Stability Regions in the Parameter Space for a Unified PID Controller

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Abstract—In this paper a unified approach is presented for finding the stability boundary and the number of unstable poles for an arbitrary order transfer function with time delay in continuous-time or discrete-time systems. These problems can be solved by finding all achievable proportional integral derivative (PID) controllers that stabilize the closed-loop system of a single-input single-output (SISO) linear time invariant (LTI) system. This method is used to predict the number of unstable poles of the closed-loop system in any region of the parameter space of a PID controller. The delta operator is used to describe the controllers because it provides not only numerical properties superior to the discrete-time shift operator, but also converges to the continuous-time case as the sampling period approaches zero.

A key advantage of this approach is that the stability boundary can be found only when the frequency response and not the parameters of the plant transfer function are known. A unified approach allows us to use the same procedure for finding the continuous-time or discrete-time stability region and the number of unstable poles of the system. If the plant transfer function is known, the stability regions can be found analytically.

I. INTRODUCTION

There has been a significant amount of research concerning the stability boundary for PID control of SISO LTI systems. Most of these methods depend on earlier developed theorems where the plant parameters must be known [1]. The method introduced by Tan in [2] broke the numerator and denominator of the plant transfer function into even and odd parts. In [3], [4], and [5] a method, which did not involve complex mathematical derivations was used to solve the problem of stabilizing an arbitrary order transfer function when only the frequency response of the plant transfer function was known. In [4], Saeki introduced a method to determine the number of unstable poles of the closed-loop system in the PI and PD planes. This method was based on the frequency response of the continuous-time system. In [6] and [7] the authors looked at the parameter space approach for PID controller design. In [8], Ou, Zhang and Yu introduced analytical solutions for stabilizing an arbitrary order SISO LTI plant with time delay by using P, PI, and PID controllers.

While most of the work in this area has been done in continuous-time, PID controllers are typically implemented in discrete-time. Digital control systems are usually described in a different framework than those used for continuous-time control systems. In [9], Middleton and Goodwin developed the delta operator, which has numerical properties superior to the discrete-time shift operator for digital controller design. In [10], the authors demonstrated that the continuous-time and discrete-time cases can be understood under a common framework through the use of delta operators. In [11], the authors of this paper extended the continuous-time stability boundaries found in [3] and [4] to the discrete-time case. The delta operator was used to obtain a unified stability boundary for PID controller of an arbitrary order transfer function with time delay. They showed that the stability boundaries can be found when only the frequency response and not the parameters of the plant transfer function were known. In this method, if the plant transfer function was known, the stability regions could be found analytically.

In this paper, we have extended the unified approach for the stability boundary of PID controllers in [11] to find the number of unstable poles for the closed-loop system in the PI and PD planes based on the boundary crossing theorem in [4]. The parameterization of the PID controller in this paper is slightly modified from [11]. This paper is organized as follows. In Section II, the design methodology is presented to find all achievable PID controllers of a proper arbitrary order transfer function with time delay. In Section III, the theorems are presented for finding the number of unstable poles in the parameter space of the PI and PD planes. In Section IV, a numerical example demonstrates the application of our theorems. Finally the conclusion is presented in Section V.

II. DESIGN METHODOLOGY

A plant transfer function is defined in continuous-time as

\[ G_p(s) = G(s)e^{-\tau s}, \]  

(1)

where \( G(s) \) is an arbitrary order proper plant transfer function and \( \tau \) is a constant time delay. The output of the plant is sampled with a zero-order hold input. The equivalent delta domain transfer function of (1) can be defined from [9] as

\[ G_p(\gamma) = \frac{\gamma}{1 + \gamma T_0} \mathcal{T}^{-1} \left[ \frac{1}{s} G_p(s) \right], \]

(2)
where \( T_0 \) is the sampling period, \( \mathcal{T} \) is the generalized transform, \( L \) is the Laplace transform, and \( \gamma \) is defined in [10] as

\[
\gamma = \begin{cases} 
    s, & T_0 = 0 \\
    \frac{e^{sT_0} - 1}{T_0}, & T_0 \neq 0
\end{cases}.
\] (3)

A standard unity feedback control system is shown in Figure 1. The PID controller is defined as

\[
G_c(\gamma) = K_p + \frac{\gamma}{\gamma^2 + \frac{1}{T_0} \gamma + K_d},
\] (4)

where \( K_p, K_i, \) and \( K_d \) are the proportional, integral, and derivative gains, respectively.

The plant transfer function and the PID controller in Figure 1 can be expressed in the frequency domain as

\[
G_p(\beta) = R_c(\beta) + jI_m(\beta),
\] (5)

\[
G_c(\beta) = K_p + \frac{\beta}{1 + T_0 \beta},
\] (6)

where \( \beta = \frac{e^{j\omega T_0} - 1}{T_0} \) \( T_0 \neq 0 \) [10].

The standard characteristic polynomial of the closed-loop system in frequency domain is written as

\[
P(\beta) = 1 + G_c(\beta)G_p(\beta).
\] (7)

Expanding the characteristic equation in (7) and writing it in terms of its real and imaginary parts yields

\[
o R_c(\beta)K_p + X_{Ri}K_i + X_{Rd}K_d = -\omega,
\] (8)

and

\[
o I_m(\beta)K_p + X_{Ri}K_i + X_{Id}K_d = 0,
\] (9)

where

\[
X_{Ri} = -\frac{T_0}{2} \omega R_c(\beta) + I_m(\beta) \left( \frac{\cos(\omega T_0)}{2\sin(\omega T_0)} + 1 \right),
\]

\[
X_{Rd} = -\frac{T_0}{2} \omega I_m(\beta) \sin(\omega T_0) \sin(\omega T_0) + \frac{1}{2} \omega^2 I_m(\beta) \sin(\omega T_0),
\]

\[
X_{li} = -\frac{T_0}{2} \omega I_m(\beta) \cos(\omega T_0) + \frac{1}{2} \omega^2 I_m(\beta) \cos(\omega T_0) + 1.
\]

This is a three-dimensional system in terms of the controller parameters \( K_p, K_i, \) and \( K_d \). First, the stability boundaries in the \((K_p, K_i)\) plane will be obtained for a fixed value of \( K_d \). After setting \( K_d \) to the fixed value \( K_d \), (8) and (9) can be re-written as

\[
\begin{bmatrix} o R_c(\beta) & X_{Ri} \end{bmatrix} \begin{bmatrix} K_p \\ K_i \end{bmatrix} = \begin{bmatrix} -\omega - X_{Ri} \tilde{K}_d \\ -X_{Ii} \tilde{K}_d \end{bmatrix},
\] (10)

For \( T_0 \neq 0 \) and \( \omega \neq 0 \), the solution to (10) is given by:

\[
K_p = \frac{-R_c(\beta)}{\left| G_p(\beta) \right|^2} \frac{I_m(\beta) \sin(\omega T_0)}{\left| G_p(\beta) \right|^2 \left( 1 + \cos(\omega T_0) \right)},
\] (11)

\[
K_i = \frac{-2\omega I_m(\beta) \sin(\omega T_0) \cos(\omega T_0)}{\left| G_p(\beta) \right|^2 \left( 1 + \cos(\omega T_0) \right)} + \frac{2 \tilde{K}_d \omega^2 \sin^2(\omega T_0)}{\left( 1 + \cos(\omega T_0) \right)}.
\] (12)

where \( \left| G_p(\beta) \right|^2 = R_c^2(\beta) + I_m^2(\beta) \). At \( \omega = 0 \), (10) simplifies to

\[
\begin{bmatrix} 0 & I_p(0) \\ 0 & R_p(0) \end{bmatrix} \begin{bmatrix} K_p(0) \\ K_i(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
\] (13)

From (13) it can be seen that \( K_p(0) \) is arbitrary and the value for \( K_i(0) \) must be zero, unless \( R_c(0) = I_m(0) = 0 \). By letting \( T_0 \rightarrow 0 \) in (11) and (12), the continuous-time stability boundaries, which are equivalent to those in [3], are found.

Next, the stability boundaries in the \((K_p, K_d)\) plane for a fixed value of \( K_i = \tilde{K}_i \) are found. The solution to (8) and (9) gives

\[
\begin{bmatrix} o R_c(\beta) & X_{Rd} \end{bmatrix} \begin{bmatrix} K_p \\ K_d \end{bmatrix} = \begin{bmatrix} -\omega - X_{Ri} \tilde{K}_i \\ -X_{Ii} \tilde{K}_i \end{bmatrix}.
\] (14)
For \( T_0 \neq 0 \) and \( \omega \neq 0 \), the solution to (14) gives the same expression as (11) for \( K_p \) and the following equation for \( K_d \):

\[
K_d = \frac{I_m(\beta) - \beta I_m(\beta) / G_p(\beta)}{\omega \left| G_p(\beta) \right|^2 \sin(\omega T_0)} + \frac{2 \omega^2 \sin^2(\omega T_0)}{\beta \left| G_p(\beta) \right|^2 \sin(\omega T_0)} + \frac{K_i(1 + \cos(\omega T_0))}{2 \omega^2 \sin^2(\omega T_0)}.
\]  

(15)

If \( \tilde{K}_i = 0 \), that is, in the case of a PD controller, it can be shown that as \( \omega \to 0 \), (14) approaches

\[
\begin{bmatrix}
R_e(0) & 0
\end{bmatrix} \begin{bmatrix}
K_i(0) \\
K_d(0)
\end{bmatrix} = \begin{bmatrix}
-1 \\
0
\end{bmatrix}.
\]

(16)

If \( I_m(0) = 0 \), \( K_d(0) \) is arbitrary and \( K_p(0) = -\frac{1}{R_e(0)} \).

By letting \( T_0 \to 0 \) in (11) and (15), the continuous-time stability boundaries, which are equivalent to those in [3], are found.

Finally, the stability boundaries in the \( (K_i, K_d) \) plane for a fixed value of \( K_p \) are obtained. After setting \( K_p \) to the fixed value \( \tilde{K}_p \), equations (8) and (9) can be rewritten as

\[
\begin{bmatrix}
x_Ri \\
x_li \\
x_Rd \\
x_id
\end{bmatrix} \begin{bmatrix}
K_i \\
K_d
\end{bmatrix} = \begin{bmatrix}
-\omega - \omega R_e(\beta) \tilde{K}_p \\
-\omega I_m(\beta) \tilde{K}_p
\end{bmatrix}.
\]

(17)

Although the coefficient matrix is singular, a solution will exist in two cases. First, at \( \omega = 0 \) \( K_d(0) \) is arbitrary and \( K_i(0) = 0 \), unless \( I_m(0) = R_e(0) = 0 \), which holds only when the plant has a zero at the origin. In such a case, a PID compensator should be avoided as the PID pole cancels the zero at the origin and the system becomes internally unstable. A second set of solutions occurs at any frequency \( \omega \) where \( K_p(\omega) \) (from (11)) is equal to \( \tilde{K}_p \). At these frequencies, \( K_d(\omega) \) and \( K_i(\omega) \) for \( T_0 \neq 0 \) and \( \omega \neq 0 \), must satisfy the following straight line equation

\[
K_d(\omega) = \frac{I_m(\beta)}{\omega^2 \left| G_p(\beta) \right|^2 \sin(\omega T_0)} + \frac{K_i(1 + \cos(\omega T_0))}{2 \omega^2 \sin^2(\omega T_0)}.
\]

(18)

By letting \( T_0 \to 0 \) in (18), the continuous-time stability boundaries, which are equivalent to those in [3], are found.

III. NUMBER OF UNSTABLE POLES OF PID CONTROLLER

Numbers of unstable poles in each region of the parameter space were determined in the PI and PD planes for a continuous-time PID controller in [4]. In this section, we seek to do the same for the unified PID controller in the delta domain.

To be stable in the delta domain the closed-loop poles must be placed inside a circle with a radius of \( \frac{1}{T_0} \) and a center at \( \frac{1}{T_0} \) as shown in Figure 2 [9]. This region is defined as \( \Gamma \) and its boundary is given as

\[
\partial \Gamma = \{ \gamma | \gamma = \beta, \ 0 \leq \omega < \infty \}.
\]

(19)

Fig. 2. Stability region in the delta domain.

The PID controller gains for which all the poles of the following characteristic equation lie in \( \Gamma \) is defined as \( K_{\Gamma} \).

We will denote the boundary of this set as \( \partial K_{\Gamma} \). The PID controller gains that satisfy the characteristic equation in (7) in the frequency domain can be written as

\[
1 + \left( \frac{K_p + \frac{K_i}{\beta} + K_d}{1 + T_0} \right) (R_e(\beta) + j I_m(\beta)) = 0.
\]

(20)

We will denote this set as \( K_{\partial \Gamma} \). It can be said that \( K_{\partial \Gamma} \) contains \( \partial K_{\Gamma} \) because of the boundary crossing theorem [7].

Next, we find the tangent vector \( \nu_\gamma \) and normal vector \( \nu_{n\gamma} \) of the contour \( \beta = \gamma \) for increasing \( \omega \). The tangent vector \( \nu_\gamma \) is calculated as

\[
\nu_\gamma = \frac{d}{d\omega}^{\gamma} = \frac{d}{d\omega} \left( e^{i\omega T_0} - 1 \right) / T_0 = d / d\omega \left( \frac{\cos(\omega T_0) - 1}{T_0} + j \sin(\omega T_0) / T_0 \right) \]

(21)

\[
= -\omega \sin(\omega T_0) + j \cos(\omega T_0).
\]

The right-hand side normal vector is \( \nu_{n\gamma} = \cos(\omega T_0) + j \sin(\omega T_0) \), as shown in Figure 2.
Now, in the delta domain, the number of unstable poles will be examined for each parameter plane. When the boundary $K_{iG}$ of each plane is drawn, it divides the parameter plane into regions. From the boundary crossing theorem, the number of unstable poles will not change in each region. Now, we will define some properties that describe how the numbers of unstable poles change as the boundary is crossed.

Lemma 1: If the parameters cross the boundary for a nonzero $\omega$, the number of unstable poles increases or decreases by two.

Proof: The characteristic polynomial will have a root $\gamma = \frac{e^{j\omega_0 T_0} - 1}{T_0}$ if the parameters cross the boundary at $\omega = \omega_0$. In the case of a single root, the number of unstable poles changes by two because there is a conjugate root. In the case of multiple roots, this point is singular because such a case where the polynomial equation has multiple roots does not occur generically. At the singular point, the number of unstable poles changes by four when the parameters cross the point. However, this contradicts the fact that the number of unstable poles does not change in each region, because the number of unstable poles changes by two at a neighborhood point, and the roots are single generically.

Lemma 2: If the parameters cross the boundary $K_i = 0$, the number of unstable poles increases or decreases by one.

Proof: This boundary occurs when $\omega = 0$. Because $\gamma = 0$ is a real root, there is no conjugate root. Thus, the number of unstable poles changes by one in the case of single root. As explained in Lemma 1, the case of multiple roots does not occur generically.

Lemma 3: Suppose $\gamma$ is the closed-loop pole for the PID controller gain on $K_{iG}$, and let it lie on a contour of $\Delta \Gamma$. Now it can be said the following equation is satisfied for a small perturbation:

$$\Delta \gamma = H(\gamma)\left[\Delta K_p \left(T_0 \gamma^2 + \gamma\right) \right] + \Delta K_i \left(T_0 \gamma + 1\right) + \Delta K_d \gamma^2),$$

(22)

where

$$H(\gamma) = \frac{G_p(\gamma)}{Q(\gamma)},$$

$$Q(\gamma) = 1 + 2T_0 \gamma + \left[ \frac{K_p \left(T_0 \gamma^2 + \gamma\right)}{K_i \left(T_0 \gamma + 1\right) + K_d \gamma^2} \right] \frac{dG_p}{d\gamma} + \left( K_p \left(2T_0 \gamma + 1\right) + K_i T_0 + 2K_d \gamma \right) G_p(\gamma).$$

Proof: If the following characteristic equation has a root $\gamma$, then

$$(T_0 \gamma + 1) \gamma + \left( \frac{K_p \left(T_0 \gamma + 1\right) + K_d \gamma^2}{K_i \left(T_0 \gamma + 1\right) + K_d \gamma^2} \right) G_p(\gamma) = 0.$$

(23)

As the PID controller gains move from $K_p$, $K_i$, $K_d$ to $K_p + \Delta K_p$, $K_i + \Delta K_i$, $K_d + \Delta K_d$, the root moves from $\gamma$ to $\gamma + \Delta \gamma$, and the plant-transfer function moves from $G_p(\gamma)$ to $G_p(\gamma + \Delta \gamma) = G_p(\gamma) + \Delta G_p(\gamma)$. Then, the following equation is satisfied:

$$\left( (T_0 \gamma + 1) \gamma + \left( \frac{K_p \left(T_0 \gamma + 1\right) + K_d \gamma^2}{K_i \left(T_0 \gamma + 1\right) + K_d \gamma^2} \right) G_p(\gamma) \right) + \left( K_p \left(2T_0 \gamma + 1\right) + K_i T_0 + 2K_d \gamma \right) \Delta G_p(\gamma) = 0.$$

(24)

The first-order approximation gives

$$\Delta \gamma = \Delta \gamma = \left( \frac{K_p \left(T_0 \gamma^2 + \gamma\right)}{K_i \left(T_0 \gamma + 1\right) + K_d \gamma^2} \right) \Delta G_p(\gamma) + \left( K_p \left(2T_0 \gamma + 1\right) + K_i T_0 + 2K_d \gamma \right) \frac{dG_p}{d\gamma} \Delta \gamma = 0.$$

(25)

Then, by substituting $\Delta G_p(\gamma) = \left( \frac{dG_p}{d\gamma} \right) \Delta \gamma$ and solving this equation with respect to $\Delta \gamma$, we get (22).

Theorem 1: As $\omega$ increases, the point $(K_p, K_i)$ moves in a particular direction along the PID stability boundary in the $(K_p, K_i)$ plane. The region on the left side of this direction has two more unstable poles than the region on the right side.

Proof: If $\gamma$ is a root, the tangent vector $\nu_t = (\Delta K_p, \Delta K_i)$ of (22) at this point on the stability boundary satisfies

$$\Delta \beta = H(\beta)\left[\Delta K_p \left(T_0 \beta^2 + \beta\right) \right] + \Delta K_i \left(T_0 \beta + 1\right).$$

(26)

Substituting $H(\beta) = c(\beta) + j d(\beta)$ into (26) and solving gives the following tangent vector $\nu_t$, as illustrated in Figure 3:
Then, the left side vector \( v_n = (\Delta K_{pn}, \Delta K_{in}) \) of \( v_t \) is obtained by

\[
\frac{dK_{pn}}{d\omega} = c \left( \cos(\omega T_0) - 1 \right) - d \sin(\omega T_0),
\]

\[
\frac{dK_{in}}{d\omega} = c T_0.
\]

When \( K_p(\beta) \) and \( K_i(\beta) \) move in the direction of \( v_n = (\Delta K_{pn}, \Delta K_{in}) \), the root at \( \gamma \) moves as

\[
\Delta \gamma_n = H(\beta) \left( \Delta K_{pn} (T \beta^2 + \beta) + \Delta K_{in} (T \beta + 1) \right) - \left( c(\beta) + jd(\beta) \right) \left( c \cos(\omega T_0) - 1 \right) - d \sin(\omega T_0) (T \beta^2 + \beta) + c T_0 (T \beta + 1)
\]

\[= a_1(\beta) + j b_1(\beta),\]

where

\[
a_1 = \frac{c^2 \left( 2 \cos^2(\omega T_0) - 3 \cos^2(\omega T_0) + 1 \right)}{T_0},
\]

\[
b_1 = \frac{2 c d \sin(\omega T_0) \cos(\omega T_0) \left( 1 - \cos(\omega T_0) \right) + d^2 \sin^2(\omega T_0) \left( 2 \cos(\omega T_0) - 1 \right) + c T_0^2 \left( c \cos(\omega T_0) - d \sin(\omega T_0) \right)}{T_0}.
\]

Now, if we examine the direction that \( \gamma \) moves in the \( \gamma \) domain due to the change of \( v_n = (\Delta K_{pn}, \Delta K_{in}) \). Taking the dot product between \( v_n \) and \( v_{ny} = \cos(\omega T_0) + j \sin(\omega T_0) \) yields

\[
\Delta \gamma_n \cdot v_{ny} = \omega^2 T_0 \left( c \left( 1 - \cos(\omega T_0) \right) + d \sin(\omega T_0) \right)^2 + c^2 T_0.
\]

Therefore,

\[
\Delta \gamma_n \cdot v_{ny} \geq 0.
\]

This means that the angle between \( \Delta \gamma \) and \( v_{ny} \) is less than \( \frac{\pi}{2} \), and the root has moved outside the stability region in Figure 2. Therefore, the number of unstable roots always increases along \( v_n \). From Lemma 1, the number of unstable roots increases by two. By letting \( T_0 \rightarrow 0 \) in (31), the continuous-time pole movement, which is equivalent to that found by Saeki [4], is obtained.

**Theorem 2:** As \( \omega \) increases, the point \((K_p, K_i)\) moves in a particular direction along the PID stability boundary in the \((K_p, K_i)\) plane. The region on the right side of this direction has two more unstable poles than the region on the left side.

**Proof:** If \( \gamma \) is a root, the tangent vector \( v_t = (\Delta K_p, \Delta K_i) \) of (22) at this point on the stability boundary satisfies

\[
\Delta \beta = H(\beta) \left( \Delta K_p (T_0 \beta^2 + \beta) + \Delta K_i \beta^2 \right).
\]

Substituting \( H(\beta) = c(\beta) + j d(\beta) \) into (34) and solving yields the tangent vector \( v_t \) as (29) and

\[
\frac{dK_d}{d\omega} = \frac{-c \sin(\omega T_0) + d \left( 1 + \cos(\omega T_0) \right)}{\omega^2 \left( \frac{c^2 + d^2}{2} \right) \left( 1 + \cos(\omega T_0) \right)}.
\]
Then, the left side vector \( \nu_n = (\Delta K_{pn}, \Delta K_{dn}) \) of \( \nu_t \) is obtained by

\[
\frac{dK_{pn}}{d\omega} = \frac{c \sin(\omega T_0) \sin(\omega T_0) + dsinc(\omega T_0)(1 + \cos(\omega T_0))}{\omega \sin^2\left(\frac{\omega T_0}{2}\right)(1 + \cos(\omega T_0))},
\]

where \( \Delta \) is greater

\[
\frac{dK_{dn}}{d\omega} = c.
\]

When \( K_p(\beta) \) and \( K_d(\beta) \) move in the direction of \( \nu_n = (\Delta K_{pn}, \Delta K_{dn}) \), the root at \( \gamma \) moves as

\[
\Delta \gamma_n = H(\beta) \left( \Delta K_{pn}(T_0\beta^2 + \beta) + \Delta K_{dn} \beta^2 \right)
\]

\[
= (c(\beta) + j d(\beta)) \left[ \frac{c \sin(\omega T_0) \sin(\omega T_0)}{\omega \sin^2\left(\frac{\omega T_0}{2}\right)(1 + \cos(\omega T_0))} \right] T \beta^2 + \beta
\]

\[
= a_2(\beta) + j b_2(\beta),
\]

(38)

and

\[
a_2 = \frac{c^2\left(2\cos^3(\omega T_0) - 3\cos^2(\omega T_0) + 1\right) + 4cd \sin(\omega T_0) \cos(\omega T_0)(1 - \cos(\omega T_0)) + d^2\left(-2\cos^3(\omega T_0) + \cos^2(\omega T_0) + 2\cos(\omega T_0) - 1\right)}{4\cos^2\left(\frac{\omega T_0}{2}\right) - 1 + 2c(c \cos(\omega T_0) - d \sin(\omega T_0))(\cos(\omega T_0) - 1)},
\]

\[
b_2 = \frac{c^2 \sin(\omega T_0)\left(2\cos^2(\omega T_0) - 3\cos(\omega T_0) + 1\right) + 2cd\left(2\cos^2(\omega T_0) - 1\right)\left(\cos(\omega T_0) - 1\right) + d^2\sin(\omega T_0)\left(-2\cos^2(\omega T_0) + \cos(\omega T_0) + 1\right)}{4\cos^2\left(\frac{\omega T_0}{2}\right) - 1 + 2c(c \sin(\omega T_0) + d \cos(\omega T_0))(\cos(\omega T_0) - 1)}.
\]

(39)

This means that the angle between \( \Delta \gamma_n \) and \( \nu_{ny} \) is greater than \( \frac{\pi}{2} \), and the root has moved inside the stability region in Figure 2. Therefore, the number of unstable roots always decreases along \( \nu_n \). From Lemma 1, the number of unstable roots decreases by two. By letting \( T_0 \to 0 \) in (38), the continuous-time pole movement, which is equivalent to that found by Saeki [4], is obtained.

Theorem 3: On the boundary \( K_i = 0 \), which corresponds to \( \omega = 0 \), the region below the line has one more unstable pole than the region above the line if

\[
\frac{G_p(0) \cos(\omega T_0)}{1 + K_p G_p(0)} > 0.
\]

(40)

Otherwise, the region above the line has one more unstable pole than the region below the line.

Proof: Substituting \( \Delta K_p = \Omega \) and \( \Delta K_d = 0 \) into (22) gives the change of the pole \( \Delta \gamma \) due to the movement of \( \Delta K_i \). The change \( \Delta \gamma \) at \( \gamma = 0 \) and \( K_i = 0 \) is given by

\[
\Delta \gamma = \frac{-G_p(0)}{1 + K_p G_p(0)} \Delta K_i.
\]

(41)

Now, we examine the movement direction of \( \Delta \gamma \) in the \( \gamma \) domain due to the change of \( \Delta K_i \). Taking the dot product between \( \Delta \gamma \) and \( \nu_{ny} \) yields

\[
\Delta \gamma \cdot \nu_{ny} = \frac{-G_p(0) \cos(\omega T_0)}{1 + K_p G_p(0)} \Delta K_i = \Delta \gamma.
\]

(42)

If \( \frac{G_p(0)}{1 + K_p G_p(0)} > 0 \) and \( \Delta K_i < 0 \), which means \( \Delta K_i \) moves below the line, (43) yields

\[
\Delta \gamma \cdot \nu_{ny} > 0.
\]

(43)
This means that the angle between $\Delta \gamma$ and $\nu_H$ is less than $\frac{\pi}{2}$, and the root has moved outside of the stability region in Figure 2. Therefore, the number of unstable roots increases. From Lemma 2, the number of unstable roots increases by one. The opposite case can be easily shown.

IV. EXAMPLE

The following transfer function was used by Saeki when determining the number of unstable poles for an infinite number of continuous-time PI controllers regions.

$$G_p(s) = \frac{1}{s+1} e^{-s}. \quad (45)$$

The goal here is to find the number of unstable poles in the discrete-time PI controller regions, when the sampling period is $T_0 = 0.1$ seconds. Using (2), the discrete-time delta-domain equivalent of (45) is given by

$$G_p(\gamma) = \frac{0.9516}{\gamma + 0.9516} \left( \frac{1}{\gamma} \right). \quad (46)$$

The boundary $K_{zT}$ in the $(K_p, K_i)$ plane with $K_d = 0$ for a discrete-time PI controller is shown in the Figure 4.

![Figure 4. Number of unstable poles on $(K_p, K_i)$ plane.](image)

The number of unstable poles is written in each region. The boundary $K_{zT}$ shows that as $\omega$ increases from 0 to $\infty$, the boundary spirals outward in a clockwise direction. Because the open-loop plant is stable, the stable region should include $K_p = 0$ and $K_i = 0$. Using Theorems 1 and 3, if the number of unstable poles for one region is known, then the number of unstable poles can be predicted for other regions. Although there are many regions in the $(K_p, K_i)$ plane, based on Theorems 1 and 3, we show that there is only one stable region. This results are analogues with those have found in [4].

V. CONCLUSION

A unified approach was introduced for finding the stability region and the number of unstable poles for an arbitrary order transfer function with time delay in continuous-time or discrete-time systems. The prediction of the number of unstable poles of the closed-loop system in the parameter space of delta-domain PID controllers is analogous with the results for continuous-time systems by Saeki in [4]. By using the delta operator, we find the unified stability boundaries of PID controllers from the frequency response of a plant without dealing with complex mathematical derivations. It was shown that the continuous and discrete cases could be understood under a common framework through the use of the delta operator. If the parameters of the plant are known, this method can be used to find the stability boundary analytically.

ACKNOWLEDGEMENTS

We extend our thanks to the reviewers of this paper for their comments. This work was supported in part by Spirit Aerosystems Inc., Boeing Integrated Defense Systems, and the Graduate School at Wichita State University. We would like acknowledge the financial support of all the sources that made this research possible.

REFERENCES