Linear Time-Varying Tracking Control With Application to Unmanned Aerial Vehicles

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Abstract—This paper presents a novel control algorithm for tracking of nonlinear dynamic systems. The tracking problem leads to linearized time-varying error equations. Hence new eigenvalue notions are introduced for linear time varying systems, and a PD eigenstructure assignment scheme is proposed for linear time-varying structure via a differential Sylvester equation. Also shown is that closed-loop systems could be stabilized by assigning PD-eigenvalues appropriately, and a desired performance could be obtained by assigning the PD-eigenvectors according to the design specifications. The present control algorithm has an internal framework for testing the controllability of the time varying system via Lyapunov transform. The algorithm proposed is very general, and in principle applicable to systems of any order of complexity with any degree or kind of nonlinearity. An Unmanned Aerial Vehicle flight control application for a real aircraft is presented to validate the proposed algorithm.

INTRODUCTION

We are primarily concerned with tracking of nonlinear dynamic systems; however, as is shown in the sequel, the resulting linearized error equations become time-varying. Hence, the discussion and literature review here is principally devoted to control of linear time-varying (LTV) systems.

There are many well developed techniques for dealing with the response of linear time invariant systems (LTI), such as Laplace and Fourier transforms. However, these techniques are not strictly valid for time-varying systems. A system undergoing slow time variation in comparison to its time constants can usually be considered to be time invariant. An example of this is the aging and wear of electronic and mechanical components, which happens on a scale of years, and thus does not result in any behavior qualitatively different from that observed in a time invariant system on a day-to-day basis. Unlike LTI systems, linear time variant systems may behave more like nonlinear systems, if the system changes quickly or is more dynamic in nature.

A large number of systems arising in practice are time-varying. Time variation is a result of system parameters changing as a function of time, such as aerodynamic coefficients of a high speed aircraft, circuit parameters in electronic circuits, mechanical parameters in machinery, and diffusion coefficients in chemical processes. In general, although in principle, all systems are time-varying, we characterize systems as time-varying if the parameter variation is happening on time scales close to that of the system dynamics. Time-variation may also be a result of linearizing a nonlinear system about a family of operating points and/or about a time-varying operating point.

Time-variation is often ignored in dealing with systems in practice. However, due to the desire to achieve better accuracy and quality in a wide range of applications, there is increasing interest to include the effects of time-variation when analyzing a system, or when designing controllers and observers. In particular, one can consider parametric models such as an input/output differential equation with time-varying coefficients [1].

In the study of the flight control of ordinary aircraft, for example, the pertinent equations of motion have coefficients which depend on flight speed. Previously, it has been the conventional practice to consider the speed as a constant, resulting in linear differential equations with constant coefficients. The relatively small acceleration experienced by subsonic aircraft rendered the assumption a reasonable one in that good results were obtained. With the much increased acceleration and velocities that characterize modern supersonic aircraft and missiles however, the parameters depending on flight velocities change at a significantly rapid rate. Further, the high rate of fuel consumption causes the mass, center of gravity, and moment of inertia of a vehicle to alter to a significant degree during the characteristic response time of the controlled motions. In addition, variations of flight conditions in rapid ascent through the atmosphere introduce time-varying parameters such that the responses are characterized by differential equations with variable coefficients.

Eigenvalues and eigenvectors play very important roles in the analysis of linear time-invariant dynamical systems. The stability and dynamic behavior of a linear time-invariant multivariable system is governed by the eigenstructure of the system. Hence the problem of simultaneous assignment of eigenvalues and eigenvectors is of great importance in control theory and applications. Generally, the speed of a response is determined by assigned eigenvalues whereas the shape of the response is determined by the assigned eigenvectors. Eigenstructure assignment is an excellent method for incorporating classical specifications on damping, settling time, and mode decoupling into a modern multivariable control framework [2], [3], [4], [5].

Many researchers have attempted to generalise the conventional notions of eigenvalues and eigenvectors for linear time-invariant system to linear time-varying systems. However, it is well known that the conventional eigenvalues of a linear time-varying system matrix $A(t)$ do not determine the stability of linear time-varying systems. Wu [6] proposed the extended-eigenvalue and extended eigenvector notion; they
are defined as a scalar function $\gamma(t)$ and a time varying vector function $u(t)$. While this extension is a great leap from the generally invalid frozen-time extension, it is not very useful, because, by the well-known existence theorem for solution of linear time-varying systems $\dot{x}(t) = A(t)x(t)$, any scalar function $\gamma(t)$ is not uniquely determined in linear time-varying systems. Therefore, the very essence of being “eigen” is lost in that attempt to extend eigenvalue concepts.

Kamen [7] developed notions on poles and zeros for linear time-varying systems, and Zhu and Morales [8] introduced a notion of co-eigenvalues. Zhu and Johnson [9] developed a new time-dependent eigenvalue theory and an associated set of matrix canonical forms for a matrix over a differential ring using the differential operator factorization developed by Cauchy and Floquet. The extended pole placement control for linear time varying plants was developed by Tsakalis and Ioannou [10].


**TRACKING CONTROL ALGORITHM**

Next we develop a novel control algorithm for tracking of nonlinear dynamic systems. We introduce new eigenvalue notions for linear time varying systems, and propose a a PD-eigenstructure assignment scheme for linear time-varying structure via a differential Sylvester equation. The present scheme of tracking control and error minimization is based on eigenvalue notions proposed by [11], [14]. We show that closed-loop systems could be stabilized by assigning PD-eigenvalues appropriately, and a desired performance could be obtained by assigning the PD-eigenvectors according to the design specifications.

In general, a dynamic system with state variables, $x$, and control input, $u$ is given by a set of $N$ equations,

$$\dot{x} = f(x, u)$$  

(1)

We would like the system to follow a specific trajectory, expressed as $x_d(t)$, where, $x_d(t)$ could be an arbitrary function of time. The essential problem is to determine the control inputs (rudder, ailerons, for an aircraft, for example) such that errors go to zero. The errors are clearly

$$\dot{x}_i = x_{di} - x_i, \quad \dot{u}_j = u_{dj} - u_j$$  

(2)

Substituting into the dynamic system, and after linearizing with the assumption that the errors stay small, we get the error equations:

$$\dot{\bar{x}} = A(t)\bar{x} + B(t)\bar{u}$$  

(3)

The objective is to get the closed loop system to behave well as determined from certain performance criteria. Note that these equations are linear but time-varying. The original problem of tracking control of the nonlinear system has now been reduced to the control of a linear time-varying system.

It is important to note that the original state variable, $x(t)$ has not been assumed to be small.

**Linear Time-Invariant Systems**

The above control problem is of course a well defined problem for time invariant systems, although there is still some research being performed in this area. In the case of time-varying systems (like we have here) however, the problem is not as well understood, and is the subject of current research. In order to explain our algorithm better, first we outline the method used for standard time-invariant systems.

Given a set of (constant coefficient) state equations,

$$\dot{x}(t) = Ax(t) + Bu(t)$$  

(4)

we seek a constant controller matrix $K$.

$$u(t) = Kx(t)$$  

(5)

This leads to

$$\dot{x}(t) = (A + BK)x(t)$$  

(6)

The essential problem is to choose the elements of $K$ in such a fashion that the performance is satisfactory. This is often accomplished by placing the poles (eigenvalues) of the close loop system at specific locations. To do this we pose the eigenvalue problem:

$$(A + BK)\phi_i = \lambda_i \phi_i$$  

(7)

where, $\phi_i$ is the right eigenvector. Next, we introduce an auxiliary parameter vector

$$h_i = K \phi_i$$  

(8)

which leads to

$$(A - \lambda_i I)\phi_i = -Bh_i$$  

(9)

Assembling all of these equations leads to the Sylvester Equation.

$$A\Phi - \Phi A = -BH$$  

(10)

Hence, the procedure can be summarized as follows.

- Choose eigenvalues $\lambda_i$
- Choose parameter matrix, $H$
- Solve for $\Phi$
- Solve for $K$ from $K\Phi = H$

This will lead to the closed loop system having the eigenvalues that we required, and hence can be expected to exhibit the desired behavior.

**Linear Time-Varying Systems**

In the case of time varying systems however, there has been very limited research partly because of the complexity of the problem. Consider a linear time-varying system,

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$  

(11)
We seek a time-varying controller matrix $K(t)$.

$$u(t) = K(t)x(t) \quad (12)$$

This leads to the closed loop system,

$$\dot{x}(t) = (A(t) + B(t)K(t))x(t) \quad (13)$$

Now, we carry out a Lyapunov transformation to convert the equations to canonical form. The problem of transforming a linear time-varying system to the companion canonical form is considered in [15]. After converting $A(t)$ into canonical form we get a companion matrix $A_c(t)$. Now we define the corresponding PD-eigenvalue problem for the system as follows:

$$(A_c(t) + \tilde{B}(t)\tilde{K})p_i(t) - \rho_i(t)p_i(t) = \rho_i(t) : \text{right} \quad (14)$$

$$(A_c(t) + \tilde{B}(t)\tilde{K})q_i(t) - \rho_i(t)q_i^T(t) = -\tilde{q}_i^T(t) : \text{left} \quad (15)$$

where $\rho_i(t)$ and $q_i(t)$ are the right and left PD-eigenvectors, respectively, corresponding to the PD-eigenvalue $\rho_i(t)$.

The objective is to find the time varying control matrix $K(t)$ in such a way that the closed loop PD-eigenvalues are archived exactly and the desired right PD-eigenvectors are assigned to the best possible set of PD-eigenvectors, at least in the least square sense.

If we define a parameter vector,

$$h_i(t) = \tilde{K}(t)p_i(t) \quad (16)$$

then the matrix form of the differential Sylvester equation can be written as

$$A_c(t)P(t) - P(t)\Upsilon(t) + \tilde{B}(t)H(t) = \dot{P}(t) \quad (17)$$

where $P(t) = [p_1(t), p_2(t), \ldots, p_N(t)]$, $\Upsilon(t) = \text{diag}[\rho_1(t), \rho_2(t), \ldots, \rho_N(t)]$, and $H(t) = [h_1(t), h_2(t), \ldots, h_N(t)]$.

Since an arbitrary selection parameter vector matrix $H(t)$ and desired PD-eigenvalues matrix $\Upsilon(t)$ is not appropriate, because it does not usually generate an attractive set of closed-loop eigenvectors, a target set of desired closed-loop right PD-eigenvectors can be selected as

$$P_d(t) = [p_{d1}, p_{d2}, \ldots, p_{dN}] \quad (18)$$

After choosing them, we calculate an auxiliary matrix, $H$.

$$\dot{H}(t) = -B^+(t)[A_c(t)P_a(t) - P_a(t)\Upsilon_a(t) - \dot{P}_a(t)] \quad (19)$$

where, $B^+$ denotes the pseudoinverse.

Upon substituting this solution for the $H(t)$ matrix and resolving (Eq. 17) for admissible right PD-eigenvectors matrix $P_a(t)$, we will find $P_a(t) \neq P_d(t)$. The resulting $P_a(t)$ matrix is as near $P_d(t)$ as possible (in the least square sense), and exactly satisfies the following equation.

$$A_c(t)P_a(t) - P_a(t)\Upsilon_a(t) + \tilde{B}\dot{H} = \dot{P}_a(t) \quad (20)$$

the vectorized form of which is

$$\dot{P}_a = (I \otimes A_c(t)\tilde{P}_a(t)) + (\Upsilon_d(t) \otimes (-I))\dot{P}_a + \tilde{B}\dot{H} \quad (21)$$

where, $\otimes$ represents the Kronecker product.

Finally, the time-varying gain matrix is given by

$$\tilde{K}(t) = \tilde{H}(t)P_a^{-1} \quad (22)$$

Now, we have the control law (for the transformed equations):

$$\tilde{u}(t) = \tilde{K}(t)\tilde{x}(t) \quad (23)$$

Calculating $K(t)$ gives the closed loop system with a satisfactory response with the error dynamics being asymptotically stable.

**UAV FLIGHT CONTROL APPLICATION**

This section presents an application of PD-eigenstructure assignment and linear time-varying control for the error equations of the UAV tracking model. An aerial vehicle has six rigid body degrees of freedom commonly called surge ($u$), sway ($v$), heave ($w$), roll ($p$), pitch ($q$) and yaw ($r$). Its motion can be defined in body-fixed coordinates or earth-fixed coordinates, and the angular orientations are defined using a sequence of Euler angles. The vehicle experiences forces and moments due to gravity, air, propulsion and actuation which will have components about all the three axes. These forces are typically nonlinear functions of the motion variables; in addition, there is always some uncertainty associated with their functional form and parametric values. The whole issue of aircraft models can be quite complex; in fact, determining the right model to use for the control process can be quite tricky. Simpler models can be derived initially for the tracking problem by including only few dynamic variables when dealing with complicated control issues. The true response will of course be closer to the full six DOF nonlinear dynamic model, and cannot be predicted accurately by simpler models; hence, it should eventually be considered.

A 3 DOF non-linear dynamic model for the Procerus unmanned aerial vehicle is as follows and includes only surge, sway and yaw, and sets the other equations to zero. Procerus UAV is a popular research vehicle [16].

$$\dot{u} = rv + F_x/m \quad (24)$$

$$\dot{v} = -ru + F_y/m \quad (25)$$

$$\dot{r} = R_4l + R_7n \quad (26)$$

$$F_x = -1/2pV^2SC_{D0} \quad (27)$$

$$F_y = 1/2pV^2SC_{L3} \tan^{-1}(v/u) \quad (28)$$

$$l = 1/2pV^2SC_{L3} \tan^{-1}(v/u) + C_{l\delta}br/(2V^2) + C_{l\delta\delta} \delta_a \quad (29)$$

$$n = 1/2pV^2SC_{n3} \tan^{-1}(v/u) + C_{n\delta}br/(2V^2) + C_{n\delta\delta} \delta_a \quad (30)$$

where,

$$R_4 = J_{xy}/R \quad (31)$$

$$R_7 = ((J_x - J_y)J_x + J_{zz}^2)/R \quad (32)$$

$$R = J_xJ_z - J_{zz}^2 \quad (33)$$

The Procerus UAV model has only ailerons $\delta_a(t)$ as the
control input.

We define a state vector,
\[
\dot{x} = \begin{bmatrix} \dot{u} & \dot{v} & \dot{r} \end{bmatrix}^T 
\]
(34)
\[
x = \begin{bmatrix} u & v & r \end{bmatrix}^T
\]
(35)

Next, we specify the desired functions of time: \(x_d(t)\) and \(\delta_{ad}(t)\).

\[
\dot{x}_d = \begin{bmatrix} \dot{u}_d & \dot{v}_d & \dot{r}_d \end{bmatrix}^T
\]
(36)
\[
x_d = \begin{bmatrix} u_d & v_d & r_d \end{bmatrix}^T
\]
(37)

The error equations are then defined by
\[
\bar{x}(t) = x_d(t) - x(t)
\]
(38)

After linearizing the above equations, the error equations can be written as linear time-varying equations as shown below,
\[
\bar{x} = A\bar{x} + b\dot{u}
\]
(39)

where, \(x_d(t)\) is a function of desired path, and \(u(t)\) is the aileron angle (control input). Now, the control algorithm described in the previous section will be applied to the linear time-varying error equations.

We apply the following procedure to the above error equations.

- Take Lyapunov transformation to transform a linear time-varying system into the companion canonical form.
- Choose the desired closed-loop PD-spectrum \(\gamma(t)\) and the desired right PD-modal matrix \(P_d(t)\) for the transformed system.
- Calculate the parameter matrix \(\bar{H}(t)\).
- Solve the differential Sylvester equation with the parameter matrix \(\bar{H}(t)\) from the above step to get the achievable right PD-modal matrix \(\bar{P}_d(t)\).
- Calculate the feedback gain matrix \(\bar{K}(t)\).
- Since \(\bar{K}(t)\) is obtained for the transformed system, we take inverse Lyapunov transformation to calculate the feedback gain matrix \(K(t)\) for the original system.

**RESULTS**

As mentioned earlier, the control algorithm derived above is very general, and is applicable to arbitrary nonlinear systems trying to follow arbitrary paths. The results shown here are for an arbitrary sinusoidal input. First, we choose the desired input trajectory using \(\delta_{ad}(t) = C \sin(t)\), where, \(C\) is a constant parameter.

We chose the following for the eigenvalues and the PD-modal matrix as discussed earlier after involved experimentation.

\[
\gamma_d(t) = \begin{bmatrix} -e^{(\alpha t)} - s_1 & -e^{(\alpha t)} - s_2 & -e^{(\alpha t)} - s_3 \end{bmatrix}
\]
(40)
\[
P_d(t) = \begin{bmatrix} -\beta & 0 & 0 \\ 0 & -\beta & 0 \\ 0 & 0 & -\beta \end{bmatrix}
\]
(41)

where, \(\alpha, s_i, \text{ and } \beta_i\) are all positive and are input parameters in the controller design process. The set of equations for \(P_a\), were solved along with the closed loop linear error equations numerically. This results in a stable gain matrix, \(K(t)\). Note again that the gain matrix itself if a function of time. It is likely that different choices of the eigenvalues and PD eigenvectors will result in varying degrees of response. The actual paths plotted in figures are obtained from subtracting the errors from the desired paths. The desired paths are assumed for the aircraft based on the application it is performing in a particular situation.

Only some sample results for the Procerus UAV model have been reported here to conserve space. The results shown here refer to an arbitrary sinusoidal aileron input. Fig. 1, Fig. 2 and Fig. 3 are errors for surge velocity, sway velocity and yaw rate respectively. It is clear from the plots that the errors are very small and asymptotically stable. The actual and desired paths are shown in Fig. 4, Fig. 5 and Fig. 6 for surge velocity, sway velocity and yaw rate respectively. The resulting plots for desired and actual paths clearly show that the actual path is very close to the desired path. And it is also observed that, even when the desired paths are very fast-changing or very dynamic in nature, the controller is able to track the desired path very accurately. The oscillations evident here in the errors (even though they are very small)
may not be desirable, and we are currently in the process of implementing a slightly modified controller to reduce them. The stability of the controlled system can be proved and is omitted here for lack of space.

CONCLUSION

In this paper, we presented new eigenvalue notions for linear time-varying systems, and developed further a PD-eigenstructure assignment scheme for tracking control of linear time-varying systems via a differential Sylvester equation. We showed that closed-loop systems could be stabilized by assigning PD-eigenstructure appropriately, and a desired performance could be obtained by assigning the right PD-eigenvectors according to the design specifications. An UAV flight control example has confirmed the utility and accuracy of the proposed tracking control algorithm.

We are currently implementing the procedure on an experimental UAV. The algorithm proposed is very general, and in principle applicable to systems of any order of complexity with any degree or kind of nonlinearity. We are hence extending the algorithm for other unmanned applications.

APPENDIX

Data for UAV model: $J_x=0.1147, J_y=0.0576, J_z=0.1712, J_{xz}=0.0015, V^2=(u^2+v^2), \rho=1.2682, S=0.2589, b=1.4224$.

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REFERENCES


811