Further Results on Disturbance Attenuation
for Multiple Input Multiple Output Nonlinear Systems

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Abstract—The problems of disturbance attenuation and almost disturbance decoupling play a central role in control theory. In this paper, by employing a recently discovered structural decomposition of multiple input multiple output nonlinear systems and the backstepping procedures that were developed based on the structure of the decomposed system, we show that these two problems can be solved for a larger class of nonlinear systems.

I. INTRODUCTION AND PROBLEM STATEMENT

In this paper, we consider the problems of disturbance attenuation and almost disturbance decoupling with internal stability for nonlinear systems affine in control,

\[
\begin{align*}
\dot{x} &= f(x) + g(x)u + p(x, w), \\
y &= h(x),
\end{align*}
\]

where \( x \in \mathbb{R}^n, u \in \mathbb{R}^m, y \in \mathbb{R}^m \) and \( w \in \mathbb{R}^q \) are the state, input, output and disturbance, respectively, and the mappings \( f, g, p \) and \( h \) are smooth with \( f(0) = 0 \) and \( h(0) = 0 \). The problem of almost disturbance decoupling was originally formulated and solved in [1] for linear systems and was later extended to single input single output (SISO) minimum phase nonlinear systems in [2–4]. It was further extended to SISO non-minimum phase systems in [5, 6].

The problem of almost disturbance decoupling is, for any \( \gamma > 0 \), to find a feedback law \( u = u(x) \) such that the \( \mathcal{L}_2 \) gain from the disturbance to the output is less than or equal to \( \gamma \). A practical solution to the almost disturbance decoupling problem would require the resulting closed-loop system to be globally or locally asymptotically stable as well. Here in this paper, we will focus on the requirement of global asymptotic stability. The problem of disturbance attenuation is a less stringent one in that it does not require the bound on the resulting \( \mathcal{L}_2 \) to be arbitrarily small. The problem of almost disturbance decoupling is a special case of disturbance attenuation.

The problem of disturbance attenuation (or almost disturbance decoupling) with stability can be solved by establishing the dissipativity of the system [7]. That is, the problem of disturbance attenuation with stability (or almost disturbance decoupling) for a given system is, for a given (arbitrarily small) scalar \( \gamma > 0 \), to find a feedback law \( u = u(x) \) such that the resulting closed-loop system is strictly dissipative with respect to the supply rate \( q(w, y) = \gamma^2 w^2 - y^2 \), which is equivalent to finding a feedback law \( u = u(x) \) such that, for some smooth, positive definite and proper function \( V(x) \), the dissipation inequality

\[
\frac{\partial V}{\partial x}(f(x) + g(x)u(x) + p(x, w)) \leq -\alpha(||x||) + \gamma^2 w^2 - h^2(x), \quad x \in \mathbb{R}^n, w \in \mathbb{R},
\]

holds for some class \( \mathcal{K}_\infty \) function \( \alpha \).

The inequality (2) guarantees that the response of the closed-loop system in the absence of disturbance is globally asymptotically stable and, with \( x(0) = 0 \),

\[
\int_0^\infty y^2(t)dt \leq \gamma^2 \int_0^\infty u^2(t)dt,
\]

for every \( \mathcal{L}_2 \) disturbance \( w \).

The solution to the problems of disturbance attenuation and almost disturbance decoupling usually resorts to transforming the nonlinear system into certain structural normal forms. In [7–9], under the assumptions that the ranks of certain matrix valued functions of the state variables are constant, a square invertible system can be transformed into the following form

\[
\begin{align*}
\dot{z} &= f_0(z), \\
\dot{\xi}_{i,j} &= \xi_{i,j+1} + \sum_{l=1}^{i-1} \delta_{i,j,l}(x)v_l, \quad j = 1, 2, \ldots, n_i - 1, \\
\dot{\xi}_{i,n_i} &= v_i, \\
\gamma_i &= \xi_{i,1}, \quad i = 1, 2, \ldots, m,
\end{align*}
\]

where \( n_1 \leq n_2 \leq \cdots \leq n_m, v_i = a_i(x) + b_i(x)u, i = 1, 2, \ldots, m \), with the matrix \( \text{col} \{b_1(x), b_2(x), \ldots, b_m(x)\} \) being smooth and nonsingular.

As pointed out in [8], when all \( \delta_{i,j,l}(x) = 0 \), the set of integers \( \{n_1, n_2, \ldots, n_m\} \) in (3) corresponds to the vector relative degrees, which represent the infinite zero structure if the system is specialized to a linear one. These integers however are not related to the infinite zero structure of linear systems when \( \delta_{i,j,l}(x) \neq 0 \), and thus cannot be viewed as the nonlinear equivalence of and expected to play a similar role as infinite zeros (see [10] for an example showing this).

In a recent paper [10], we studied the structural properties of affine-in-control nonlinear systems beyond the case of square invertible systems. We proposed an algorithm that identifies a set of integers that are equivalent to the infinite zero structure of linear systems and leads to a normal form that corresponds to these integers as well as to the system invertibility structure. The new normal form for a square
invertible system is
\[
\begin{cases}
\dot{z} = f_0(x), \\
\dot{\xi}_{i,j} = \xi_{i,j+1} + \sum_{l=1}^{i-1} \delta_{i,j,l}(x)v_{l,j} = 1, \ldots, \alpha(\cdot),
\end{cases}
\]
where \(q_1 \leq q_2 \leq \cdots \leq q_m, v_i = a_i(x) + b_i(x)u,\) with the matrix \(\{b_1(x), b_2(x), \ldots, b_m(x)\}\) being of full row rank and smooth, and
\[
\delta_{i,j,l}(x) = 0, \quad \text{for } j < q_i, i = 1, 2, \ldots, m.
\]

Normal form (4)-(5) is the same as (3) except for the additional structural property (5). The \(\dot{\xi}_{i,j}\) equation in (4) displays a triangular structure of the control inputs that enter the system. Property (5) imposes additional structure within each chain of integrators on how control inputs enter the system. With this additional structural property, the set of integers \(\{q_1, q_2, \ldots, q_m\}\) indeed represent an infinite zero structure when the system is specialized to a linear one. It has been shown in [11] that the new normal form allows the development of some new backstepping design procedures, referred to as the level-by-level backstepping and mixed chain-by-chain and level-by-level backstepping, which lead to the stabilization of a larger class of systems that the conventional chain-by-chain backstepping design procedure cannot stabilize. The objective of this paper is to show that the backstepping design procedures of [11] can also be utilized to solve the problems of disturbance attention and almost disturbance decoupling for a larger class of multiple input multiple output systems.

II. PRELIMINARY RESULTS

We first recall the follow result on disturbance attenuation with stability from [7]. This result will serve as a building block in our design procedures.

**Lemma 2.1:** Consider a system described by
\[
\begin{cases}
\dot{z} = f_0(z, \xi) + p_0(z, w), \\
\dot{\xi} = u + f_1(z, \xi) + p_1(z, \xi, w), \\
y = h(z, \xi),
\end{cases}
\]
where \((z, \xi) \in \mathbb{R}^n \times \mathbb{R}, f_0(0,0) = 0 \text{ and } f_1(0,0) = 0.\) Assume that
\[
\|p_0(z, w)\| \leq R_0(z)|w|, \quad \forall z, w, \\
\|p_1(z, \xi, w)\| \leq R_1(z, \xi)|w|, \quad \forall z, \xi, w,
\]
for some smooth real-valued functions \(R_0(z)\) and \(R_1(z, \xi).\) Suppose that there exist a number \(\gamma > 0,\) a smooth real-valued function \(v(z)\) with \(v(0) = 0,\) a smooth positive definite and radially unbounded function \(V(z),\) and a class \(\mathcal{K}_\infty\) function \(\alpha_0(\cdot)\) such that
\[
\frac{\partial V}{\partial z} [f_0(z, v(z)) + p_0(z, w)] \\
\leq -\alpha_0(\|z\|) + \gamma^2 w^2 - h^2(z, v(z)), \quad \forall z, \xi, w,
\]
that is, there exists a smooth \(v(z)\) such that the subsystem
\[
\begin{cases}
\dot{z} = f_0(z, v(z)) + p_0(z, w), \\
y = h(z, v(z)),
\end{cases}
\]
is strictly dissipative with respect to the supply rate \(q(w, y) = \gamma^2 w^2 - y^2.\)

Then, for every \(\epsilon > 0,\) there exist a smooth feedback law \(u = \epsilon(z, \xi),\) a smooth positive definite and radially unbounded function \(W(z, \xi),\) and a class \(\mathcal{K}_\infty\) function \(\alpha(\cdot)\) such that
\[
\frac{\partial W}{\partial z} (f_0(z, \xi) + p_0(z, w)) \\
+ \frac{\partial W}{\partial \xi} (u(z, \xi) + f_1(z, \xi) + p_1(z, \xi, w)) \\
\leq -\alpha(\|\xi\|) + (\gamma + \epsilon)^2 w^2 - h^2(z, \xi), \quad \forall z, \xi, w,
\]
or equivalently, there exists a smooth feedback law \(u = \epsilon(z, \xi)\) such that the resulting closed-loop system is strictly dissipative with respect to the supply rate \(q(w, y) = (\gamma + \epsilon)^2 w^2 - y^2.\)

In [7], the possibility of fulfilling the main condition (7) in Lemma 2.1 is discussed. In the context of the almost disturbance decoupling problem, suppose that the \(z\)-subsystem
\[
\dot{z} = f(z, \xi_1, w)
\]
can be decomposed as
\[
\begin{cases}
\dot{z}_1 = f_1(z_1, z_2, \xi_1, w), \\
\dot{z}_2 = f_2(z_2, \xi_1),
\end{cases}
\]
where \(z_1\) represents “stable component” and \(z_2\) represents “unstable but stabilizable component.”

**Lemma 2.2:** Consider system (8) which can be decomposed as (9). Suppose that

1) there exists a smooth positive definite and radially unbounded function \(V_1(z_1)\) such that
\[
\frac{\partial V_1}{\partial z_1} f_1(z_1, z_2, \xi_1, w) \\
\leq -\alpha_1(\|z_1\|) + \gamma_0^2 w^2 + \gamma_0^2 \|z_2\|^2 + \gamma_3 \xi_1^2,
\]
for some \(\mathcal{K}_\infty\) function \(\alpha_1\) and some \(\gamma_0 > 0,
\]
2) there exist a smooth real-valued function \(v_2(z_2)\) with \(v_2(0) = 0\) and a smooth positive definite and radially unbounded function \(V_2(z_2)\) such that
\[
\frac{\partial V_2}{\partial z_2} f_2(z_2, v_2(z_2)) + v_2^2(z_2) \leq -\alpha_2(\|z_2\|),
\]
for some \(\mathcal{K}_\infty\) function \(\alpha_2.\)

Then, for every \(\gamma > 0,\) there exist a smooth \(v(z)\) with \(v(0) = 0\) and a smooth positive definite and radially unbounded function \(V(z)\) such that
\[
\frac{\partial V}{\partial z} f(z, v(z), w) \leq -\alpha(\|z\|) + \gamma^2 w^2 - v^2(z),
\]
for some \(\mathcal{K}_\infty\) function \(\alpha(\cdot),\)

Next, we recall from [7, 11–13] some results on the backstepping design methodology for the stabilization of
systems in the forms of (3) and (4)-(5) under the following assumptions.

Assumption 1: The dynamics \( z \) is driven only by \( \xi_{i,1}, \ i = 1, 2, \ldots, m, \) i.e.,

\[
\dot{z} = f_0(z, \xi_{1,1}, \xi_{2,1}, \ldots, \xi_{m,1}),
\]

and there exist smooth functions \( \phi_{i,1}(z) \), with \( \phi_{i,1}(0) = 0, \)
\( i = 1, 2, \ldots, m, \) such that \( \dot{z} = f_0(z, \phi_{1,1}(z), \phi_{2,1}(z), \ldots, \phi_{m,1}(z)) \) is globally asymptotically stable at its equilibrium \( z = 0. \)

Assumption 2: The functions \( \delta_{i,j,l} \) depend only on variables \( z \) and \( \xi_{l,p}, \xi_{b} \), with
\[
1 \leq \ell_p \leq m \quad \text{and} \quad \ell_b = 1; \quad \text{or,}
\]
\[
\ell_p \leq i - 1; \quad \text{or,}
\]
\[
\ell_p = i \quad \text{and} \quad \ell_b \leq j.
\]

Assumption 3: The functions \( \delta_{i,j,l} \) depend only on variables \( z \) and \( \xi_{l,p}, \xi_{b} \), with
\[
1 \leq \ell_p \leq m \quad \text{and} \quad \ell_b = 1; \quad \text{or,}
\]
\[
\ell_p \leq j - 1; \quad \text{or,}
\]
\[
\ell_b = j \quad \text{and} \quad \ell_p \leq i.
\]

Then, there exists a feedback \( v = v(z, \xi) \) that globally asymptotically stabilizes the system at \( (z, \xi) = 0. \)

III. DISTURBANCE ATTENUATION AND ALMOST DISTURBANCE DECOPULING WITH STABILITY

As in the literature on the problems of disturbance attenuation and almost disturbance decoupling for SISO systems, we assume that the zero dynamics is driven only by the states on the top of the \( m \) chains of integrators, \( \xi_{i,1}, \ i = 1, 2, \ldots, m. \)

To apply the level-by-level backstepping [11], we also assume that the coefficients \( \delta_{i,j,l}, p_{i,j}(z, \xi, w), j = 1, 2, \ldots, q_i, \ i = 1, 2, \ldots, m, \) satisfy the level-by-level triangular dependency on state variables. We have following result on the problem disturbance attenuation with stability.

Theorem 3.1: Consider a system given by

\[
\begin{align*}
\dot{z} &= f_0(z, \xi_{1,1}, \xi_{2,1}, \ldots, \xi_{m,1}) + p_0(z, w), \\
\dot{\xi}_{i,j} &= \xi_{i,j+1} + \sum_{l=1}^{i-1} \delta_{i,j,l}(z, \xi) v_l + p_{i,j}(z, \xi, w), \\
\dot{\xi}_{i,q_i} &= v_i + p_{i,q_i}(z, \xi) w_i,
\end{align*}
\]

where \( \xi = \text{col} \{ \xi_1, \xi_2, \ldots, \xi_m \}, \)
\( \xi_i = \text{col} \{ \xi_{i,1}, \xi_{i,2}, \ldots, \xi_{i,q_i} \}, \)
\( v = \text{col} \{ v_1, v_2, \ldots, v_m \}, \)
\( q_1 \leq q_2 \leq \cdots \leq q_m, \)
\( \text{and functions } f_0, p_0, \delta_{i,j,l}, p_{i,j}, j = 1, 2, \ldots, q_i, \ i = 1, 2, \ldots, m, \) are smooth with \( f_0(0, 0, \ldots, 0) = 0. \) Assume that

\[
\| p_0(z, w) \| \leq R_0(z) \| w \|, \quad \forall z, w,
\]

\[
\| p_{i,j}(z, \xi, w) \| \leq R_{i,j}(z, \xi) \| w \|, \quad \forall z, \xi, w, j = 1, 2, \ldots, q_i, \ i = 1, 2, \ldots, m,
\]

for some smooth functions \( R_0(z) \) and \( R_{i,j}(z, \xi), j = 1, 2, \ldots, q_i, \ i = 1, 2, \ldots, m. \) Suppose that

1) there exist a number \( \gamma > 0, \) smooth function \( \phi_{i,1}(z), \)
\( \phi_{i,1}(0) = 0, \ i = 1, 2, \ldots, m, \) a smooth positive definite and radially unbounded function \( V(z), \)
\( \text{and a class } K_\infty \) function \( \alpha_0(\cdot) \) such that

\[
\frac{\partial V}{\partial z} [f_0(z, \phi_{1,1}(z), \phi_{2,1}(z), \ldots, \phi_{m,1}(z)) + p_0(z, w)] \leq -\alpha_0(\| z \|) + \gamma^2 \| w \|^2
\]

\[
-\| \text{col} \{ \phi_{1,1}(z), \phi_{2,1}(z), \ldots, \phi_{m,1}(z) \} \|^2,
\]

for all \( z \) and \( w. \)

2) the functions \( \delta_{i,j,l}(z, \xi) \) and \( p_{i,j}(z, \xi, \cdot) \) depend only on variables \( z \) and \( \xi_{l,p}, \xi_{b}, \) with
\( a) \ 1 \leq \ell_p \leq m \quad \text{and} \quad \ell_b = 1; \quad \text{or,}
\]
\( b) \ \ell_b \leq j - 1; \quad \text{or,}
\]
\( c) \ \ell_b = j \quad \text{and} \quad \ell_p \leq i. \)

Then, for every \( \epsilon > 0, \) there exist smooth feedback laws \( v_i = v_i(z, \xi), \ i = 1, 2, \ldots, m, \) such that the resulting closed-loop system is strictly dissipative with respect to the supply rate \( q(w, y) = (\gamma + \epsilon)^2 \| w \|^2 + \| y \|^2. \)

Proof: The theorem can be proven by using the level-by-level backstepping design procedure [11]. In each step of the procedure, we use Lemma 2.1. Let \( n_d = \sum_{l=1}^{m} q_l. \)
We start the backstepping with
\[
\begin{align*}
\dot{z} &= f_0(z, \xi_{1,1}, \phi_{2,1}(z), \phi_{3,1}(z), \ldots, \phi_{m,1}(z)) + p_0(z, w), \\
\dot{\xi}_{1,1} &= \phi_{1,2}(z, \xi_{1,1}, \phi_{2,1}(z), \phi_{3,1}(z), \ldots, \phi_{m,1}(z), w), \\
y_1 &= \phi_{1,1}(z),
\end{align*}
\]
and
\[
y_i = \phi_{1,i}(z), \quad i = 2, 3, \ldots, m.
\]
(13)

Here, \(\xi_{1,2}\) is viewed as a virtual input. By Lemma 2.1, for every \(\epsilon > 0\), there exist a smooth feedback law \(\xi_{1,2} = \phi_{1,2}(z, \xi_{1,1})\), a smooth positive definite and radially unbounded function \(W_{1,1}(z, \xi_{1,1})\), and a class \(\mathcal{K}_\infty\) function \(\alpha_{1,1}(\cdot)\) such that
\[
\begin{align*}
\frac{\partial W_{1,1}}{\partial z}[f_0(z, \xi_{1,1}, \phi_{2,1}(z), \phi_{3,1}(z), \ldots, \phi_{m,1}(z)) + p_0(z, w)] \\
+ \frac{\partial W_{1,1}}{\partial \xi_{1,1}}[\phi_{1,2}(z, \xi_{1,1}) + p_{1,1}(z, \xi_{1,1}, w)] \\
\leq -\alpha_{1,1}(\|\{z, \xi_{1,1}\}\| + (\gamma + \epsilon/n_d)^2\|w\|^2 - \|y\|^2),
\end{align*}
\]
(14)

for all \(z, \xi_{1,1}\) and \(w\). That is, subsystem (13) with the feedback \(\xi_{1,2} = \phi_{1,2}(z, \xi_{1,1})\) is strictly dissipative with respect to the supply rate \(q(w, y) = (\gamma + \epsilon/n_d)^2\|w\|^2 - \|y\|^2\).

Next, consider the subsystem
\[
\begin{align*}
\dot{z} &= f_0(z, \xi_{2,1}, \phi_{3,1}(z), \phi_{4,1}(z), \ldots, \phi_{m,1}(z)) + p_0(z, w), \\
\dot{\xi}_{1,1} &= \phi_{1,2}(z, \xi_{2,1}, \phi_{3,1}(z), \phi_{4,1}(z), \ldots, \phi_{m,1}(z), w), \\
y_1 &= \phi_{1,1}(z),
\end{align*}
\]
and
\[
y_i = \phi_{1,i}(z), \quad i = 3, \ldots, m,
\]
where \(\xi_{2,2}\) is viewed as a virtual input. By Lemma 2.1, and in view of (14), there exist a smooth feedback law \(\xi_{2,2} = \phi_{2,2}(z, \xi_{1,1}, \xi_{2,1})\) such that the resulting closed-loop system is strictly dissipative with respect to the supply rate \(q(w, y) = (\gamma + 2\epsilon/n_d)^2\|w\|^2 - \|y\|^2\).

Similarly, we step back from the remaining states in the first-level, and obtain \(v_i = v_i(z; \xi_{1,1}, \xi_{2,1}, \ldots, \xi_{i,1})\), \(i = 1, 2, \ldots, b_1\), where \(b_1\) is the number of chains that contain exactly one integrator, i.e., \(q_1 = q_2 = \cdots = q_{b_1} = 1\).

For chains that contain more than one integrator, we have \(\xi_{i,2} = \phi_{i,2}(z; \xi_{1,1}, \xi_{2,1}, \ldots, \xi_{i,1})\), \(i = b_1 + 1, b_1 + 2, \ldots, m\). Thus, the following subsystem
\[
\begin{align*}
\dot{z} &= f_0(z, \xi_{1,1}, \xi_{2,1}, \ldots, \xi_{m,1}) + p_0(z, w), \\
\dot{\xi}_{1,1} &= v_i + p_{i,1}(z, \xi_{1,1}, \xi_{2,1}, \ldots, \xi_{m,1}, w), \\
y_i &= \xi_{i,1},
\end{align*}
\]
and
\[
y_i = \phi_{1,i}(z), \quad i = 1, 2, \ldots, m,
\]
is strictly dissipative with respect to the supply rate \(q(w, y) = (\gamma + m\epsilon/n_d)^2\|w\|^2 - \|y\|^2\).

To proceed backstepping on the first state in the second-level, we view \(\xi_{b_1+1,3}\) as a virtual input of the following subsystem, which consists of (16) and the dynamics
\[
\dot{\xi}_{b_1+1,2} = \xi_{b_1+1,3} + p_{b_1+1,2}(z, \xi_{1,1}, \xi_{2,1}, \ldots, \xi_{m,1}, \xi_{b_1+1,2}, w).
\]

Again, by Lemma 2.1, there exists a smooth feedback law \(\xi_{b_1+1,3} = \phi_{b_1+1,3}(z, \xi_{1,1}, \xi_{2,1}, \ldots, \xi_{m,1}, \xi_{b_1+1,2})\), such that the resulting closed-loop subsystem is strictly dissipative with respect to the supply rate \(q(w, y) = (\gamma + (m + 1)\epsilon/n_d)^2\|w\|^2 - \|y\|^2\).

Continuing in this way, we finally obtain
\[
v_i = v_i(z; \xi_{1,1}, \xi_{2,1}, \ldots, \xi_{m,1}; \xi_{1,q_m-1}, \xi_{2,q_m-1}, \ldots, \xi_{m,q_m-1}; \xi_{1,q_m}),
\]
for chains that contain \(q_m\) integrators, such that the closed-loop system is strictly dissipative with respect to the supply rate \(q(w, y) = (\gamma + \epsilon)^2\|w\|^2 - \|y\|^2\).

The level-by-level backstepping procedure enlarges the class of systems for which the disturbance attenuation problem can be solved. The triangular dependency requirement in Theorem 3.1 can be further weakened if we mix the chain-by-chain backstepping and the level-by-level backstepping and implement it on a same system. The following result includes the chain-by-chain backstepping and level-by-level as special cases.

**Theorem 3.2:** Consider a system in the form
\[
\begin{align*}
\dot{z} &= f_0(z, \xi_{1,1}, \xi_{2,1}, \ldots, \xi_{m,1}) + p_0(z, w), \\
\dot{\xi}_{i,j} &= \xi_{i,j+1} + \sum_{l=1}^{i-1} \delta_{i,j,l}(z, \xi)v_l + p_{i,j}(z, \xi, w), \\
y_i &= \xi_{i,1},
\end{align*}
\]
where \(\xi = \text{col}\{\xi_1, \xi_2, \ldots, \xi_m\}\), \(\xi_i = \text{col}\{\xi_{i,1}, \xi_{i,2}, \ldots, \xi_{i,q_i}\}\), \(q_1 \leq q_2 \leq \cdots \leq q_m\), and functions \(f_0, p_0, \delta_{i,j,l}\) and \(p_{i,j}\), \(i = 1, 2, \ldots, m\) are smooth. Assume that \(\|p_0(z, w)\| \leq R_0\|z\|\), \(\forall z, w\), \(\|p_{i,j}(z, w)\| \leq R_{i,j}\|z, \xi\|\), \(\forall z, \xi, w, j = 1, 2, \ldots, q_i, i = 1, 2, \ldots, m\), for some smooth functions \(R_0(z)\) and \(R_{i,j}(z, \xi), j = 1, 2, \ldots, q_i, i = 1, 2, \ldots, m\), Suppose that

1) **Condition 1** in Theorem 3.1 holds,

2) there exists an ordered list \(\kappa\) containing all variables of \(\xi\) such that, for \(j = 1, 2, \ldots, q_i - 1, l = 1, 2, \ldots, i - 1, i = 1, 2, \ldots, m\),

a) \(\xi_{i,j}\) appears earlier than \(\xi_{i,j+1}\) in \(\kappa\);

b) for \(\delta_{i,j,l} \neq 0\), the variables \(\xi_{i,j}\) appear earlier than \(\xi_{i,j+1}\) in \(\kappa\);

c) \(\delta_{i,j,l}(z, \xi)\) and \(p_{i,j}(z, \xi, \cdot)\) depend only on \(z, \xi_{l,1}, \ell = 1, 2, \ldots, m\), and the variables that appear no later than \(\xi_{i,j}\) in \(\kappa\).

Then, for every \(\epsilon > 0\), there exist smooth feedback laws \(v_i = v_i(z, \xi)\), \(i = 1, 2, \ldots, m\), such that the resulting closed-loop system is strictly dissipative with respect to the supply rate \(q(w, y) = (\gamma + \epsilon)^2\|w\|^2 - \|y\|^2\), where \(y = \text{col}\{y_1, y_2, \ldots, y_m\}\).

**Proof:** The backstepping can be carried out one state by one state in the order of the list \(\kappa\). Suppose that, after backstepping \(\ell\) state variables, we want to backstep from \(\xi_{i,j}\),
the $\ell+1$-th element in the list $\kappa$, to $\xi_{i,j+1}$. Let $n_d = \sum_{l=1}^{m} q_l$. Denote all the state variables that come before $\xi_{i,j}$ in the list $\kappa$ as $Z$. By Condition 2), we can describe the subsystem of $Z$ and $\xi_{i,j}$ as
\[
\begin{cases}
\dot{Z} = F_0(Z, \xi_{i,j}) + P_0(Z, w), \\
\dot{\xi}_{i,j} = \xi_{i,j+1} + \sum_{l=1}^{i-1} \delta_{i,j,l}(Z, \xi_{i,j}) v_l(Z) + P_{i,j}(Z, \xi_{i,j}, w), \\
y_i = \xi_{i,1}, \ i = 1, 2, \ldots, m,
\end{cases}
\tag{18}
\]
where $\xi_{i,j+1}$ is viewed as a virtual input. By Lemma 2.1, there exists a smooth function
\[
\xi_{i,j+1} = \phi_{i,j+1}(Z, \xi_{i,j})
\]
such that the resulting closed-loop subsystem is strictly dissipative with respect to the supply rate $q(w, y) = [\gamma + (\ell + 1)\varepsilon/n_d]^2 \|w\|^2 - \|y\|^2$. By backstepping through all the state variables in the list $\kappa$, we can find the desired feedback laws $v_i = v_i(z, \xi)$, $i = 1, 2, \ldots, m$. \hfill \square

As the problem of almost disturbance decoupling is a special case of the problem of disturbance attenuation, the following result on almost disturbance decoupling with stability is a corollary to Theorem 3.2.

**Corollary 3.1:** Consider a system in the form
\[
\begin{cases}
\dot{z} = f_0(z, \xi_1, \xi_2, \ldots, \xi_m, w), \\
\dot{\xi}_{i,j} = \xi_{i,j+1} + \sum_{l=1}^{i-1} \delta_{i,j,l}(z, \xi_{i,j}) v_l(z) + p_{i,j}(z, \xi, w), \\
y_i = \xi_{i,1}, \ i = 1, 2, \ldots, m,
\end{cases}
\tag{19}
\]
where $\xi = \col \{\xi_1, \xi_2, \ldots, \xi_m\}$, $\xi_i = \col \{\xi_{1,i}, \xi_{2,i}, \ldots, \xi_{m,i}\}$, $q_1 \leq q_2 \leq \ldots \leq q_m$, and functions $f_0, \delta_{i,j,l}$ and $p_{i,j}$, $i = 1, 2, \ldots, m$, are smooth. Assume that
\[
|p_{i,j}(z, \xi, w)| \leq R_{i,j}(z, \xi)|w|, \ \forall z, \xi, w, \\
j = 1, 2, \ldots, q_i, \ i = 1, 2, \ldots, m,
\]
for some smooth functions $R_{i,j}(z, \xi)$, $j = 1, 2, \ldots, q_i$, $i = 1, 2, \ldots, m$. Suppose that

1) for every $\gamma_0 > 0$, there exist smooth $\phi_{i,1}(z)$ with $\phi_{i,1}(0) = 0$, $i = 1, 2, \ldots, m$, and a smooth positive definite and radially unbounded function $V_1(z)$ such that
\[
\frac{\partial V_1}{\partial z} f_0(z, \phi_{i,1}(z), \phi_{2,1}(z), \ldots, \phi_{m,1}(z), w) \\
\leq -\alpha(\|z\|) + \gamma_0^2 \|w\|^2 \\
-\|\col \{\phi_{1,1}(z), \phi_{2,1}(z), \ldots, \phi_{m,1}(z)\}\|^2, \ \forall z, w,
\]
for some $K_\infty$ function $\alpha(\cdot)$.

2) Condition 2) in Theorem 3.2 holds.

Then, for every $\gamma > 0$, there exist smooth feedback laws $v_i = v_i(z, \xi)$, $i = 1, 2, \ldots, m$, such that the resulting closed-loop system is strictly dissipative with respect to the supply rate $q(w, y) = \gamma^2 \|w\|^2 - \|y\|^2$, where $y = \col \{y_1, y_2, \ldots, y_m\}$.

We next consider further the fulfillment of Condition 1) in Corollary 3.1. It is a generalization of Lemma 2.2 to multiple input multiple output systems. Suppose that the $z$-subsystem
\[
\dot{z} = f_0(z, \xi_{1,1}, \xi_{2,1}, \ldots, \xi_{m,1}, w)
\tag{20}
\]
can be decomposed as
\[
\begin{cases}
\dot{z}_1 = f_1(z_1, z_2, \xi_{1,1}, \xi_{2,1}, \ldots, \xi_{m,1}, w), \\
z_2 = f_2(z_2, \xi_{1,1}, \xi_{2,1}, \ldots, \xi_{m,1}),
\end{cases}
\tag{21}
\]
where $z_1$ represents “stable component” and $z_2$ represents “unstable but stabilizable component.” We have the following result.

**Corollary 3.2:** Consider system (20) which can be decomposed as (21). Suppose that

1) there exists a smooth positive definite and radially unbounded function $V_1(z_1)$ such that
\[
\frac{\partial V_1}{\partial z_1} f_1(z_1, z_2, \xi_{1,1}, \xi_{2,1}, \ldots, \xi_{m,1}, w) \\
\leq -\alpha_1(\|z_1\|) + \gamma_0^2 \|z_2\|^2 + \gamma_0^2 \|w\|^2 \\
+\gamma_0^2 \|\col \{\xi_{1,1}, \xi_{2,1}, \ldots, \xi_{m,1}\}\|^2, \ \forall z_1, w,
\]
for some $K_\infty$ function $\alpha_1$ and some $\gamma_0 > 0$, and

2) there exist smooth functions $\bar{v}_i(z_2)$ with $\bar{v}_i(0) = 0$ for $i = 1, 2, \ldots, m$, and a smooth positive definite and radially unbounded function $V_2(z_2)$ such that
\[
\frac{\partial V_2}{\partial z_2} f_2(z_2, \bar{v}_1(z_2), \bar{v}_2(z_2), \ldots, \bar{v}_m(z_2)) \\
+\|\col \{\bar{v}_1(z_2), \bar{v}_2(z_2), \ldots, \bar{v}_m(z_2)\}\|^2 \leq -\alpha_2(\|z_2\|)
\]
for some $K_\infty$ function $\alpha_2$.

Then, for every $\gamma > 0$, there exist smooth $v_i(z)$ with $v_i(0) = 0$, $i = 1, 2, \ldots, m$, and a smooth positive definite and radially unbounded function $V(z)$ such that
\[
\frac{\partial V}{\partial z} f_0(z, v_1(z), v_2(z), \ldots, v_m(z), w) \leq -\alpha(\|z\|) \\
+\gamma^2 \|w\|^2 - \|\col \{v_1(z), v_2(z), \ldots, v_m(z)\}\|^2, \ \forall z, w,
\]
for some $K_\infty$ function $\alpha(\cdot)$.

Corollary 3.2 provides, for the system in Corollary 3.1, a starting point from which the backstepping can be carried out. We next use a numerical example with unstable zero dynamics to illustrate Corollary 3.2.

**Example 3.1:** Consider a system in the form of (19) with $q_1 = 1$ and $q_2 = 2$,
\[
\begin{cases}
\dot{z} = z + \xi_{1,1} + \xi_{2,1}, \\
\dot{\xi}_{1,1} = v_1 + \xi_{2,1} w, \\
\dot{\xi}_{2,1} = \xi_{2,1} + z w, \\
\dot{\xi}_{2,2} = v_2 + (\cos \xi_{1,1}) \sin w, \\
y_1 = \xi_{1,1}, \\
y_2 = \xi_{2,1}.
\end{cases}
\tag{22}
\]
Note that the dependency requirement in Corollary 3.1 holds and zero dynamics satisfies the conditions in Corollary 3.2. The zero dynamics $\dot{z} = z$ is unstable. View $\xi_{1,1}$ and $\xi_{2,1}$ as virtual input of
\[
\dot{z} = z + \xi_{1,1} + \xi_{2,1},
\tag{23}
\]
Then $\xi_{1,1} = \phi_{1,1}(z) = -2z$ and $\xi_{2,1} = 0$ stabilizes (23) with Lyapunov function $V_0 = z^2/2$.

Condition 2) of Corollary 3.1 holds with $\kappa = \{\xi_{2,1}, \xi_{1,1},\xi_{2,2}\}$.

To begin the mixed chain-by-chain and level-by-level backstepping procedure, we consider the subsystem
\[
\begin{cases}
\dot{z} = -z + \xi_{2,1}, \\
\xi_{2,1} = \xi_{2,2} + zw, \\
\xi_{1,1} = v_1 + \xi_{2,1}w,
\end{cases}
\]
and view $\xi_{2,2}$ as a virtual input. Consider the Lyapunov function $V_1 = V_0 + \xi_{2,1}^2/2 = z^2/2 + \xi_{2,1}^2/2$. Its time derivative is given by
\[
\dot{V}_1 = \dot{V}_0 + z\xi_{2,1} + \xi_{2,1}\xi_{2,2} + \xi_{2,1}zw.
\]
Let
\[
\xi_{2,2} = \phi_{2,2}(z, \xi_{2,1}) = -z - \xi_{2,1} - \frac{3}{4\gamma^2}\xi_{2,1}(1 + z^2),
\]
which renders
\[
\dot{V}_1 \leq -\xi_{2,1}^2 + \frac{\gamma^2}{3}\|w\|^2 - z^2.
\]

We next consider
\[
\begin{cases}
\dot{z} = z + \xi_{1,1} + \xi_{2,1}, \\
\xi_{2,1} = -z - \xi_{2,1} - \frac{3}{4\gamma^2}\xi_{2,1}(1 + z^2) + zw, \\
\xi_{1,1} = v_1 + \xi_{2,1}w,
\end{cases}
\]
Letting $V_2 = V_1 + (\xi_{1,1} + 2z)^2/2$, we have
\[
\dot{V}_2 \leq \dot{V}_1 + (\xi_{1,1} + 2z)(v_1 + 3z + \xi_{1,1} + \xi_{2,1} + \xi_{2,1}w).
\]
Let
\[
v_1 = -\frac{11}{3} - \frac{4}{3}\xi_{1,1} - \xi_{2,1} - \frac{3}{4\gamma^2}(\xi_{1,1} + 2z)(1 + \xi_{2,1}^2).
\]
We have
\[
\dot{V}_2 \leq -\xi_{2,1}^2 + \frac{2\gamma^2}{3}w^2 - z^2 - (\xi_{1,1} + 2z)^2/3 \leq -\xi_{2,1}^2 + \frac{2\gamma^2}{3}w^2 - 4z^2 - \xi_{1,1}^2.
\]
Finally, consider
\[
\begin{cases}
\dot{z} = z + \xi_{1,1} + \xi_{2,1}, \\
\xi_{1,1} = v_1 + \xi_{2,1}w, \\
\xi_{2,1} = \xi_{2,2} + zw, \\
\xi_{2,2} = v_2 + (\cos\xi_{2,1})\sin w,
\end{cases}
\]
for which we let $V_3 = V_2 + (\xi_{2,2} - \phi_{2,2})^2/2$. Thus,
\[
\dot{V}_3 = \dot{V}_2 + (\xi_{2,2} - \phi_{2,2}) (\xi_{2,1} + v_2 + (\cos\xi_{1,1})\sin w - \phi_{2,2}) = \dot{V}_2 + (\xi_{2,2} - \phi_{2,2}) (v_2 + \Psi + \Phi w + (\cos\xi_{1,1})\sin w),
\]
where
\[
\begin{align*}
\Phi &= z + \frac{3}{4\gamma^2}(z + z^3), \\
\Psi &= z + \xi_{1,1} + 2\xi_{2,1} + \xi_{2,2} + \frac{3}{4\gamma^2}\xi_{2,2}(1 + z^2) + \frac{3}{2\gamma^2}\xi_{2,1}z(z + \xi_{1,1} + \xi_{2,1}).
\end{align*}
\]
Let
\[
v_2 = -\xi_{2,2} + \phi_{2,2} - \Psi - \frac{3}{4\gamma^2}(\xi_{2,2} - \phi_{2,2}) (1 + (|\Phi| + |\cos\xi_{1,1}|)^2).
\]
We have
\[
\dot{V}_3 \leq -\xi_{2,2}^2 - \xi_{2,1}^2 + \gamma^2 w^2 - 4z^2 - \xi_{1,1}^2 \leq -\xi_{2,2}^2 - \xi_{2,1}^2 + \gamma^2 w^2,
\]
from which we have
\[
\int_0^t \|y(\tau)\|^2 d\tau \leq \gamma^2 \int_0^t w(\tau)^2 d\tau + V_3(x(0)).
\]

IV. Conclusions

In this paper, we have revisited the problems of disturbance attenuation and almost disturbance decoupling for nonlinear systems and showed how a recently developed structural decomposition of multiple input multiple output systems and the new backstepping design procedures it motivates can lead to the solution of the these two problems for a larger class of systems.

REFERENCES


