Abstract—This paper presents the pedagogy of digital control design and experiment for a magnetic bearing system as part of a graduate level digital control class. The system identification verifies the system model structure, where the unstable MIMO system is kinematically de-coupled into SISO systems with model uncertainty bounds characterized for robustness analysis. Low order SISO plant models are used for control design while a high order MIMO model is used for verification before the implementation. Following a simple lead compensator, model based control design, analysis, verification, and implementation include state estimator feedback control and its augmentation with integrator and oscillator internal models; all stabilizing control and minimum variance control; approximate plant inversion for feedforward tracking and repetitive control.

I. INTRODUCTION

Graduate students majored in control oftentimes do not gain from formal classes the experience of conducting the entire cycle of control system modeling, design, and implementation and making the connection to the control theory they learned. First year graduate control classes typically cover state space linear system and control theory, in addition to the classical control theory covered in the undergraduate level. Based on such background a class aims at the complete control design and implementation cycle would be beneficial to the pedagogical learning for course work only M.S. level students before wrapping up for degrees as well as thesis option M.S. and Ph. D. students before engaging in research.

The physical system used for such a class should be related to typical text book example physical systems, such as double integrator for a motor or a oscillator for a mechanical structure [1] so that theory and methods can be related to. On the other hand it should be somewhat different so that scrutiny in modeling and analysis is required in order to apply the theory. The system should also be reasonably challenging, demonstrating the necessity of model based design that would otherwise be improbable or difficult if performed by simple control parameter tuning.

Possible examples that are common in text books [1] are the inverted pendulum and electromagnetically suspended steel ball, both of which are open loop unstable SISO systems. In addition to being open loop unstable, we look for a system that can be justified for the need of the control approach covered in the first year graduate classes.

A magnetic bearing system, which is an MIMO open loop unstable system, appears to be an appropriate candidate for this purpose.

This paper presents the use of a magnetic bearing system for the aforementioned purpose. The MBC500 system available from Magnetic Moments Inc. now doing business as Launch Point Technologies Inc. was developed by Professor Brad Paden at University of California Santa Barbara. A number of teaching resources have been available for the modeling and control of MBC 500 [2], [3], [4], [5], so are research literature based on the same systems [6], [7]. The specific one used by this paper is a special high-speed version of MBC 500, which has a shorter shaft length than the standard version.

The remainder of this paper is organized as follows: Section 2 describes the magnetic bearing system. Section 3 presents the analytical modeling of magnetic bearing system. Section 4 shows the system identification from frequency response data. Section 5 presents the studied digital control algorithms with both simulation and experimental results. With concluding remarks in section 6.

II. DESCRIPTION AND MODELING OF THE MBC500 MAGNETIC BEARING SYSTEM

The MBC500 magnetic bearing system [3] consists of two active radial magnetic bearings which support a rotor, as shown in Figure 1. The rotor shaft is actively suspended in the radial directions at the shaft ends by magnetic forces. The system is equipped with 4 hall effect sensors and employs 4 internal linear current amplifiers. Four internal analog compensators are available to stabilize the system. In this paper, the analog compensators are replaced by digital controller implemented by National Instruments LabVIEW Real-Time and/or Mathworks xPC Targets. Both systems use the processors in dedicated target computers for realizing digital control.

In order to get an intuitive understanding of the Magnetic bearing system, a model is derived from rigid body dynamics [4]. For simplicity, assume that motion in the $x$ (vertical) and $y$ (horizontal) directions are identical and independent. As seen in Figure 2 the magnetic bearing applies forces $F_1$ and $F_2$ at $x_1$ and $x_2$ respectively. Positions $x_1$ and $x_2$ are measured to be a distance of $l_1$ from each end of the shaft. The sensors measure at positions $X_1$ and $X_2$, which are at a distance of $l_2$ from each end of the shaft.

The differential equations governing the rigid body motion are obtained by summing the forces and moments around the
The hall effect sensors can be linearized as
\[ V_{x,j} = k_x X_j \]  

Because the model was derived from rigid body dynamics, we know system states should be able to be split into translational and rotational modes allowing a single 2-input 2-output system to be split into 2 simpler single-input single-output systems. \( \bar{x} = [x_0, x_0, \dot{\theta}, \dot{\theta}, i_1, i_2]^T \).

Define a new state transformation \( x = [x_0, x_0, (i_1 + i_2), \dot{\theta}, \dot{\theta}, (i_1 - i_2)] \), and use a simple transformation on the inputs and outputs for decoupling purpose: \( u = TV, y = V_T T^{-1} \).

\[
T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{bmatrix}, \quad T^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{bmatrix}
\]  

which can be described as two independent SISO systems.

### III. System Identification and Decoupling

The analytical model provides the model structure of AMB system. Since it is unstable according to the analytic modeling, closed-loop system identification technique [8] is performed.

\[
\begin{align*}
\dot{x} &= \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \\
\end{align*}
\]

Experimental frequency response for the system is obtained in the following manner. The entire magnetic bearing system then has 4 inputs and 4 outputs, and we split it into 16 single input single output (SISO) channels. For each channel, we use a Hewlett Packard 3562A Dynamic Signal Analyzer to collect experimental data. Specifically, the signal analyzer’s swept sine function is used to obtain the frequency response of each input-output channel. In Figure 3, we have

\[
\begin{align*}
T_{yr}(s) &= (I + CG)^{-1}G \\
T_{ur}(s) &= (I + CG)^{-1}
\end{align*}
\]

Note that \( r_p, u_p \) and \( y_p \) are \( 4 \times 1 \) vectors and \( C \) and \( G \) are \( 4 \times 4 \) transfer function matrices. By taking inverse of \( T_{ur}(s) \)
and then multiplying with \( T_{yr}(s) \), we may get the open-loop model

\[
G(s) = T^{-1}_{ur} T_{yr}
\]  

(11)

After the above procedures performed, the experimental frequency responses for AMB are shown in Figure 4. In

\[
\begin{align*}
G_{x1,x1}(s) & = \frac{1}{s^2+10s+100} \\
G_{x1,x2}(s) & = \frac{1}{s^2+10s+100} \\
G_{x1,y1}(s) & = \frac{1}{s^2+10s+100} \\
G_{x1,y2}(s) & = \frac{1}{s^2+10s+100} \\
G_{y1,x1}(s) & = \frac{1}{s^2+10s+100} \\
G_{y1,x2}(s) & = \frac{1}{s^2+10s+100} \\
G_{y1,y1}(s) & = \frac{1}{s^2+10s+100} \\
G_{y1,y2}(s) & = \frac{1}{s^2+10s+100} \\
G_{y2,x1}(s) & = \frac{1}{s^2+10s+100} \\
G_{y2,x2}(s) & = \frac{1}{s^2+10s+100} \\
G_{y2,y1}(s) & = \frac{1}{s^2+10s+100} \\
G_{y2,y2}(s) & = \frac{1}{s^2+10s+100} \\
\end{align*}
\]

Fig. 4. Experimental Frequency Responses (only magnitude ratios in dB are shown)

Figure 4, the off block diagonal terms are -40dB lower than the block diagonal terms. It agrees with analytical model that x-axis and y-axis are independent. Then, the \( 4 \times 4 \) transfer matrix can be approximated by two more manageable \( 2 \times 2 \) transfer matrices: one for x-axis and the other is for y-axis.

Let us define the upper left \( 2 \times 2 \) block of Figure 4 as \( G_{raw,x} \) and lower right \( 2 \times 2 \) block as \( G_{raw,y} \). Both \( G_{raw,x} \) and \( G_{raw,y} \) are coupled systems, the MIMO system system identification is performed using MATLAB System Identification Toolbox [9]. The subspace identification method, \( N4SID \) function is used and a 10th order MIMO transfer matrix is derived for each axis. This high-order MIMO model is used to represent the “actual system” for verification by simulation, which reduces the possibility of the damage to the hardware by implementation of an unstable or unacceptable control system.

However, the high order MIMO model is not convenient for control design. The system can be decoupled with the aforementioned transition matrix \( T \) in the transformed domain

\[
\begin{align*}
P_{raw,x} &= TG_{raw,x} T^{-1} \\
P_{raw,y} &= TG_{raw,y} T^{-1}
\end{align*}
\]  

(12) (13)

Figure 5(a) and 5(b) show the experimental frequency response data after decoupling for x-axis and y-axis, respectively. It is seen that the coupling between translation and rotation are reduced by about 20dB for each axis and become negligible. Hence, the original MIMO system is decoupled into 4 SISO subsystem.

Finally, each of the four decoupled frequency responses

\[
P_{raw} \quad \text{is curve fit by a 3rd order transfer function using the} \quad \text{invfreqs function in Matlab. The transfer functions we get} \quad \text{are considered as simplified low-order models for magnetic bearing system. Since the lower frequency range data is more} \quad \text{accurate, weighting functions are used. Figure 6 are curve fitting results for each SISO system, which agrees with the} \quad \text{structure of the analytical model having three states and an} \quad \text{unstable real pole.}
\]

\[
\begin{align*}
P_{x} & \quad \text{Fig. 5. Experimental Frequency Response Data after decoupling (only magnitude ratios in dB are shown)} \\
P_{y} & \quad \text{Fig. 6. Experimental Frequency Response Curve Fit}
\end{align*}
\]

The differences between the data and curve fitted model are considered as \textit{model uncertainties}. The multiplicative error upper bounds, which are referred as \( W_r(s) \), are important for robust stability analysis and may be obtained as shown in Figure 7.

\[
\begin{align*}
P_{xt} & \quad \text{Fig. 7. Curve Fit Errors and Bounds}
\end{align*}
\]
plant model is also constructed by balanced realization and model reduction of the more accurate high order plant model.

IV. CONTROL DESIGN

A series of direct digital control design methods, which are based on the discrete plant model with zero order hold equivalence, is introduced in pedagogical order. The cycle includes control design and time and frequency domain performance and robustness analysis based on the low order plant model, simulation verification based on the high order MIMO model, and implementation and experimental data analysis. The robustness check includes finding phase, gain, and vector margins, and checking the inequality $|TW_r| \leq 1$, where $T$ is the complementary sensitivity and $W_r$ is the multiplicative uncertainty bound.

A. Lead Lag Compensator

A simple lead lag compensator of the form

$$C = k \frac{z - \alpha}{z - \beta}$$

are designed, typically by the classical root locus method.

![Sensitivity Function](image1)

![Robustness Check](image2)

Fig. 8. Lead Lag Sensitivity and Robustness Checks

Figure 8(a) and 8(b) show one such design that give step response with large steady state error in Figure 9(a) and 9(b). The experimental response agrees with simulated verification except for a constant offset. From now, only translational responses will be shown, since rotational responses are similar.

![Translational](image3)

![Rotational](image4)

Fig. 9. Lead Lag Step Responses

B. State Space Control with Internal Model

In this design, a full order Luenberger observer is designed to estimate the states and a state feedback control is designed for both the low order MIMO and de-coupled SISO plant models individually. The design for MIMO plant model does not exploit the knowledge of de-coupling but demonstrates the advantage of state space approach in this case. To check robust stability, the loop gain of closed-loop system is then calculated to determine the stability margins. The purpose of this calculation is two fold. It serves to make a connection between the state space design and frequency domain measures from classical control. It is also used to illustrate by examples that state estimator feedback tends to reduce the generous stability margins that full state feedback control usually enjoys and robust stability is verified.

In order to eliminate the steady state error, an integrator can be added and the states are augmented accordingly as [1]. The augmented system can be described as

$$\begin{bmatrix} x(k+1) \\ y(k) \end{bmatrix} = \begin{bmatrix} A & 0 \\ C & 1 \end{bmatrix} \begin{bmatrix} x(k) \\ y(k) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} r(k) + \begin{bmatrix} 0 \\ -1 \end{bmatrix} e(k)$$

as can be seen in Figure 10.

![Block Diagram](image5)

Fig. 10. Block diagram of state estimator with internal model

The robust stability is confirmed by comparing with $1/W_r$. As seen in Figure 11, the step response matches the simulation data quite well. Also all steady state error is eliminated. Students are also assigned to expand the integrator internal model to also include an oscillator, in order to track or reject a single frequency harmonic signal.

![Simulation Results](image6)

Fig. 11. Simulation and experimental results of state estimator with internal model

C. Youla-Parameterization and $H_2$ Control

Youla-parameterization (or $Q$-parameterization) [10], [11] provides an affine description of the set of all achievable closed loop transfer functions as function of an arbitrary stable proper transfer function. Using Youla-parameterization, feedback control design can be converted to $Q$ filter design. Suppose there exists a controller $C_0 = U_0/V_0$, which stabilizes the system. Let $P = B/A$, then the whole space of stabiling controller can be expressed by the free parameter $Q$, as shown in Figure 12.
The closed-loop sensitivity and complementary sensitivity function can be also be expressed by $Q$:

$$
S = \frac{1}{1 + CG} = AV_0 - BAQ = t_1 - t_2Q \quad (16)
$$

$$
T = 1 - S = (1 - AV_0) + BAQ \quad (17)
$$

where $t_1$ and $t_2$ are stable filters. Equation (17) shows that the feedback control design has become a model matching problem.

The general model matching problem [12], [13] is to find stable $F$, such that

$$
\min_{F \in RH_\infty} \|M - GF\|_2 \quad (18)
$$

Then the goal becomes to minimize $H_2$ norm of sensitivity function, shown in Equation (18), that is,

$$
J_2 = \inf_{F \in RH_\infty} \|M - GF\|_2 \quad (19)
$$

This problem can be solved by spectral factorization, and the solutions are shown as follows

$$
J_{2,\text{min}} = \|(G_1^* M)_-\|_2 \quad (20)
$$

$$
F_{\text{opt}} = G_0^{-1}(G_1^* M)_+ \quad (21)
$$

where $G = G_1G_0$, where $G_1$ represents the non-minimum phase part of system and $G_0$ is the minimum phase part. $(G_1^* M)_+$ and $(G_1^* M)_-$ are the causal and anti-causal parts of system, respectively.

In general, the weighting function $W_p$ may also be included, i.e.,

$$
\min_{F \in RH_\infty} \| (M - GF)W_p \|_2 \quad (22)
$$

In the design, more emphasis is put on low frequency range. Notice that since there is no integrator embedded, the steady state error will not be zero, which is confirmed by simulation and experimental results in Figure 13.

D. Zero-Phase-Error-Tracking-Control (ZPETC) based feedforward control

The model matching problem introduced in the feedback control context can be easily cast for feedforward tracking control design problem, where the plant dynamics are to be inverted. Non-causal approximate stable inversion of the plant can be formulated and solved as 19 or 22 [13]. Another simple way to parameterize an approximate stable inversion is the zero-phase-error-tracking feed-forward control (ZPETC) [14]. Given the stable plant $P = B/A$, factor $B$ into

$$
B = B^+B^- \quad (23)
$$

where $B^+$, $B^-$ represent stable and unstable zeros, respectively. The ZPETC causal compensator $F_{ZPC}$ is defined as follows,

$$
F_{ZPC} = \gamma z^{-(n_u+d)} \frac{A(z^{-1})B^- (z)}{B^+(z^{-1})} \quad (24)
$$

where $\gamma$ is a positive real design parameter usually selected to make unity DC gain, $n_u$ is the number of unstable zeros.

A preview (look ahead) of the reference trajectory by $\nu+d$ steps cancels the delay and make the zero-phase input-output relation for all frequencies. An accurate model would render decent feedforward tracking performance. The simulation and experimental results shown in Figure 14 confirm this.

E. Repetitive Control

Repetitive control [15], based on the Internal Model Principle (IMP) [16], deals with periodic exogenous signals to exactly track or reject periodic signals asymptotically. A prototype discrete-time repetitive controller design was proposed by [17] using zero phase error tracking compensation (ZPETC) technique. To improve the robust stability, zero-phase low-pass filter $Q(z)$ was introduced in the prototype repetitive control scheme [18]. The tradeoff between robust-
ness stability and disturbance rejection performance is made in the design of $Q$ filter.

Figure 15 shows repetitive control with ZPETC compensator, where $G_{cl}$ is the stabilized plant, $F_{ZPETC}$ is the zero-phase compensator, which is illustrated in the previous section. The internal model consists of $Q(z^{-1})$, $z^{-(d+n_u)}$ and $z^{-N+d+n_u+n_s}$, where $Q(z^{-1})$ is the zero-phase low-pass filter.

![Fig. 15. Block Diagram of ZPETC type Repetitive Control](image)

The sensitivity function of ZPETC type repetitive control system is shown in Figure 16, where $N = 400$. The repetitive control system can track 10Hz periodic signal and its harmonics. The simulation and experimental results are shown in Figure 17. Compared with feedforward control shown in Figure 14, the tracking error has been reduced further.

![Fig. 16. Sensitivity function of ZPETC based repetitive control](image)

![Fig. 17. Simulation and experimental results for ZPETC based repetitive control: the simulated step response (dashed line) are very close to the experimental result (solid line)](image)

V. CONCLUSION

The magnetic bearing system provided a complex but manageable and rewarding testbed for digital control education. By analyzing the system from first principles, student were able to decompose the complex MIMO system into multiple simpler SISO systems using only a linear transform. Using that information, the students were able to identify the systems and begin to design control. Once comfortable with the hardware and software, the students progressed to more complex control algorithms. With the basic tools, students were able to tackle state-space feedback with internal models and Youla parameterization of stabilizing control is then introduced to facilitate the $H_2$ optimal control solved as a model matching problem. Connecting to the model matching problem, approximate stable inversion compensation is introduced and applied to the zero phase error feedforward tracking and repetitive control with very effective experimental results. With the issues of coupling and non-linearities not fully addressed, the test bed also allows for the students the opportunity for further exploration in their term projects.

REFERENCES