Stability of Stochastic Systems with Probabilistic Mode Switchings and State Jumps

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Abstract—In this paper, we discuss the stability of continuous-time stochastic systems with probabilistic mode switchings and state jumps. Occurrences of mode transitions and state jumps are modeled with independent Poisson processes. We use multiple Lyapunov functions to derive sufficient conditions for stability in probability and moment exponential stability both for linear and nonlinear stochastic switching systems. Furthermore, we provide numerical examples to demonstrate the efficacy of our results.

I. INTRODUCTION

Stochastic switching systems are capable of describing real life processes that are subject to noise and random changes in their dynamics. The research in stochastic switching systems have found applications in diverse fields such as finance, population dynamics, flexible manufacturing, and fault tolerant control systems [1].

In the literature, researchers have proposed different models to describe stochastic switching systems. The place where randomness arises is the main difference between these models. First model introduces randomness in the dynamics of each mode of the switching system. In this type, subsystem dynamics are described by stochastic differential equations. Another model for stochastic switching systems introduces randomness in the mode transition. In this model, the switching signal, which manages the transition between the subsystems of a switching system, is probabilistic. If this probabilistic switching signal is the state of a finite-state Markov chain, then the overall system is called a Markov jump system. There has been considerable amount of research on Markov jump linear systems [2], [3], [4]. It is asserted in [5] that more general stochastic switching system models can be adopted by combining probabilistic switching signals with subsystems that have stochastic dynamics. These systems have been investigated under the name “switching diffusion processes”. Switching diffusion processes have found applications in power management [6], population studies [7], [8], [9], and finance [7], [10]. There has been increasing amount of research devoted on the stability of switching diffusions [11], [12], [13], [14]. In addition, recently some researchers have investigated the stability of Markovian switching diffusions with state jumps [15], [16], [17].

In this paper, we discuss the stability of stochastic switching systems with probabilistic mode transition and state jump. The systems that are of concern in this study differ from Markovian switching diffusion processes in terms of the switching models that describe the transition between subsystems. We model the occurrences of switchings between subsystems with a counting (Poisson) process. Similar switching models have been used in the literature. For example, consensus problem for stochastic switched linear systems with Poisson switchings is investigated in [18]. Furthermore, stability of asynchronous systems with Poisson switchings are discussed in [19], [20]. Stability analysis and stabilization of systems with deterministic subsystems and probabilistic switchings are discussed in [21], [22] where the authors show that if all the subsystems of a switching system are stable satisfying a common stability margin and mode transitions occur sufficiently rarely, then the overall system is stable.

In this work, we describe the subsystems by stochastic differential equations that are subject to state jumps at random instances. Models with random mode switchings and state jumps can describe systems that face randomly occurring sharp and sudden changes in their dynamics. This kind of dynamics are inherent in many systems from finance, biology, physics, and engineering. As a consequence, examination of stochastic system models with probabilistic mode switchings and state jumps have great significance. We employ independent Poisson processes for modeling the occurrences of random state jumps and mode switchings. Moreover, in our analysis, we use multiple Lyapunov functions which are discussed in [5], [23]. We derive sufficient conditions of stability in probability and moment exponential stability for both linear and nonlinear cases. Furthermore, we show the stability of stochastic switching systems without state jumps as a special case of our main result.

The paper is organized as follows. In Section II, the notation used in the paper is explained; moreover, a review of Poisson processes and definitions of stability in probability and moment exponential stability are given. We give the problem setup for continuous-time stochastic switching systems with state jumps in Section III and present sufficient conditions of stability for both linear and nonlinear stochastic switching systems. Several numerical examples are provided in Section IV to demonstrate the utility of the results. Finally, we conclude the paper in Section V.

II. MATHEMATICAL PRELIMINARIES

The notation used in this paper is fairly standard. Specifically, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^n$ denotes the set of $n \times 1$ real column vectors, $\mathbb{R}^{n \times m}$ is the set of $n \times m$
real matrices, \( N \) and \( N_0 \) respectively denote positive and nonnegative integers. \( \| \cdot \| \) is the Euclidean vector norm. Furthermore, we write \((\cdot)^T\) for transpose and \(\text{tr}(\cdot)\) for trace of a matrix, \(\lambda_{\text{min}}(\cdot)\) and \(\lambda_{\text{max}}(\cdot)\) for the minimum and maximum eigenvalues of a Hermitian matrix, \(I_n\) for the identity matrix of dimension \(n\). A function \(V : \mathbb{R}^n \rightarrow \mathbb{R}\) is said to be positive definite if \(V(x) > 0, \ x \neq 0\), \(V(0) = 0\) and proper if \(\lim_{\|x\| \rightarrow \infty} V(x) = \infty\). Finally, \(\nabla V\) denotes the vector of the first order spatial derivatives of a twice continuously differentiable scalar \(V\), that is, \(\nabla V = \left[\frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}, \ldots, \frac{\partial V}{\partial x_n}\right]\) and \(\nabla(\nabla V)\) denotes the matrix of the second-order spatial derivatives of \(V\), that is,

\[
\nabla(\nabla V) = \begin{bmatrix}
\frac{\partial^2 V}{\partial x_1^2} & \cdots & \frac{\partial^2 V}{\partial x_1 \partial x_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial^2 V}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 V}{\partial x_n^2}
\end{bmatrix}.
\]

Let \((\Omega, \mathcal{F}, P)\) be a probability space. A filtration \(\{\mathcal{F}_t\}_{t \geq 0}\) on this probability space is a family of \(\sigma\)-algebras such that

\[
\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}, \quad 0 \leq s < t.
\]

A stochastic process \(\{x(t)\}_{t \geq 0}\) is adapted to the filtration \(\{\mathcal{F}_t\}_{t \geq 0}\) if the random variable \(x_t : \Omega \rightarrow \mathbb{R}^n\) is \(\mathcal{F}_t\)-measurable, that is,

\[
\{\omega \in \Omega : x_t(\omega) \in B\} \in \mathcal{F}_t
\]

for all Borel sets \(B \subset \mathbb{R}^n\). \(P[\cdot]\) and \(E[\cdot]\) denote the probability and expectation, respectively.

### A. Poisson Processes

A Poisson process is a continuous-time stochastic process counting the number of occurrences of some events. Mathematically it is the collection of \(N_0\)-valued, \(\mathcal{F}_t\)-adapted random variables \(\{N(t)\}_{t \geq 0}\) with \(N(0) = 0\). Here \(N(t)\) denotes the number of events that occur up to time \(t\). In a Poisson process, length of intervals between consecutive events have exponential distribution. The “independent increments” property of Poisson processes suggests that the numbers of events occurring in non-overlapping intervals are independent. Moreover, a Poisson process has the “stationary increments” property; the number of events in any time interval is distributed with Poisson distribution depending only on the length of the interval. The number of occurrences in the interval \((t,t+\tau)\) is distributed by the Poisson distribution,

\[
P[N(t+\tau) - N(t) = k] = \frac{e^{-\lambda \tau} (\lambda \tau)^k}{k!},
\]

where \(\lambda\) denotes the intensity of occurrences. For Poisson processes, only one event occurs at a time and the number of events occurring in finite time intervals is finite, almost surely.

In this study, the mode switchings between subsystems of a dynamical system as well as the occurrences of state jumps are modeled with Poisson processes.

### B. Stochastic Stability Definitions

We adopt the stability in probability notion used in [5]. The zero solution \(x(t) \equiv 0\) of a stochastic system is stable in probability if for any \(\epsilon > 0\),

\[
\lim_{x(0) \rightarrow 0} P[\sup_{t \geq 0} \|x(t)\| > \epsilon] = 0.
\]

Additionally, the zero solution \(x(t) \equiv 0\) of a stochastic system is \(p\)-moment exponentially stable if there exist positive constants \(C\) and \(a\) such that

\[
E[\|x(t)\|^p] \leq C(\|x(0)\|^p e^{-at}, \ t \geq 0.
\]

### III. Stability of Stochastic Switching Systems with State Jumps

#### A. Stability of Nonlinear Stochastic Switching Systems with State Jump

Consider the continuous-time nonlinear stochastic switching system with state jumps given by

\[
dx(t) = f_r(t)(x(t))dt + G_r(t)(x(t))dW(t), \quad t \neq \tilde{t}_k (2) \\
x(t) = J(x(t^-)), \quad t = \tilde{t}_k,
\]

for \(t \geq 0\) with initial conditions \(x(0) = x_0, r(0) = q_0\), where \(\{x(t)\}_{t \geq 0}\) is an \(\mathbb{R}^n\)-valued \(\mathcal{F}_t\)-adapted stochastic process, \(\{W(t)\}_{t \geq 0}\) is an \(\mathbb{R}^l\)-valued \(\mathcal{F}_t\)-adapted Wiener process. The switching system described by (2), (3) is assumed to have \(M\) number of subsystems. The switching signal \(\{r(t)\}_{t \geq 0}\) is a right-continuous stochastic process that takes values from the index set \(Q \triangleq \{1, 2, \ldots, M\}\). \(f_q : \mathbb{R}^n \rightarrow \mathbb{R}^n\) and \(G_q : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times l}\) are vector- and matrix-valued functions with \(f_q(0) = 0, G_q(0) = 0, q \in Q\), respectively, \(J : \mathbb{R}^n \rightarrow \mathbb{R}^n\) is a vector-valued function, \(\{\tilde{t}_k > 0, k \in \mathbb{N}\}\) is the sequence of state jump times. Length of intervals between consecutive jumps are distributed independently and identically, by exponential distribution with parameter \(\lambda_j\). Hence, \(\{N_j(t)\}_{t \geq 0}\) is an \(\mathcal{F}_t\)-adapted Poisson process that counts the number of jumps occurring in the time interval \((0, t]\). Note that the probability distribution of \(N_j(t)\) is given by

\[
P[N_j(t) = k] = \frac{e^{-\lambda_j t} (\lambda_j t)^k}{k!}.
\]

The probability distribution of the number of switchings occurring in any time interval is given by

\[
P[N_s(t) = k] = \frac{e^{-\lambda_s t} (\lambda_s t)^k}{k!},
\]

where \(\{N_s(t)\}_{t \geq 0}\) is an \(\mathcal{F}_t\)-adapted counting process that counts the number of mode switchings occurring in the time interval \((0, t]\) and \(\lambda_s\) denotes the intensity of the Poisson process.

We assume that the stochastic processes \(\{f_q(x(t))\}_{t \geq 0}\) and \(\{G_q(x(t))\}_{t \geq 0}\) are adapted to the filtration \(\{\mathcal{F}_t\}_{t \geq 0}\) to ensure that the Ito integrals including these terms are well defined. Furthermore, \(\{W(t)\}_{t \geq 0}\), \(\{N_j(t)\}_{t \geq 0}\) and \(\{N_s(t)\}_{t \geq 0}\) are assumed to be mutually independent stochastic processes.

**Theorem 3.1:** Consider the nonlinear stochastic switching system with state jumps given by (2), (3). If there exist twice continuously differentiable, positive definite, and
proper functions $V_q : \mathbb{R}^n \to \mathbb{R}$, $q \in Q$, and scalars $\beta \geq 1$, $\mu \geq 0$, $\zeta$, such that
\[
LV_q(x) \triangleq \nabla V_q(x)f_q(x) + \frac{1}{2} \text{tr} \left( G_q(x)G_q^T(x)\nabla^2 V_q(x) \right) \leq \zeta V_q(x), \quad q \in Q, \tag{4}
\]
\[
V_p(x) \leq \beta V_q(x), \quad p, q \in Q, \tag{5}
\]
\[
V_q(J(x)) \leq \mu V_q(x), \quad q \in Q, \tag{6}
\]
\[
(\mu - 1)\lambda_3 + (\beta - 1)\lambda_8 + \zeta \leq 0, \tag{7}
\]
then the zero solution $x(t) \equiv 0$ of the system described by (2), (3) is stable in probability.

**Proof:** Each subsystem of the switched system (2), (3) is described by a multi-dimensional Ito stochastic differential equation. Thus, multi-dimensional Ito formula can be employed for computing the Lyapunov derivative $dV_q$ for the $q$th system as
\[
dV_q(x(t)) = \left( \nabla V_q(x(t))f_q(x(t)) + \frac{1}{2} \text{tr} \left( G_q(x(t))G_q^T(x(t))\nabla^2 V_q(x(t)) \right) \right) dt
+ \nabla V(x(t))G_q(x(t))dW(t)
= LV_q(x(t))dt
+ \nabla V(x(t))G_q(x(t))dW(t). \tag{8}
\]

Let $\{t_1, t_2, \ldots, t_s\}$ be the sequence of time points when a switching between subsystems occurs before time $t$. In this case $t_s$ will denote the time instance of the last switching. We will denote the active subsystem index by $q_s$ for the time interval $t_s \leq t < t_{s+1}$. Additionally, let $\{t_1, t_2, \ldots, t_k\}$ be the sequence of time points when state jumps occur after the last switching instance $t_s$ and before $t$. Hence, $k = N(t - t_s) \geq 0$ denotes the number of jumps in the interval $(t_s, t)$. After the last jump that occurs at time $\bar{t}_k$, the system evolves according to the $q_s$th stochastic differential equation (2). As a result, we can use the integral form of Ito formula (8) to compute $V_{q_s}(x(t))$ as
\[
V_q(x(t)) = V_{q_s}(x(\bar{t}_k)) + \int_{\bar{t}_k}^t LV_{q_s}(x(\tau))d\tau
+ \int_{\bar{t}_k}^t \nabla V(x(\tau))G_q(x(\tau))dW(\tau). \tag{9}
\]

Let $\mathcal{H}_t \subset \mathcal{F}_t$ be the $\sigma$-algebra generated by $N(t)$ and $N(t)$, $\tau \leq t$. We introduce the conditional expectation of $V_q$, given $\mathcal{H}_t$ at time $t$ as
\[
\mathbb{E}[V_q(x(t))|\mathcal{H}_t] = \mathbb{E}[V_{q_s}(x(\bar{t}_k)) + \int_{\bar{t}_k}^t LV_{q_s}(x(\tau))d\tau
+ \int_{\bar{t}_k}^t \nabla V(x(\tau))G_q(x(\tau))dW(\tau)|\mathcal{H}_t]
= \mathbb{E}[V_{q_s}(x(\bar{t}_k))|\mathcal{H}_t]
+ \mathbb{E}\left[ \int_{\bar{t}_k}^t LV_{q_s}(x(\tau))d\tau|\mathcal{H}_t \right]. \tag{10}
\]

Note that $W(\tau), \tau \leq t$ is independent of $\mathcal{H}_t$ and $\mathbb{E}\left[ J_{\bar{t}_k}^t \nabla V(x(\tau))G_q(x(\tau))dW(\tau)|\mathcal{H}_t \right] = 0$. Then by using (6), it follows that
\[
\mathbb{E}[V_q(x(t))|\mathcal{H}_t] \leq \mathbb{E}\left[ \mu V_{q_s}(x(\bar{t}_k))|\mathcal{H}_t \right]
+ \mathbb{E}\left[ \int_{\bar{t}_k}^t LV_{q_s}(x(\tau))d\tau|\mathcal{H}_t \right]
= \mu \mathbb{E}[V_{q_s}(x(\bar{t}_k))|\mathcal{H}_t]
+ \mathbb{E}\left[ \int_{\bar{t}_k}^t LV_{q_s}(x(\tau))d\tau|\mathcal{H}_t \right]. \tag{11}
\]

By (4) and the integral form of Gronwall’s inequality, we arrive at
\[
\mathbb{E}[V_q(x(t))|\mathcal{H}_t] \leq \mu \mathbb{E}[V_{q_s}(x(\bar{t}_k))|\mathcal{H}_t]
+ \int_{\bar{t}_k}^t \zeta \mathbb{E}[V_{q_s}(x(\tau))|\mathcal{H}_t]d\tau
\leq \mu \mathbb{E}[V_{q_s}(x(\bar{t}_k))|\mathcal{H}_t]e^{\zeta(\bar{t}_k - \bar{t}_k)}. \tag{12}
\]

Furthermore, since,
\[
\mathbb{E}[V_{q_s}(x(\bar{t}_k))|\mathcal{H}_t] = \mathbb{E}[V_{q_s}(x(\bar{t}_{k-1}))|\mathcal{H}_t]
+ \mathbb{E}\left[ \int_{\bar{t}_{k-1}}^{\bar{t}_k} LV_{q_s}(x(\tau))d\tau|\mathcal{H}_t \right], \tag{13}
\]

after operations similar to (11) and (12), it follows that
\[
\mathbb{E}[V_{q_s}(x(\bar{t}_{k-1}))|\mathcal{H}_t] \leq \mu \mathbb{E}[V_{q_s}(x(\bar{t}_{k-2}))|\mathcal{H}_t]e^{\zeta(\bar{t}_{k-2} - \bar{t}_{k-1})}. \tag{14}
\]

Substituting (14) into (12) yields
\[
\mathbb{E}[V_{q_s}(x(t))|\mathcal{H}_t] \leq \mu^2 \mathbb{E}[V_{q_s}(x(\bar{t}_{k-1}))|\mathcal{H}_t]e^{\zeta(\bar{t}_{k-1} - \bar{t}_{k-1})}. \tag{15}
\]

Now consider $\mathbb{E}[V_{q_s}(x(t))|\mathcal{H}_t]$. Employing operations similar to (11) and (12), we find the following inequality
\[
\mathbb{E}[V_{q_s}(x(\bar{t}_{k-1}))|\mathcal{H}_t] \leq \mu \mathbb{E}[V_{q_s}(x(\bar{t}_{k-2}))|\mathcal{H}_t]e^{\zeta(\bar{t}_{k-2} - \bar{t}_{k-2})}. \tag{16}
\]

Moreover, after substituting (16), (15) becomes
\[
\mathbb{E}[V_{q_s}(x(t))|\mathcal{H}_t] \leq \mu^3 \mathbb{E}[V_{q_s}(x(\bar{t}_{k-2}))|\mathcal{H}_t]e^{\zeta(\bar{t}_{k-2} - \bar{t}_{k-2})}. \tag{17}
\]

Thus, if we find inequalities for $\mathbb{E}[V_{q_s}(x(t))|\mathcal{H}_t]$, $i = 1, 2, \ldots, k - 1$, and substitute them back into the inequality for $\mathbb{E}[V_{q_s}(x(t))|\mathcal{H}_t]$, the final expression will be
\[
\mathbb{E}[V_{q_s}(x(t))|\mathcal{H}_t] \leq \mu^k \mathbb{E}[V_{q_s}(x(t))|\mathcal{H}_t]e^{\zeta(t - s)}
= \mu^N(t - s) \mathbb{E}[V_{q_s}(x(t))|\mathcal{H}_t]e^{\zeta(t - s)}. \tag{18}
\]

In inequality (18), $t_s$ is an instance of a switching. Hence, from (5),
\[
\mathbb{E}[V_{q_s}(x(t))|\mathcal{H}_t] \leq \mu^N(t - t_s) \mathbb{E}[V_{q_{s-1}}(x(t))|\mathcal{H}_t]e^{\zeta(t - s)}
\leq \mu^N(t - t_s) \beta \mathbb{E}[V_{q_{s-1}}(x(t))|\mathcal{H}_t]e^{\zeta(t - s)}. \tag{19}
\]
Here we form an inequality for $E[V_{qs}(x(t))|H_t]$ similar to the inequality (18) as

$$E[V_{qs}(x(t))|H_t] \leq \mu N_j(t-t_{s-1}) \cdot E[V_{qs-1}(x(t_{s-1}))|H_t] e^{\xi(t-t_{s-1})}. \quad (20)$$

Noting that $N_j(t-t_{s-1}) = N_j(t-t_s) + N_j(t_s-t_{s-1})$, substitution of (20) into (19) yields

$$E[V_{qs}(x(t))|H_t] \leq \mu N_j(t-t_{s-1}) \cdot E[V_{qs-1}(x(t_{s-1}))|H_t] e^{\xi(t-t_{s-1})}. \quad (21)$$

After repetitive calculation and substitution of sub-inequalities similar to (21), we obtain

$$E[V_{qs}(x(t))|H_t] \leq \mu N_j(t)\beta^qE[V_{qs}(x_0)|H_t] e^{\xi t}, \quad (22)$$

and since $s = N_S(t)$ and $V_{qs}(x_0)$ is a constant,

$$E[V_{qs}(x(t))|H_t] \leq \mu N_j(t)\beta^q V_{qs}(x_0) e^{\xi t}, \quad (23)$$

from which we obtain an inequality for $E[V_{qs}(x(t))]$ as

$$E[V_{qs}(x(t))] \leq \mu \beta^q V_{qs}(x_0) e^{\xi t}.$$ 

Note that we used Taylor series formula of exponential function for obtaining (24), that is, $e^{\mu \lambda t} = \sum_{k=0}^{\infty} \frac{\mu^k \lambda^k}{k!}$ and $e^{\beta \lambda t} = \sum_{k=0}^{\infty} \frac{\beta^k \lambda^k}{k!}$.

Let $\{\tau^0_q, \tau^1_q, \ldots\}$ be the sequence of endpoints of the intervals in which the $q$th system is active. Since we assumed $(\mu - 1)\lambda_1 + (\beta - 1)\lambda_S + \zeta < 0$, from (24),

$$E[V_q(x(t))] \leq V_{qs}(x_0), \quad t \geq 0, \quad q \in Q. \quad (25)$$

Consequently,

$$E[V_q(x(\tau^j_{qs}))] \leq E[V_q(x(\tau^j_{qs-1}))] \leq V_{qs}(x_0), \quad j \in \mathbb{N}_0. \quad (26)$$

Hence, the Lyapunov functions $V_q, q \in Q$ are supermartingales satisfying the supermartingale inequality

$$P[sup_{\tau^j_{qs}} V_q(x(t)) \geq \epsilon_q] \leq \frac{1}{\epsilon_q} (E[V_q(x(\tau^j_{qs}))] - E[V_q(x(\tau^j_{qs-1}))]) \leq \frac{1}{\epsilon_q} E[V_q(x(\tau^j_{qs}))] \leq \frac{1}{\epsilon_q} V_{qs}(x_0), \quad \epsilon_q > 0, \quad (27)$$

where $T^j_q$ denotes the interval $\tau^j_{qs} \leq t \leq \tau^j_{qs+1}$ and $Q^{-}(x(\tau^j_{qs+1})) = \max\{-V_q(x(\tau^j_{qs+1})), 0\}$. Note that since $V_q$ is positive definite, $V_q(x(\tau^j_{qs+1})) = 0$. We extend (27) for all intervals in which the $q$th subsystem is active.

$$P[\sup_{t \in \mathbb{T}^q} V_q(x(t)) \geq \epsilon_q] \leq \frac{1}{\epsilon_q} V_{qs}(x_0), \quad \epsilon_q > 0, \quad (28)$$

where $\mathbb{T}^q = \bigcup_{t \in \mathbb{N}_0} T^j_q$. Since $V_q(x(t))$ are proper functions, for an arbitrary $\epsilon > 0$, there exist $\epsilon_q > 0$ for every $q \in Q$ such that $\|x(t)\| \geq \epsilon$ implies $V_q(x(t)) \geq \epsilon_q$. Therefore,

$$P[\sup_{t \geq 0} \|x(t)\| \geq \epsilon] \leq \frac{1}{\epsilon_q} V_{qs}(x_0). \quad (29)$$

If we extend (29) for $t \geq 0$, we obtain

$$P[\sup_{t \in \mathbb{T}^q} \|x(t)\| \geq \epsilon] \leq \max_{q \in Q} \frac{1}{\epsilon_q} V_{qs}(x_0). \quad (30)$$

As $x_0$ approaches 0, the probability value on the left hand side of (30) approaches 0. Thus, the zero solution of the system given by (2), (3) is stable in probability.

**Remark 3.2:** Consider the nonlinear stochastic switching system with state jumps described by (2), (3). If there exist quadratic positive definite functions $V_q(x), q \in Q$, such that $(\mu - 1)\lambda_1 + (\beta - 1)\lambda_S + \zeta < 0$, then the zero solution to the system given by (2), (3) is second moment exponentially stable. Specifically, if $V_q(x), q \in Q$, are quadratic positive definite functions, then there exist positive constants $a_1^q$ and $a_2^q$ such that

$$a_1^q \|x\|^2 \leq V_q(x) \leq a_2^q \|x\|^2, \quad q \in Q. \quad (31)$$

From (31), we obtain

$$\min_{q \in Q} a_1^q \|x\|^2 \leq V_q(x) \leq \max_{q \in Q} a_2^q \|x\|^2, \quad q \in Q. \quad (32)$$

Using inequalities (24) and (32),

$$E[\|x(t)\|^2] \leq \frac{E[V_q(x(t))]}{\min_{q \in Q} a_1^q} \leq \frac{V_{qs}(x_0)}{\min_{q \in Q} a_1^q} e^{((\mu - 1)\lambda_1 + (\beta - 1)\lambda_S + \zeta)t} \leq \max_{q \in Q} a_2^q \|x(0)\|^2 e^{((\mu - 1)\lambda_1 + (\beta - 1)\lambda_S + \zeta)t}. \quad (33)$$

Thus, since $(\mu - 1)\lambda_1 + (\beta - 1)\lambda_S + \zeta < 0$, the zero solution $x(t) \equiv 0$ to (2), (3) is second moment exponentially stable, that is, the inequality (1) holds with $C = \max_{q \in Q} a_2^q$ and $a = (\mu - 1)\lambda_1 + (\beta - 1)\lambda_S + \zeta$.

We can describe a stochastic switching system without state jumps as a special case of the system (2), (3) by assuming $J : \mathbb{R}^n \to \mathbb{R}^n$ is the identity function and $\{\eta_k > 0, k \in \mathbb{N}\}$ are random time instances that do not have any effect on the system given by

$$dx(t) = f_{\eta(t)}(x(t))dt + G_{\eta(t)}(x(t))dW(t). \quad (34)$$

**Corollary 3.3:** Consider the stochastic switching system given by (34). If there exist twice continuously differentiable,
positive definite, and proper functions $V_q : \mathbb{R}^n \rightarrow \mathbb{R}$, $q \in Q$, and scalars $\beta \geq 1$, $\zeta$, such that (4), (5), and

$$(\beta - 1)\lambda_S + \zeta \leq 0,$$  

(35)

hold, then the zero solution $x(t) \equiv 0$ of the system (34) is stable in probability.

Proof: The result is a direct consequence of Theorem 3.1 with $\mu = 1$. \hfill \Box

B. Stability of Linear Stochastic Switching Systems with State Jump

In this section we present the linear version of the stability conditions of continuous-time linear stochastic switching systems with state jumps described by

$$\begin{align*}
dx(t) &= A_r(t)x(t)dt + B_{r(t)}x(t)dW(t), \quad t \neq \hat{t}_k, \\
x(t) &= Jx(t^-), \quad t = \hat{t}_k
\end{align*}$$

(36)  
(37)

for $t \geq 0$ with initial conditions $x(0) = x_0$ and $r(0) = q_0$ where $\{x(t)\}_{t \geq 0}$ is an $\mathbb{R}^n$-valued $F_t$-adapted stochastic process, $\{W(t)\}_{t \geq 0}$ is an $\mathbb{R}$-valued $F_t$-adapted Wiener process, $A_q$, $B_q \in \mathbb{R}^{n \times n}$ are subsystem matrices, $J \in \mathbb{R}^{n \times n}$ is the state jump matrix, $\{\hat{t}_k > 0, k \in \mathbb{N}\}$ is the sequence of time points when a jump in the state occurs.

Corollary 3.4: Consider the linear stochastic switching system with state jumps given by (36), (37). If there exist Hermitian matrices $P_q > 0$, $q \in Q$, and $\zeta \in \mathbb{R}$ such that

$$0 \geq A_q^T P_q + P_q A_q + B_q^T P_q B_q - \zeta P_q, \quad q \in Q,$$  

(38)

$$(\mu - 1)\lambda_J + (\beta - 1)\lambda_S + \zeta \leq 0,$$  

(39)

with

$$\mu = \frac{\max_{q \in Q}(\lambda_{\max}(P_q))\lambda_{\max}(x^T J)}{\min_{q \in Q}(\lambda_{\min}(P_q))}$$

and $\beta = \frac{\max_{q \in Q}(\lambda_{\max}(P_q))}{\min_{q \in Q}(\lambda_{\min}(P_q))}$, then the zero solution $x(t) \equiv 0$ of the system described by (36), (37) is stable in probability.

Proof: The result is a direct consequence of Theorem 3.1 with $V_q(x) = x^T P_q x$, $f_q(x) = A_q x$, and $G_q(x) = B_q x$, $q \in Q$. \hfill \Box

For strictly negative values of $(\mu - 1)\lambda_J + (\beta - 1)\lambda_S + \zeta$, we can also show the second moment exponential stability of the zero solution $x(t) \equiv 0$ of (36), (37) (see Remark 3.2).

Consider the linear stochastic switching system described by (36), (37). We note that for the case where $J^T J \leq I_n$, the state jump (37) reduces the magnitude of the state $x(t)$, that is,

$$x^T(\hat{t}_k)x(\hat{t}_k) = x^T(\tilde{t}_k)J^T J x(\tilde{t}_k) \leq x^T(\tilde{t}_k)x(\tilde{t}_k).$$

(40)

In this case, at state jump instances, the state is driven closer to the origin regardless of the dynamics of the active mode. Hence, the state of a linear stochastic switching system with unstable modes may as well converge to the origin with this type of state jumps.

IV. ILLUSTRATIVE NUMERICAL EXAMPLES

In this section, we present two numerical examples to demonstrate the effectiveness of our results.

Example 4.1: Consider the 2-dimensional linear stochastic switching system (36), (37) with $M = 4$ modes described by subsystem matrices

$$\begin{align*}
A_1 &= \begin{bmatrix} -1.5 & -6 \\ 6 & -1.5 \end{bmatrix}, & B_1 &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\
A_2 &= \begin{bmatrix} -1.3 & 5 \\ -7 & -1.4 \end{bmatrix}, & B_2 &= \begin{bmatrix} 1.1 \\ 0 \end{bmatrix}, \\
A_3 &= \begin{bmatrix} -1.4 & -8 \\ 6 & -1.6 \end{bmatrix}, & B_3 &= \begin{bmatrix} 0.9 \\ 0 \end{bmatrix}, \\
A_4 &= \begin{bmatrix} -1.7 & -6 \\ 8 & -1.3 \end{bmatrix}, & B_4 &= \begin{bmatrix} 1.1 \\ 0 \end{bmatrix},
\end{align*}$$

the state jump matrix

$$J = \begin{bmatrix} 1.4 & 0 \\ 0 & 1.4 \end{bmatrix},$$

and counting process intensity parameters $\lambda_J = 0.5$, $\lambda_S = 0.5$.

Note that

$$\begin{align*}
P_1 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, & P_2 &= \begin{bmatrix} 1 & 0.71 \\ 0 & 0 \end{bmatrix}, \\
P_3 &= \begin{bmatrix} 0.75 & 0 \\ 0 & 1 \end{bmatrix}, & P_4 &= \begin{bmatrix} 1 & 0 \\ 0 & 0.75 \end{bmatrix},
\end{align*}$$

and scalar $\zeta = -1.13$ satisfy (38). Furthermore, (39) holds with $\mu = 2.75$ and $\beta = 1.4$. Therefore, it follows from Corollary 3.4 that the zero solution of the system given by (36), (37) is stable in probability. Moreover, since $(\mu - 1)\lambda_J + (\beta - 1)\lambda_S + \zeta$ is strictly negative, the zero solution is also second moment exponentially stable (see Remark 3.2).

With initial conditions $x(0) = [1, 1]^T$ and $r(0) = 1$, we obtain a sample path of $x(t)$ which we present in Fig. 1.

![State trajectory versus time](image_url)
Example 4.2: Now, we consider a 2-dimensional linear stochastic switching system with unstable modes where $M = 2$. Assume that we are given the following subsystem matrices

$$A_1 = \begin{bmatrix} 1.2 & -0.2 \\ 0.1 & 0.3 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1.2 & 0 \\ 0 & 1 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0.3 & -0.1 \\ 0.2 & 1.2 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1.1 & 0 \\ 0 & 1.1 \end{bmatrix},$$

the state jump matrix

$$J = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix},$$

and counting process intensity parameters $\lambda_1 = 5$, $\lambda_3 = 0.5$.

Note that

$$P_1 = \begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix},$$

and scalar $\zeta = 1.92$ satisfy the inequalities (38) and (39) with $\mu = 0.5$ and $\beta = 2$. As a consequence, it follows from Corollary 3.4 that the zero solution $x(t) \equiv 0$ is stable in probability. Furthermore, since $(\mu - 1)\lambda_1 + (\beta - 1)\lambda_3 + \zeta$ is strictly negative, the zero solution is second moment exponentially stable (see Remark 3.2). We note that although the subsystems are individually unstable, the overall system is stable.

We obtain a sample path of the state $x(t)$ with the initial conditions $x(0) = [1, 1]^T$ and $r(0) = 1$. The sample path is presented in Fig. 2.

V. CONCLUSION

The stability of continuous-time stochastic systems with probabilistic mode switchings and state jumps has been investigated. Occurrences of mode transitions and state jumps are modeled with independent Poisson processes. First, multiple Lyapunov functions have been used to derive sufficient conditions of stability in probability and exponential stability of continuous-time nonlinear stochastic switching systems. Second, sufficient matrix-algebraic conditions for stability of linear stochastic switching systems have been obtained. Moreover, we showed the stability of stochastic switching systems without state jumps as a special case of our main result.

REFERENCES