a more general notion of dissipativity introduced by Willems[2]. A net increase of energy stored into a dissipative system is never greater than the amount of energy supplied to the system in a given time interval.

Inventory control[3] is based on the idea of rendering a system passive. We choose input-output pairs so that the mapping between the input and output is passive. In this paper, we show that different input-output pairings for inventory control can lead to very different performance and that stability of the internal dynamics can be guaranteed only if the inventories are chosen so that the zero dynamics are stable.

Byrnes[4] shows that if and only if a system is minimum phase, it can be feedback equivalent to a passive system with a storage function which is positive definite. Whether a system is minimum phase or not depends on the stability of zero dynamics[5]. A Lyapunov-like function is required in order to show the global stability of zero dynamics. A universal storage function for chemical process system called an availability function, developed by Alonso[6], is employed in this paper to check the global stability of zero dynamics under various input output pair synthesis.

Particulate processes are well known for their complexities. The limited number of input variables and distributed properties impose great challenges for control engineers [7]. The first controllability analysis for particulate process was achieved by Semino and Ray [8]. Christofides [9] applied nonlinear output feedback control on a crystallization process. Diez et al. [7] showed that inventory control provides a promising methodology to control particulate systems. Reaction kinetics network theory was used to demonstrate that the particulate system is controllable by inventory control. We propose a novel method to show the controllability of particulate system by examining the stability of the zero dynamics of the system.

The paper is organized as follows: Section 2 describes the basic ideas of inventory control. The significance of input-output pairing for inventory control is demonstrated by two typical problems in Section 3. In Section 4, the importance of the zero dynamics is highlighted. It is shown that stability of the zero dynamics guarantees the stability of inventory control. In addition the stability analysis of two examples are shown to support the results. An industrial application is introduced in Section 5 to extend the proposed method to particulate systems. Mathematical proofs are omitted due to space limitation.

II. INVENTORY CONTROL

The conservation laws used to develop dynamic models of chemical processes balance storage, production and flows of inventories in a given control volume. Such balances are written in mathematical form so that:

$$\frac{dZ}{dt} = p + \phi(m)$$

(1)

The variable $Z$ represents the inventory at time $t$ in a subsection of the process; $p$ is the rate of production and $\phi(m)$ is the net flow rate from the surroundings, $m$ is the control variable. Positive flow and production increase the inventory whereas negative flow and consumption decrease the inventory. The inventory is invariant when flow and production are balanced so that $\frac{dZ}{dt} = 0$. Inventories are nonnegative and additive which means the total inventory of a system is equal to the sum of the inventories of all its parts. The states of typical process models are represented by inventories like internal energy, volume and mass or moles of individual species.

Inventory control is based on the idea of manipulating flows so that the inventories follow their setpoints $Z^\ast$. Equation (1) implies controllers must be written so that solving for $m$:

$$p + \phi(m) = -C(e) + \frac{dZ^\ast}{dt}$$

(2)
where $C$ is the feedback controller and $e = Z - Z^*$ is the error which measures the difference of measured inventory and its reference. The expression $\phi(m)$ has to be invertible with respect to $m$ in the domain of interest to implement the control law. If the condition is satisfied then we get the closed loop expression:

$$\frac{dZ}{dt} = -C(e) + \frac{dZ^*}{dt}$$

(3)

If the process model has no model mismatch, a proportional control law is appropriate for inventory control.

$$C(e) = Ke$$

(4)

where $K > 0$ is the proportional gain.

Farschman et al.[3] proved that for the synthetic input and output pair:

$$u = p + \phi(m) - \frac{dZ^*}{dt}$$

$$y = Z - Z^*$$

(5)

where $Z^*$ are arbitrary vector of constant set points, the mapping

$$(\phi + p) \rightarrow (Z - Z^*)$$

(6)

is passive with storage function $\Psi = 1/2(Z - Z^*)^T(Z - Z^*)$.

III. INPUT-OUTPUT PAIRING OF INVENTORY CONTROL

A. Input Multiplicity

Input multiplicity means that there exists more than one input for a given process output [10]. The serial reaction in an isothermal continuous stirred tank reactor (CSTR) illustrates such behavior. We have

$$A \begin{bmatrix} A \end{bmatrix} B \begin{bmatrix} \frac{k_1}{k_2} \end{bmatrix} C$$

The inventories are moles of component A and B. Thus the dynamics constrained by (1) are given by the mass balance

$$\frac{dM_A}{dt} = -k_1 M_A + F(c_{A}^{in} - M_A/V)$$

$$\frac{dM_B}{dt} = k_1 M_A - k_2 M_B - \frac{F}{V} M_B$$

(7)

We assume values of the constants shown in table I[11]. The

<table>
<thead>
<tr>
<th>Table I</th>
<th>Nomenclature and Parameters of Isothermal CSTR</th>
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<tbody>
<tr>
<td>$F$</td>
<td>mol/s</td>
</tr>
<tr>
<td>$M_A$</td>
<td>mol</td>
</tr>
<tr>
<td>$M_B$</td>
<td>mol</td>
</tr>
<tr>
<td>$k_1$</td>
<td>0.8293 min$^{-1}$</td>
</tr>
<tr>
<td>$k_2$</td>
<td>0.8293 min$^{-1}$</td>
</tr>
<tr>
<td>$R$</td>
<td>8.314 J/K/mol</td>
</tr>
<tr>
<td>$V$</td>
<td>0.001 m$^3$</td>
</tr>
<tr>
<td>$c_{A}^{in}$</td>
<td>5.1 mol/L</td>
</tr>
</tbody>
</table>

steady state behavior of the system is shown in Figure 1. We choose steady state I ($M_A = 1.37, M_B = 1, F = 0.3046$) to illustrate the control capabilities. Figure 1 shows that $M_A$ in the reactor increases with $F$. The other state, $M_B$ exhibits input multiplicity. Steady states I and II

Fig. 1. Steady state behavior of isothermal CSTR system.

Fig. 2. Different control behavior of isothermal CSTR system at steady state I.

Fig. 3. Different control behavior of isothermal CSTR system at steady state I.

Fig. 4. Control behavior of isothermal CSTR system at steady state II.

Fig. 5. Control behavior of isothermal CSTR system at steady state II.
(MA = 3.73, MB = 1, F = 2.2579) on the figure show input multiplicity.

**Case I: Measure MA and manipulate F**

We first choose MA as system measurement and manipulate flow rate F. The inventory control strategy is described by equation (8) with y = MA and u = F and gives:

\[-k_1M_A + F(c_A^a - M_A/V) = -K(M_A - M_A^*)\]  

(8)

Here, \( K > 0 \) is the feedback gain which adjusts how quickly the system converges. The equation is now rearranged for the control variable \( F \):

\[ F = \frac{k_1M_A - K(M_A - M_A^*)}{c_A^m - M_A/V} \]  

(9)

This choice of inventory control gives stable control. Figure 2 shows that the controller allows both \( M_A \) and \( M_B \) to converge to their setpoints.

**Case II: Measure MB and input F pair is chosen**

Another control scheme is based on the measurement of the concentration of B. With \( y = M_B \) as the output, we use feedback inventory control to adjust the input parameters, the flow into the system \( F \). This leads to the feedback control strategy

\[ k_1M_A - k_2M_B - \frac{F}{V}M_B = -K(M_B - M_B^*) \]  

(10)

Control action \( F \) is calculated by:

\[ F = \frac{V}{M_B}[k_1M_A - k_2M_B + K(M_B - M_B^*)] \]  

(11)

The results in Figure 3 demonstrate that the system can not be controlled since \( M_A \) drifts away from its desired point of operation.

**Case III: Steady state II as setpoint**

However if we choose steady state II as the desired point, then both input-output pairs described above have the capability to maintain the system at steady state II. Figure 4 and 5 illustrate the capability of two different inventory controllers.

**B. Output Multiplicity**

Output multiplicity means that the system has multiple outputs for the same input. An exothermic CSTR can exhibit output multiplicity with three different steady states for the same input. The intermediate steady state is typically unstable while the other two are stable. The behavior is illustrated with a jacketed CSTR with a first order chemical reaction: \( A \rightarrow B \).[12] The dynamics of the stirred tank reactor are governed by the following mass and energy balances.

\[ \frac{dM_A}{dt} = F_{Ac} - F_A - rV \]
\[ \frac{dM_B}{dt} = -F_B + rV \]
\[ \frac{du}{dt} = \lambda(T_w - T) + F_{Ac}h_A - F_Ah_A - F_Bh_B \]  

(12)

The notation and parameters of non isothermal CSTR are shown in table II.[12]

<table>
<thead>
<tr>
<th>TABLE II</th>
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<tbody>
<tr>
<td><strong>NONISCORAL CSTR</strong></td>
</tr>
<tr>
<td>( F_{Ac} )</td>
</tr>
<tr>
<td>( F_A )</td>
</tr>
<tr>
<td>( F_B )</td>
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<td>( F )</td>
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<td>( M_B )</td>
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<td>( U )</td>
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<tr>
<td>( T )</td>
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<td>( Q )</td>
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<tr>
<td>( h_A )</td>
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<tr>
<td>( h_B )</td>
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<td>( M_{tot} )</td>
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<tr>
<td>( C_{pA} )</td>
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<tr>
<td>( C_{pB} )</td>
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<tr>
<td>( h_{Aref} )</td>
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<tr>
<td>( h_{Bref} )</td>
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<tr>
<td>( k )</td>
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<tr>
<td>( E )</td>
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<tr>
<td>( R )</td>
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<tr>
<td>( \lambda )</td>
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<td>( v )</td>
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<td>( V )</td>
</tr>
<tr>
<td>( s_{Aref} )</td>
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<tr>
<td>( s_{Bref} )</td>
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<tr>
<td>( T_e )</td>
</tr>
<tr>
<td>( T_{ref} )</td>
</tr>
</tbody>
</table>

The steady state behavior shown in Figure 6 demonstrate the existence of three steady states. Our control goal is to stabilize the unstable steady state II. Two input-output pairings are used to control the unstable steady state.

**Case I: Measure MA and manipulate inlet flow rate \( F_{Ac} \)**

We use inventory control to adjust the flow rate \( F_{Ac} \), so the output converges to the desired setpoint, \( M_A^* \). This leads to the feedback control strategy

\[ F_{Ac} - F_A - rV = -K(M_A - M_A^*) \]  

(13)

Here, \( K > 0 \) is the feedback gain which adjusts how quickly the system converges. The expression of the input that achieves the feedback control is obtained by rearranging (13).

\[ F_{Ac} = \frac{ke^{(-\frac{E}{RT})}M_A - K(M_A - M_A^*)}{1 + \frac{F}{F_A}} \]  

(14)

Simulation results for the closed loop system are shown in Figure 7. We can see that the mass of each component and the internal energy converge to their setpoints.
Case II: Measure U and manipulate input Q
We now see inventory control to adjust the heat flow Q, so that the internal energy $U$ converges to the desired set point $U^*$. This leads to the feedback control strategy

$$Q + F_A e_A - F_A h_A - F_B h_B = -K(U - U^*)$$

(15)

The corresponding input is obtained by rearranging (15).

$$Q = \frac{v}{V} F_A M_A h_A + \frac{v}{V} F_A M_B h_B - F_A e_A h_A - K(U - U^*)$$

(16)

Simulation results are shown in Figure 8. We see that $M_A$ and $M_B$ drift away from their setpoints while $U$ converges. As $U$ is a function of $M$ and $T$, When $U$ converges to $U^*$ via inventory control, there is no guarantee that $M_A$ converges to stationary value as well.

Case III: multi-variable control structure
We now use two states $M_A$ and $U$ to develop a multi-input and multi-output control strategy. The control variables are $F_A e$ and $Q$ and the corresponding outputs are $M_A$ and $U$. Inventory control laws are applied to each inventory and we get the control laws as:

$$\frac{dM_A}{dt} = F_A e - F_A h_A - rV = -K(M_A - M_A^*)$$

(17)

$$\frac{dU}{dt} = Q + F_A e_A h_A - F_A h_A - F_B h_B = -K(U - U^*)$$

(18)

The simulation results in Figure 9 demonstrate that the multi-variable inventory strategy follows their respective setpoints. The corresponding evolutions of input variables are shown in Figure 10.

IV. THE ROLE OF THE ZERO DYNAMICS
We propose that the behavior of the zero dynamics of inventory control system can be used to explain the behavior above and is critical to judge whether the input-output pairing is effective or not. The zero dynamics [1] are the internal dynamics of the system under the constraints that the output is identically zero and the input is designed to enforce the output to be zero. Thus the zero dynamics determine the dynamics of the system after the controlled variable has converged to its setpoint. The behavior of the system is divided into two parts: the external part is enforced by the inventory controller; the other part, so called zero dynamics remains unaffected directly by the inventory control system. If the zero dynamics have a unique asymptotically stable equilibrium point, then the uncontrolled inventories converge to stationary points and the overall system is stable. If the zero dynamics at the desired operating point are unstable, then the uncontrolled inventories diverge and the system may converge to undesirable or ambiguous points. A system with stable zero dynamics is said to be minimum phase, otherwise it is non-minimum phase.

An energy function was developed by Ydstie et al [13] to check if the zero-dynamics of a chemical process system are stable or not. This energy function is defined so that:

$$W = A + \Psi$$

(19)

$$A(Z) = S(Z^*) + w^T (Z - Z^*) - S(Z)$$

(20)

$$\Psi = \frac{1}{2} (Z_c - Z_c^*)^T (Z_c - Z_c^*)$$

(21)

The function $A$ is used to analyze stability of the zero dynamics whereas $\Psi$ analyze stability of the inventory controller. By differentiating the energy function we obtain:

$$\frac{dW}{dt} = (w^* - w)^T \frac{dZ}{dt} + (Z_c - Z_c^*) \frac{dZ_c}{dt}$$

(22)

$$= (w^* - w)^T (p + \phi) + eC(e)$$

(23)
where \( Z \) is the vector of all inventories used to describe the states of the system, \( Z_c \) is the vector of controlled inventories, \( S(Z) \) is the entropy function, \( w \) is the vector of intensive variables and \( w^* \) is a vector of the intensive variables at their setpoints.

If the zero dynamics are asymptotically stable, then \( \frac{dW}{dt} < 0 \) for \( Z \neq Z^* \) and \( \frac{dW}{dt} = 0 \) for \( Z = Z^* \), which implies stability.

If the zero dynamics are unstable, then \( \frac{dW}{dt} > 0 \) for some \( Z \neq Z^* \), which implies instability.

Revisiting the two illustrative examples described in the previous section, the proposed method is validated by two examples. For the example of input multiplicity, if the input-output pair is selected as \( M_A \) and \( F \), then the zero dynamics is expressed as:

\[
\frac{dM_B}{dt} = k_1M_A - k_2M_B - \frac{k_1M_AM_B - K(M_A - M_A^*)M_B}{V_c^A - M_A} \tag{24}
\]

If the input-output pair is changed to \( M_B \) and \( F \), then the zero dynamics is written as:

\[
\frac{dM_A}{dt} = -k_1M_A + \frac{V}{M_B}(k_1M_A - k_2M_B) + K(M_B - M_B^*)(c_A^A - M_A/V) \tag{25}
\]

For the example of output multiplicity, if measure \( M_A \) and input \( F_{Ae} \) is pairing, then the zero dynamics is in the form as:

\[
\frac{dU}{dt} = Q + (h_{Ae} - \frac{v}{V}h_A)M_A - \frac{v}{V}h_BM_B \tag{26}
\]

\[
\times \exp(-\frac{E}{RT})M_A - K(M_A - M_A^*) \frac{1}{1 - \frac{v}{V}M_A}
\]

However, if measure \( U \) and input \( Q \) is selected instead, then the zero dynamics is described as:

\[
\frac{dM_A}{dt} = F_{Ae} - F_A - rV \tag{27}
\]

Applying stability analysis via the storage function \( W \) on the zero dynamics, the results are shown in Figure 11 and 12. From Figure 11 we can see zero-dynamics are stable in case I for both examples while they are unstable for case II as a result of the energy function behavior. A complete analysis of the zero dynamics using analytical tools rather than simulations is provided by [14].

V. INDUSTRIAL APPLICATION

Silicon based solar cells are expected to play an important role in meeting future energy demand and new processes have been introduced to increase production rate and reduce cost. Thermal decomposition of silane (\( SiH_4 \)) in a fluidized bed reactor is one new technology for solar grade silicon production. Silane and hydrogen gases enter at the bottom of the reactor with sufficient velocity to fluidize the silicon particles. When silane gas is heated by wall heaters, it decomposes to silicon and hydrogen gas. Most of the silicon deposits on the surface of the particles which cause particles to grow. The gas exits at the top of the reactor together with solar silicon dust. The objective of the control system is to stabilize the reactor so that silicon loss is minimized. During continuous production, silicon product is removed from the bottom of the reactor and seeds are added to ensure constant average particle size. Particle size is an important property of silicon product and hence it is necessary to predict and control particle size during operations. White etc al. [15] developed a size distribution model in a discrete form.

Fig. 10. Inventory control law for multi-variable inventory control

Fig. 11. The stability of zero dynamics in the first example.

Fig. 12. The stability of zero dynamics in the second example.
model represents mass balances over size intervals such that:

\[
\frac{dM_i}{dt} = f_{i-1} - f_i + r_i + q_i + a_i \tag{28}
\]

The rate of material transfer from the continuous phase to the particle is represented by \( r_i \). The rate of transition of particles from one size interval to the next, caused by particle growth, is represented by \( f_{i-1} \) for flow into interval \( i \) and \( f_i \) for flow out of interval \( i \). The external flow rate is \( q_i \), \( a_i \) represents the rate of agglomeration or breakage of particles. This model has the same form as equation (1) and hence it is suited for inventory control. The control law is expressed as:

\[
g_i + \sum_{\gamma} q_{i,\gamma} = -C(e) \tag{29}
\]

where \( g_i \) represents production and internal flow terms (between size intervals), and \( q_{i,\gamma} \) represents external particle flow terms. Total mass hold up is maintained constant via inventory control to keep the change of volume of solid phase as zero. Proportional control law is performed over all size intervals (1 to \( N \)) to derive the product flow rate (\( P \)) required to maintain a constant hold-up in the reactor (\( M_{\text{total}}^* \)).

\[
P = -\sum_{i=1}^{N} g_i - K_i \left( \sum_{i=1}^{N} M_i - M_{\text{total}}^* \right) \tag{30}
\]

To achieve the convergence of particle size distribution, the summation over the seed size intervals is performed to determine the seed flow rate (\( S \)) required to maintain a constant mass of seed particles in the reactor (\( M_{\text{seed}}^* \)).

\[
S = -\sum_{i=1}^{I_s} g_i - K_s \left( \sum_{i=1}^{I_s} M_i - M_{\text{seed}}^* \right) \tag{31}
\]

Simulation of controlling the total and seed particle hold-up is shown in Figures 13. The product and seed flow rates required to achieve the control are also shown. The particle size distribution achieved during each steady state are shown in Figure 13. The stability analysis of the size distribution model is achieved using the same energy function as before. We get

\[
\frac{dW}{dt} = R \ln \left( \frac{M}{M^*} \right) \sum \frac{dM_i}{dt} + (M - M^*) \frac{dM}{dt} \tag{32}
\]

The mathematical analysis is achieved as well based on the proposed method. Due to limit space the complete analysis is not presented here. The result from the simulation is shown in Figure 13.

VI. Conclusions and Future Work

This work shows the significance of input-output pairing in the design of inventory control. An energy function is introduced to investigate stability. The proposed method is applied to a particle growth process. This example shows that inventory control can be applied to industrial applications. Two different examples, one with input multiplicity and the other with output multiplicity, illustrate that the inventory control system works only if the manipulated and controlled variables are properly paired. It is demonstrated that the stability of the control system can be related to the stability of the zero dynamics. The stability result has been verified by mathematical analysis using a new storage function based on energy of the system.

References