State estimation and output feedback stabilization of a class of upper-triangular systems using a homogeneous observer

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Abstract—This paper considers the problem of state estimation and output feedback control of a class of upper-triangular systems with unmeasurable states. Due to the presence of the high-order nonlinearities, most of the existing state estimation methods can only yield local results. In this paper, we will show that the newly developed homogeneous observers can be used to estimate the states of some nonlinear systems in larger regions than a small neighborhood of the origin. When the initial error is small, we can design an asymptotically convergent high-order observer for the upper-triangular system. The observer is also combined with the output feedback controller and Least Square Estimation method to stabilize the upper-triangular systems.

I. INTRODUCTION

In this paper, we are interested in designing state estimator for a class of upper-triangular nonlinear systems described by

\[
\dot{x}_1 = x_2 + f_1(x_3, \cdots, x_n) \\
\vdots \\
\dot{x}_{n-2} = x_{n-1} + f_{n-2}(x_n) \\
\dot{x}_{n-1} = x_n \\
\dot{x}_n = u \\
y = x_1
\]

(1.1)

where \( x = (x_1, \cdots, x_n)^T \in \mathbb{R}^n \), \( u \in \mathbb{R} \) are system states and control input, respectively. The nonlinear functions \( f_i(x_{i+2}, \cdots, x_n), \ i = 1, \cdots, n-2 \) are smooth. System (1.1) includes an important class of nonlinear systems known as upper-triangular or feed-forward systems. The state feedback stabilization of system (1.1) has been well-studied and a number of interesting results have been established using nested-saturation ([1],[2]) or forwarding method ([5],[11]).

Compared to the massive results achieved for lower-triangular systems, in the literature there are fewer results on the global stabilization of upper-triangular systems due to the difficulty caused by the upper-triangular nonlinearities. Among the very few existing results in the literature, the work [3] designed an output feedback stabilizer using the homogeneous domination approach, which has been proved to be globally and finite-time convergent. Later the paper [4] used an arbitrarily bounded controller which is accomplished via nested saturation linear control and dynamic compensator to stabilize the upper-triangular systems.

This paper aims to apply high-order homogeneous observer to upper-triangular system, and we will prove the homogeneous observer is asymptotically convergent to zero to the real systems when the initial error is small. Secondly, to get the required initial values, we introduce a new method based on the least square estimation (LSE) method ([13]), which can provide us unknown state estimations used as the initial values for the homogeneous observers. And we will also demonstrate that our combined design has the property of disturbance rejection.

The rest of the paper is organized as follows: In Section II, we introduce some useful lemmas and existing homogeneous design methods. In Section III, we first design a homogeneous observer which will be convergent when the initial estimations are close to the real states. Then this observer is further combined with the widely used least square estimation technique and a state feedback controller to stabilize a class of upper-triangular systems with polynomial nonlinearities. Our conclusion is included in Section IV.

II. PRELIMINARIES OF HOMOGENEOUS OBSERVERS

In this section, we introduce some useful definitions and lemmas from homogeneous system theory which will be constantly used in proving the main results.

Listed below are the definitions of homogeneous functions and homogeneous systems with weighted dilation.

A function \( f \) is said to be homogeneous if there exists \( (r_1, r_2, \cdots, r_n) \in ((0, +\infty))^n \) and \( \tau \in \mathbb{R} \) such that \( \forall x = (x_1, x_2, \cdots, x_n)^T \in \mathbb{R}^n \forall \varepsilon > 0, \) and

\[
f_i(\varepsilon^{r_1}x_1, \cdots, \varepsilon^n x_n) = \varepsilon^{r_i+\tau} f_i(x), \ i = 1, \cdots, n
\]

homogeneous domination theorem is based on the innovative idea of homogeneity and the nice properties of homogeneous systems (refer to References [6]-[9] for details).

Weighted homogeneity: For fixed coordinates \( (x_1, \cdots, x_n) \in \mathbb{R}^n \) and real numbers \( r_i > 0, \) for \( i = 1, 2, \cdots, n, \)

- the dilation \( \Delta_\varepsilon(x) = (\varepsilon^{r_1}x_1, \cdots, \varepsilon^n x_n), \forall \varepsilon > 0, \) with \( r_i \) being called as the weights of the coordinates.
• a function $V \in C(\mathbb{R}^n, \mathbb{R})$ is said to be homogeneous of degree $\tau$ if there is a real number $\tau \in \mathbb{R}$ such that
\[
\forall x \in \mathbb{R}^n \setminus \{0\}, \varepsilon > 0, V(\Delta_\varepsilon(x)) = \varepsilon^\tau V(x_1, x_2, \ldots, x_n)
\]
• a vector $V \in C(\mathbb{R}^n, \mathbb{R}^n)$ is said to be homogeneous of degree $\tau$ if there is a real number $\tau \in \mathbb{R}$ such that for $i = 1, 2, \ldots, n$
\[
\forall x \in \mathbb{R}^n \setminus \{0\}, \varepsilon > 0, f_1(\Delta_\varepsilon(x)) = \varepsilon^{r_i+1} V(x_1, x_2, \ldots, x_n)
\]
• a homogeneous $p$-norm is defined as
\[
\|x\|_{\Delta, p} = \left(\sum_{i=1}^{n} |x_i|^{\frac{p}{\tau_i}}\right)^{\frac{1}{p}}, \forall x \in \mathbb{R}^n
\]
for a constant $p \geq 1$. For the simplicity, in this paper, we choose $p = 2$, and write $\|x\|_{\Delta}$ for $\|x\|_{\Delta, 2}$.

**Lemma 2.1:** If the trivial solution $x = 0$ of the $\Delta_x$-homogeneous system
\[
\dot{x} = f(x), \quad f(0) = 0
\]
is globally stable, there exists a $\Delta_x$-homogeneous Lyapunov function $V$, which is positive definite and proper, such that
\[
\dot{V} = \frac{\partial V}{\partial x} f(x) < 0, \forall x \neq 0.
\]

**Lemma 2.2:** Given a dilation weight $\Delta = (\tau_1, \tau_2, \ldots, \tau_n)$, suppose $V_1(x)$ and $V_2(x)$ are respectively homogeneous functions of degree $\tau_1$ and $\tau_2$. Then $V_1(x)V_2(x)$ is also homogeneous with respect to the same dilation weight $\Delta$. Moreover, the homogeneous degree of $V_1(x)V_2(x)$ is $\tau_1 + \tau_2$.

**Lemma 2.3:** Assume $V: \mathbb{R}^n \to \mathbb{R}$ is a homogeneous function of degree $\tau$ with respect to the dilation weight $\Delta$. Then the following holds: (A) $\frac{\partial V}{\partial x}$ is homogeneous of degree $\tau - \tau_i$ with $\tau_i$ being the homogeneous weight of $x_i$. (B) There is a constant $c$ such that
\[
V(x) \leq c \|x\|_{\Delta}^\tau.
\]
Moreover, if $V(x)$ is positive definite, $\exists \|x\|_{\Delta} \leq V(x)$ for a constant $c > 0$. 

Next we will review an existing result on homogeneous observers.

The paper [10] solved the problem of global output feedback stabilization for a class of nonlinear systems whose nonlinearities are bounded by both low-order and high-order terms. The paper [10] also introduced a homogeneous observer estimating the high-order parts of unmeasurable states. In what follows, we list the high-order homogeneous observer for the following system:
\[
\dot{x}_1 = x_2, \ldots, \dot{x}_{n-1} = x_n, \quad \dot{x}_n = u, \quad y = x_1.
\]

The high-order observer for the former system is described in the form of:
\[
\dot{\hat{x}}_1 = \hat{x}_2 + d_1(x_1 - \hat{x}_1)^{r_2}
\]
\[
\dot{\hat{x}}_2 = \hat{x}_3 + d_2(x_1 - \hat{x}_1)^{r_3}
\]
\[
\vdots
\]
\[
\dot{\hat{x}}_{n-1} = \hat{x}_n + d_{n-1}(x_1 - \hat{x}_1)^{r_n}
\]
\[
\dot{\hat{x}}_n = u + d_n(x_1 - \hat{x}_1)^{r_{n+1}}
\]

where $r_1 = 1, r_{i+1} = r_i + 1$, and $d_i$ are appropriate constants for $r_1 > 0, i = 1, 2, \ldots, n$. Defining $e_i = x_i - \hat{x}_i, i = 1, 2, \ldots, n$, the error dynamics can be described as follows:
\[
\dot{e}_1 = e_2 - d_1(x_1 - \hat{x}_1)^{r_2}
\]
\[
\dot{e}_2 = e_3 - d_2(x_1 - \hat{x}_1)^{r_3}
\]
\[
\vdots
\]
\[
\dot{e}_{n-1} = e_n - d_{n-1}(x_1 - \hat{x}_1)^{r_n}
\]
\[
\dot{e}_n = -d_n(x_1 - \hat{x}_1)^{r_{n+1}}.
\]

It was also proved in [10] that the high-order error dynamic is also globally A.S. for appropriate constants $d_i$.

### III. HOMOGENEOUS OBSERVER FOR (1.1)

This section proposes a method to stabilize the upper-triangular systems by designing a homogeneous observer. We construct the following observer for system (1.1) based on the high-order homogeneous observer described in the former section:
\[
\dot{\hat{x}}_1 = \hat{x}_2 + f_1(\hat{x}_3, \hat{x}_4, \ldots, \hat{x}_n) + d_1(x_1 - \hat{x}_1)^{r_2}
\]
\[
\dot{\hat{x}}_2 = \hat{x}_3 + f_2(\hat{x}_4, \hat{x}_5, \ldots, \hat{x}_n) + d_2(x_1 - \hat{x}_1)^{r_3}
\]
\[
\vdots
\]
\[
\dot{\hat{x}}_{n-2} = \hat{x}_{n-1} + f_{n-2}(\hat{x}_n) + d_{n-2}(x_1 - \hat{x}_1)^{r_{n-1}}
\]
\[
\dot{\hat{x}}_{n-1} = \hat{x}_n + d_{n-1}(x_1 - \hat{x}_1)^{r_n}
\]
\[
\dot{\hat{x}}_n = u + d_n(x_1 - \hat{x}_1)^{r_{n+1}}.
\]

for $i = 1, 2, \ldots, n$, we define error $e_i = x_i - \hat{x}_i$, where $r_1 = 1, r_i > 0, r_{i+1} = r_i + 1$. Note that $r_{i+1}$ should be odd for keeping the sign of the error term, and $d_i$’s are appropriate constants. Thus the error dynamics can be described as following:
\[
\dot{e}_1 = e_2 - d_1 e_1^{r_2} + f_1(\hat{x}_3, \hat{x}_4, \ldots, \hat{x}_n) - f_1(\hat{x}_3, \hat{x}_4, \ldots, \hat{x}_n)
\]
\[
\dot{e}_2 = e_3 - d_2 e_1^{r_3} + f_2(\hat{x}_4, \hat{x}_5, \ldots, \hat{x}_n) - f_2(\hat{x}_4, \hat{x}_5, \ldots, \hat{x}_n)
\]
\[
\vdots
\]
\[
\dot{e}_{n-2} = e_{n-1} - d_{n-2} e_1^{r_{n-1} - 1} + f_{n-2}(\hat{x}_n) - f_{n-2}(\hat{x}_n)
\]
\[
\dot{e}_{n-1} = e_n - d_{n-1} e_1^{r_n}
\]
\[
\dot{e}_n = -d_n e_1^{r_{n+1}}.\]

#### Theorem 3.1: Suppose that $K$ is a compact, positively invariant set for the system (1.1). For the initial condition $\hat{x}(0)$ of equation (3.1) close to $x(0)$ in (1.1), the error dynamics (3.2) are asymptotically stable.

**Proof.** The error dynamics can be written as:
\[
\dot{\hat{e}} = \begin{pmatrix}
\dot{e}_1 \\
\dot{e}_2 \\
\vdots \\
\dot{e}_{n-2} \\
\dot{e}_{n-1} \\
\dot{e}_n
\end{pmatrix} = \begin{pmatrix}
e_2 - d_1 e_1^{r_2} \\
e_3 - d_2 e_1^{r_3} \\
\vdots \\
e_{n-2} - d_{n-2} e_1^{r_{n-1} - 1} \\
e_{n-1} - d_{n-1} e_1^{r_n} \\
-d_n e_1^{r_{n+1}}
\end{pmatrix}
\]
Combining (3.6)-(3.5), the derivative of the homogeneous Lyapunov function along the system (3.3) is

\[ \dot{V} \left( (3.3) \right) \leq -\omega(e) + \sum_{i=1}^{n-2} \left( \frac{r_{i+1}}{r_i} \right) e_i \left( e_i \left| e_i \right|^{r_i+1} + \cdots + \left| e_n \right|^{-r_n+1} \right) \]

On the other hand, by Lemma 2.2 the following term

\[ \left| \frac{\partial V}{\partial e_i} \right| \left( e_i \left| e_i \right|^{r_i+1} + \cdots + \left| e_n \right|^{-r_n+1} \right) \]

is homogeneity of degree \( k + \tau_1 \). Thus, by Lemma 2.3, we can find a constant \( c \) such that

\[ \left| \frac{\partial V}{\partial e_i} \right| \left( e_i \left| e_i \right|^{r_i+1} + \cdots + \left| e_n \right|^{-r_n+1} \right) \leq c \cdot \omega(e) \]

for the positive definite homogeneous term \( \omega(e) \).

Substituting (3.9) into (3.7) yields

\[ \dot{V} \left( (3.3) \right) \leq -\omega(e) \left[ 1 - c \sum_{i=1}^{n-2} H_i(x_{i+2}, \cdots, x_n) \left| e_i \left| e_i \right|^{r_i+1} + \cdots + \left| e_n \right|^{-r_n+1} \right) \right] \]

Since \( r_{n+1} > r_n > \cdots > r_2 > r_1 = 1 \), we have \( 0 < \frac{r_{i+1}}{r_i} < 1 \), for \( i = 1, 2, \cdots, n-2 \), and the power \( 1 - \frac{r_{i+1}}{r_i} > 0 \) for \( i = 1, 2, \cdots, n-2 \). Consequently, when the error \( e_{i+1} \) is small enough, the term

\[ c \sum_{i=1}^{n-2} H_i(x_{i+2}, \cdots, x_n) \left| e_i \left| e_i \right|^{r_i+1} + \cdots + \left| e_n \right|^{-r_n+1} \right) < 1 \]

for bounded \( x \) in \( K \). Clearly, we have proven (3.10) is negative definite due to the boundedness of \( x \), and we can conclude that the error dynamics (3.2) is A.S. as long as \( \| e \| \) is small enough.

We now combine the homogeneous observer developed in the previous section with the initial state calculation by using the LSE method.

According to the previous section, we know that when \( \| e(0) \| \) is small, the error \( e(t) \rightarrow 0 \) as \( t \rightarrow \infty \). There exist various methods of state estimation, which can make the initial error small. Next we will use Least Square Estimation (LSE) method which is the most widely used method in estimating constants to estimate the initial values for an upper-triangular system.

**Initial state estimation:** In order to estimate the initial condition based on the output \( y = x_1 \), in a short time period \( t \), we set the controller \( u = 0 \). Then we can take integration...
of (1.1), and get the following equations:
\[ x_n(t) = x_n(0) \]
\[ x_{n-1}(t) = x_{n-1}(0) + x_n(0)t \]
\[ x_{n-2}(t) = x_{n-2}(0) + x_{n-1}(0)t + \frac{x_n(0)t^2}{2} + tf_{n-2}(x_n(0)) \]
\[
\vdots
\]
\[ x_1(t) = x_1(0) + \int_0^t [x_2(s) + f_1(x_3(s), \ldots, x_n(s))]ds \tag{3.12} \]

**Remark 3.1:** If the integration term in (3.12) can be written in the form of
\[ \int_0^t [x_2(s) + f_1(x_3(s), \ldots, x_n(s))]ds \]
\[ = [g_1(x_2(0), \ldots, x_n(0)), \ldots, g_{n-1}(x_n(0))] \begin{pmatrix} h_1(t) \\ \vdots \\ h_{n-1}(t) \end{pmatrix} \tag{3.13} \]

where the functions \( g_i(\cdot) \) and \( h_i(\cdot) \) are functions of \( x_2(0), \ldots, x_n(0) \) and \( t \), respectively. We can estimate \( g_1(x_2(0), \ldots, x_n(0)), \ldots, g_{n-1}(x_n(0)) \) and consequently \( x_2(0), \ldots, x_n(0) \) based on \( x_1(t) \). As a matter of fact, since the state \( x_1 \) is measurable, in (3.12), we can use the last equation to compute the other unknown states by using the LSE method [13]. First we set \( z(t) = x_1(t) - x_1(0) \), thus the above equations yield the following results:
\[ z(t) = X^T(t)\theta \tag{3.14} \]

where
\[ z(t) = x_1(t) - x_1(0) \]
\[ \theta = \begin{pmatrix} g_1(x_2(0), \ldots, x_n(0)) \\ \vdots \\ g_{n-1}(x_n(0)) \\ h_1(t) \\ \vdots \\ h_{n-1}(t) \end{pmatrix} \]
\[ X(t) = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \]

Thus we can apply the LSE method equation to solve the unknown \( n - 1 \) states with the help with \( n - 1 \) times of measurements.
\[ \hat{\theta} = (\phi^T\phi)^{-1}\phi^TZ \]
\[ \phi = \begin{pmatrix} X^T(t_1) \\ \vdots \\ X^T(t_{n-1}) \end{pmatrix}, \quad Z = \begin{pmatrix} z(t_1) \\ \vdots \\ z(t_{n-1}) \end{pmatrix}. \tag{3.15} \]

The relation (3.13) actually can be done if \( f_i(\cdot) \)'s are polynomials.

**Remark 3.2:** Even though we set \( u = 0 \) during the estimation period, the upper-triangular system will stay bounded for any small period thanks to the upper-triangular structure. It is not necessary to set \( u = 0 \) during the estimation, since \( u \) is known, and according to the upper-triangular system, this implies that we have the last state available.

**Combined with homogeneous observer:** After we get the initial values of the unknown states \((x_2(0), \ldots, x_n(0))\), we can plug them back into the equations (3.12) to compute the estimation values of these unknown states after the short period, and we can use them as the initial values for the homogeneous observer (3.1). Technically, the estimation value should be exactly the same as the real states, which means the initial error is supposed to be zero.

Next, we use an example to show how our method works.

**Example 3.1:** Consider the following upper-triangular system:
\[ \dot{x}_1 = x_2 + x_3^2, \quad \dot{x}_2 = x_3, \quad \dot{x}_3 = u, \quad y = x_1 \tag{3.16} \]

where the controller \( u \) is designed to be a saturation controller [12] for getting the global results to be designed later. Then we proceed this system in the method which we described above:
\[ x_3(t) = x_3(0) \]
\[ x_2(t) = x_2(0) + x_3(0)t \]
\[ x_1(t) = x_1(0) + \int_0^t x_2(s)ds + \int_0^t x_3^2(0)ds \]
\[ x_1(t) - x_1(0) = x_2(0)t + x_3(0)\frac{t^2}{2} + x_3^2(0)t \]
\[ = \left( t, \frac{t^2}{2}, x_3(0) \right) \begin{pmatrix} x_2(0) + x_3^2(0) \\ x_3(0) \end{pmatrix} \tag{3.17} \]

where \( x_2(0) + x_3^2(0) = g_1(x_2(0), x_3(0)), x_3(0) = g_2(x_3(0)) \).

To estimate unknown \( x_2(0) \) and \( x_3(0) \), we first let the controller \( u = 0 \) in a short period when the system starts, for example, first 0.4 seconds. After having gathered enough data in the first beginning 0.4 seconds, we then use the LSE method to compute \( x_2(0) \) and \( x_3(0) \), from which we can predict the states \( \hat{x}_2(0.4) \) and \( \hat{x}_3(0.4) \) at the time \( t = 0.4 \text{sec} \). During this whole 0.4 seconds, the system is running without controller \( (u = 0) \). Then we start our high-order observer for this system at 0.4sec, with initial states using the estimated states \( \hat{x}_2(0.4) \) and \( \hat{x}_3(0.4) \). Fig. 1 shows how we combine the LSE predictor and observer together to estimate and control the system.

![Fig. 1 Block diagram of LSE + homogeneous observer](image_url)
According to the principle of the high-order homogeneous observer, we can choose: \( \tau_1 = \frac{2}{7}, r_2 = r_1 + \tau_1 = \frac{9}{7}, r_3 = r_2 + \tau_1 = \frac{14}{7} \). In this way, our high-order observer is described as:

\[
\begin{align*}
\dot{\hat{x}}_1 &= \hat{x}_2 + \hat{x}_3^2 + 3\hat{x}_1^2 \\
\dot{\hat{x}}_2 &= \hat{x}_3 + 3\hat{x}_1^2 \\
\dot{\hat{x}}_3 &= u + e_1^{\frac{14}{7}}. 
\end{align*}
\]

Then we use the estimated states from (3.18) in the saturation controller [12]

\[
S(x, L) = \begin{cases} 
  x, & \text{for } |x| \leq L \\
  L\text{sign}(x), & \text{for } |x| > L
\end{cases}
\]

\[
\begin{align*}
y_1 &= x_1 + 2\hat{x}_2 + \hat{x}_3 \\
y_2 &= \hat{x}_2 + \hat{x}_3 \\
y_3 &= \hat{x}_3 \\
u &= S(y_3 + S(y_2 + S(y_1, L_1), L_2), L_3)
\end{align*}
\]

and the parameters are chosen as:

\[
\begin{align*}
L_1 &= 0.64 \\
L_2 &= 1.6 \\
L_3 &= 4.
\end{align*}
\]

As we see in Fig. 2, after the first 0.4 seconds, the LSE method provides perfect estimation of this system without controller. Then we use this estimation as the initial states for the high-order homogeneous observer, and we also use this estimation to design a saturation controller. Since \( x_2(0.4) = \hat{x}_2(0.4) \) and \( x_3(0.4) = \hat{x}_3(0.4) \) the estimation and real states are exactly all the same.

Remark 3.3: Fig. 2 LSE + homogeneous observer

As seen in Fig. 2, we can just use the system model to get all the states by using the initial value provided by LSE. In the ideal case, the system model running in the computer will produce \( \hat{x}(t) \) which are supposedly identical to the real states \( x(t) \) if \( \hat{x}(0) = x(0) \). This observer can be viewed as an open-loop observer. However, in practice, there exist various unexpected disturbances, which will make the open-loop observer not be able to keep track of the changes.

To demonstrate the robustness of our proposed high-order homogeneous observer, we add some disturbances which force the states jump at \( t = 10\text{sec} \). With the help of the homogeneous correction terms, the observer will quickly respond to the disturbances and make \( \hat{x}(t) \to x(t) \) as \( t \to \infty \). As we can see in Fig. 3 our system and observer
react very fast to reduce the effect of the disturbances and all the trajectories still converge to zero.

\[ \begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 + x_1x_4 \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= u \\
y &= x_1
\end{align*} \]

(3.20)

In this case, we can construct a very similar observer like the former example as following:

\[ \begin{align*}
\dot{x}_1 &= \hat{x}_2 + d_1(x_1 - \hat{x}_1)^2 \\
\dot{x}_2 &= \hat{x}_3 + \hat{x}_1^2 + d_2(x_1 - \hat{x}_1)^2 \\
\dot{x}_3 &= \hat{x}_4 + d_3(x_1 - \hat{x}_1)^2 \\
\dot{x}_4 &= u + d_4(x_1 - \hat{x}_1)^2 \\
y &= x_1
\end{align*} \]

(3.21)

where \( r_1 = 1, \tau_1 > 0, \tau_{i+1} = \tau_i + \tau_1 \), for \( i = 1, 2, \ldots, n \), and the controller can still be chosen in the form of saturation controller.

IV. CONCLUSION

In this paper, for a class of upper-triangular systems, we designed a high-order homogeneous observer which is asymptotically convergent when the initial conditions of the observer are close to the initial conditions of the system. Then we developed a method of calculating initial states based on the least square estimation method, and it shows this method can provide the estimation of the initial values. It was also shown by a simulation example that the homogeneous observer is robust to disturbances.

REFERENCES


