Output feedback synthesis for sampled-data system with input saturation

Dan Dai, Tingshu Hu, Andrew R. Teel, and Luca Zaccarian

Abstract—In this paper we address the sampled-data regional and global $H_\infty$ synthesis problem for a class of linear plants subject to input saturation, where the sampling and hold rates are synchronous. Such a sampled-data system is expressed as a jump system, which is further described as a class of hybrid systems. Based on Lyapunov theorems for hybrid systems, a Lyapunov function is constructed and it is proved that the $H_\infty$ problem is equivalent to a purely discrete-time synthesis problem. The proposed synthesis approach is cast as a convex optimization over Linear Matrix Inequalities (LMIs), which leads to an output feedback controller with an internal deadzone loop, achieving stability and desired performance. The effectiveness of the proposed techniques is illustrated by one example consisting in a mechanical system with different sample-and-hold rates.

Keywords: sampled-data system, input saturation, nonlinear $L_2$ gain, jump linear system, LMIs

I. INTRODUCTION

In this paper we bring the hybrid control technique to bear on the direct design synthesis problem for sampled-data systems with input saturation. In this control scenario, a continuous-time plant is controlled by a digital controller. The controller samples the plant’s measurement output, computes a control signal, and sends it to the actuator. Since the system is subject to input saturation, the plant’s control input is updated by the saturated version of the computed value. The controller’s output remains constant between updates. Sampling and hold devices perform analog-to-digital and digital-to-analog conversions. Therefore, the sampled-data system operates in two different time domains, namely, in the space of piecewise continuous vector valued functions and the space of vector valued sequences. The two time domains are joined via the synchronized sample-operator and zero-order hold.

For the stability analysis and the $L_2$ gain estimation of the sampled-data system, the lifting technique is widely used in the literature (see e.g. [2]). In particular, a continuous-time signal can be lifted into a discrete-time representation where the respective inner product and $L_2$ norm follow the same rules and have the same value as those of the continuous-time counterpart. Sampled-data anti-windup analyses are also presented in [1] and [13], where the lifting technique is used to transpose the problem into the purely discrete domain.

Various other methods have been developed for sampled-data systems. For example, [14] used a hybrid approach to develop robust sampled-data controllers for a class of uncertain nonlinear systems; in [19], a hybrid model for the sample-and-hold implementation of a nonlinear controller is presented. Systems with a hybrid controller is proposed, and its stability is studied; in [6], an input delay approach is applied for sampled-data stabilization of linear systems.

In this paper, instead of the lifting technique to deal with the sampled-data nature of the system, we firstly adopt the approach used in [5] and [16] to express the sampled-data problem as a synthesis problem for jump systems. Then, we describe the jump system as a hybrid systems, following the approach of [17] and [8]. The hybrid system directly describes the dynamics of the closed-loop system experiencing continuous behaviors (or flows), during the hold interval, and discontinuous behaviors (or jumps), at the sample times.

Based on the hybrid framework, our objective is to present regional and global $H_\infty$ output feedback design with an internal deadzone loop. The direct design problem is a particular control design task that takes saturation directly into account in the controller design and no specification or constraint is imposed on the behavior of the closed-loop for small signals. The synthesis problem has been considered in [4], [18] for continuous-time systems, and proved to be an effective control strategy to deal with systems with input saturation. In the hybrid description of the sampled-data closed-loop adopted here, the synthesis becomes a direct digital controller design procedure without approximation. The sample and hold devices are directly taken into account in the control design. In particular, we construct a Lyapunov function from a Hamiltonian matrix. Based on the Lyapunov theorems on hybrid systems given in [8], the Lyapunov function is designed to decrease along flows, and required to decrease along jumps. It is further proved that by using this Lyapunov function the sampled-data $H_\infty$ problem is equivalent to a purely discrete-time synthesis problem. Based on the general stability properties of hybrid systems in [8] and the regional analysis tool to describe the deadzone function in [15], sufficient conditions are derived for the existence of an output feedback controller with an internal deadzone loop, with guaranteed regional stability and minimized upper bound on the regional input-output $L_2$ gain for a class of norm bounded input. The overall synthesis of the sampled-data controller is cast as an optimization over LMIs. To the best of our knowledge, there is no existing direct design results available in the literature on regional synthesis for sampled-data systems with input saturation.

The rest of the paper is organized as follows: In Section II we formulate the sampled-data system as a jump system and further as a hybrid system; in Section III, preliminary results on hybrid systems from [8], [19] are presented; in Section IV, a Lyapunov function is constructed and the analysis and synthesis results are stated; in Section V, one example is used to illustrate the synthesis results.

Notation: For $Q = Q^T > 0$, denote $\mathcal{E}(Q) := \{ x : x^T Q x \leq 1 \}$. For a given matrix $\Theta$, define $\mathcal{L}(\Theta) := \{ x : \| U^{-1} \Theta x \|_\infty \leq 1 \}$, where $U := \text{diag}(\bar{u}_1, \bar{u}_2, \cdots, \bar{u}_n)$, and $\bar{u}_i > 0$ is the saturation level of the $i$th component of sat$(\cdot)$. For a given function $V$, its sub-level set is given by $L_V(c) := \{ x : V(x) \leq c^2 \}$. To save space, we use $0_n$ to denote a square.
zero matrix of dimensions \( a \times a \), and \( 0_{a \times b} \) for a zero matrix of dimensions \( a \times b \).

II. PROBLEM FORMULATION

The synthesis problem under consideration is a sampled-data problem. The plant is a continuous-time linear time-invariant system subject to input saturation. The system is described as:

\[
P = \begin{cases} \dot{x}_p = A_p x_p + B_p u + B_p w \\ y = C_p y + D_p u + D_p w \\ z = C_z x + D_z w \end{cases}
\]

where \( x_p \in \mathbb{R}^{n_p} \) is the plant state, \( u \in \mathbb{R}^{n_u} \) is the control input, \( w \in \mathbb{R}^{n_w} \) is the exogenous input (possibly containing disturbance, reference and measurement noise), \( y \in \mathbb{R}^{n_y} \) is the measurement output and \( z \in \mathbb{R}^{n_z} \) is the performance output. Assume that \((A_p, B_p)\) is stabilizable and \((C_p, A_p)\) is detectable. We assume \( D_{p,y} = 0 \) in the following content because this is a necessary assumption to guarantee that the controller introduced next induces a finite \( L_2 \) gain on the sampled-data closed-loop (this is trivially shown by picking a disturbance with zero \( L_2 \) norm which is zero almost everywhere except for the sampling instants).

A. Sampled-data controller

The goal of this paper is the synthesis of an output feedback controller with internal deadzone loops in the sampled-data scheme.

The sampled-data controller performs the actual computation and updates the state of the hold device. We consider the hold device to be of the zero-order type. To model this system, \( \tau \in \mathbb{R}_{\geq 0} \) is defined as the timer that after every \( h \) units of time triggers the computation of the control algorithm and the update of the hold device. Let’s define \( x_c \in \mathbb{R}^{n_c} \) as the state of the controller and \( u \in \mathbb{R}^{n_u} \) as the state of the zero-order holder. Then, when \( \tau \in [0, h] \), the controller has the continuous-time dynamics given by

\[
\begin{align*}
\dot{x}_c &= 1 \\
\dot{u}_c &= 0
\end{align*}
\]  

and, when \( \tau = h \), the controller has the discrete-time dynamics given by

\[
\begin{align*}
\tau^+ &= 0 \\
x^+_c &= A_c x_c + B_c y + \Lambda_1 dz(\tilde{u}) \\
u^+ &= \text{sat}(\tilde{u})
\end{align*}
\]  

(2b)

where (2b) is determined by solving \( \tilde{u} \in \mathbb{R}^{n_u} \) from the algebraic loop at time \( \tau = h \):

\[
\tilde{u} = C_c x_c + D_c y + \Lambda_2 dz(\tilde{u}).
\]

In (2b) and (2c), \( \text{sat}() : \mathbb{R}^{n_u} \rightarrow \mathbb{R}^{n_u} \) is the symmetric decentralized saturation function having saturation levels \( \bar{u}_i, \ldots, \bar{u}_{n_u} > 0 \) with its \( i \)-th component depending only on the \( i \)-th input component \( u_i \) as follows:

\[
\text{sat}(\tilde{u}_i) := \text{sign}(\tilde{u}_i)\min\{\bar{u}_i, |\tilde{u}_i|\}, i = 1, \ldots, n_u,
\]

and \( dz() : \mathbb{R}^{n_u} \rightarrow \mathbb{R}^{n_u} \) is defined as \( dz(\tilde{u}) = \tilde{u} - \text{sat}(\tilde{u}) \). The order of the controller is chosen as \( n_c = n := n_p + n_u \). For convenience, we define

\[
\Omega = \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix}, \quad \Lambda = \begin{bmatrix} \Lambda_1 \\ \Lambda_2 \end{bmatrix}.
\]

In the following section, we re-define the state of the plant as \( \xi \), and the order of the controller is chosen as the same as the new state in order to cast the synthesis into convex optimization problem.

B. Construction of the jump system

To model the sampled-data system, define \( \xi = \begin{bmatrix} x^T_p & u^T \end{bmatrix}^T \). A similar formulation involving linear systems with jumps is used in [5] and [16], where asynchronous multi-rate sample and hold is considered. Then, we define the state \( x = \begin{bmatrix} \xi^T & x_c^T \end{bmatrix}^T \), such that \( x \in \mathbb{R}^{2n} \). When \( \tau \in [0, h] \), the closed-loop system follows the continuous dynamics given by

\[
\begin{align*}
\dot{x} &= Ax + Bu \\
\dot{\tau} &= 1 \\
z &= C_z x + D_z w
\end{align*}
\]  

(5a)

where

\[
\begin{bmatrix} A & B \\ C_z & D_z \end{bmatrix} = \begin{bmatrix} \hat{A}_p & 0_n \\ 0_n & 0_n \end{bmatrix} \begin{bmatrix} 0_{(n_x \times n_x)} \\ 0_{(n_x \times n_x)} \end{bmatrix} + \begin{bmatrix} \hat{B}_{p,w} \\ D_{p,w} \end{bmatrix}
\]

with

\[
\begin{bmatrix} \hat{A}_p & \hat{B}_{p,w} \\ \hat{C}_z & D_{p,z} \end{bmatrix} = \begin{bmatrix} A_p & 0_{(n_u \times n_u)} \\ 0_{(n_u \times n_u)} & 0_{n_u} \end{bmatrix}, \quad \begin{bmatrix} \hat{B}_{p,w} \\ D_{p,w} \end{bmatrix} = \begin{bmatrix} B_{p,w} \\ D_{p,w} \end{bmatrix}
\]

When \( \tau = h \), the closed-loop follows the discrete dynamics given by

\[
\begin{align*}
x^+ &= J x + F dz(\tilde{u}) \\
\tau^+ &= 0 \\
u^+ &= C_y x + D_y dz(\tilde{u}),
\end{align*}
\]

(5b)

where

\[
\begin{bmatrix} J \\ C_y \\ D_{yq} \end{bmatrix} = \begin{bmatrix} J_0 & 0_{(n_x \times 2n)} \\ 0_{(n_u \times 2n)} & 0_{n_u} \end{bmatrix} + \begin{bmatrix} \Omega & \Lambda \\ \Omega y_2 & I_{n_u} \end{bmatrix} \begin{bmatrix} 0_{(n_x \times 2n)} \\ 0_{n_u} \end{bmatrix}
\]

with the following matrices partitioned by the dimensions of \( \xi \) and \( x_c \):

\[
\begin{align*}
J_0 &= \begin{bmatrix} J_0 & 0_{n \times n_u} \\ 0_n & 0_{n_u} \end{bmatrix}, \quad \tilde{J}_0 = \begin{bmatrix} I_{n_p} & 0_{(n_p \times n_u)} \\ 0_{(n_p \times n_u)} & 0_{n_u} \end{bmatrix}, \\
F_0 &= \begin{bmatrix} \tilde{F}_0 \\ 0_{(n_u \times n_u)} \end{bmatrix}, \quad \tilde{F}_0 = \begin{bmatrix} 0_{(n_u \times n_u)} \end{bmatrix}, \\
Y_1 &= \begin{bmatrix} 0_{(n \times n_u)} \\ 0_{(n \times n_u)} \end{bmatrix}, \quad Y_2 = \begin{bmatrix} 0_{(n \times n_u)} \end{bmatrix}, \\
Y_3 &= \begin{bmatrix} 0_{(n \times n_p)} \\ 0_{(n \times n_u)} \end{bmatrix}, \quad C_y = \begin{bmatrix} I_{n_u} \\ D_{p,w} \end{bmatrix}
\end{align*}
\]

When \( D_{yq} \neq 0 \), namely \( \Lambda_2 \neq 0 \), a nonlinear algebraic loop is imposed by the third equation in (5b). This algebraic loop is well-posed if \( \tilde{u} \) is uniquely determined from \( x \) in this equation. By [15], a sufficient condition for the well-posedness is that there exists a diagonal matrix \( \mathcal{X} > 0 \) satisfying \( 2\mathcal{X} - D_{yq} \mathcal{X} - \mathcal{X} D_{yq}^T > 0 \).

C. A hybrid jump linear system

The sampled-data system (5) given in the previous section operates in the space of piecewise continuous vector-valued functions and in the space of vector valued sequences. The two spaces are joined via the following hybrid approach used in [8]. Such a system can be easily described as a class
of hybrid systems $\mathcal{H}(C, F, D, G)$ with states $\eta \in \mathbb{R}^{2n+1}$, decomposed as $\eta = (x, \tau)$, where $x \in \mathbb{R}^{2n}$, and data

$$
\mathcal{H} \begin{cases}
C := \{ \eta : \tau \in [0, h] \} \\
F(\eta, w) := \left[ \begin{array}{c}
A_{x} + B_{w} w \\
1
\end{array} \right], w \in \mathbb{R}^{n_w}, \forall \eta \in C \\
D := \{ \eta : \tau = h \} \\
G(\eta) := \left[ \begin{array}{c}
J_{x} + F dz(\tilde{u}(x)) \\
0
\end{array} \right], \forall \eta \in D,
\end{cases}
$$

(6)

where $\tilde{u}(x)$ is the implicit solution of the last equation in (5b). The analysis and synthesis will be conducted on the description of the hybrid system $\mathcal{H} = (C, F, D, G)$ in (6).

### III. Preliminary Results on Hybrid Systems

In this section, we summarize some preliminary results regarding the class of hybrid systems $\mathcal{H}$ with data $(C, F, D, G)$ defined in (6).

The Basic Assumptions are the following three conditions on the data $(C, F, D, G)$ of a hybrid system $\mathcal{H}$:

- A1. the sets $C$ and $D$ are closed sets in $\mathbb{R}^{m}$;
- A2. $F : C \to \mathbb{R}^{m}$ is a continuous function;
- A3. $G : D \to \mathbb{R}^{m}$ is a continuous function.

These conditions guarantee good structural properties for the solutions of $\mathcal{H}$; see [10].

Solutions to a hybrid system are given on hybrid time domains by hybrid arcs. A set $H_{\text{dom}}$ is a hybrid time domain if for all $(t, J) \in H_{\text{dom}}$, $H_{\text{dom}} \cap ([0, T] \times (0, 1, \ldots))$ is a compact hybrid time domain, i.e., it can be written as $\bigcup_{j=0}^{J-1} ([t_j, t_{j+1}], j)$ for some finite sequence of times $0 = t_0 \leq t_1 \leq \ldots \leq t_J$. A hybrid arc $\eta$ is a function defined on a hybrid time domain $\eta$ mapping to $\mathbb{R}^{m}$ such that $\eta(t, j)$ is locally absolutely continuous in $t$ for each $(t, j) \in \text{dom} \ A$. A hybrid arc $\eta$ is a solution to the hybrid system $\mathcal{H}$ if $\eta(0, 0) \in C \cap D$ and

(S1) For all $j \in \mathbb{N}$ and almost all $t$ such that $(t, j) \in \text{dom} \ A$, $\eta(t, j) \in C$, $\dot{\eta}(t, j) = F(\eta(t, j), w(t))$.

(S2) For all $(t, j) \in \text{dom} \ A$ such that $(t, j+1) \in \text{dom} \ A$, $\eta(t, j+1) = G(\eta(t, j))$.

The following results are taken from [8, Theorem 23]. In the following corollary, we define $V$ to be positive definite with respect to $A$ as $V(x) = 0$ for $x \in \text{int} \ A$, and $V(x) > 0$ for $x \notin A$.

**Corollary 1:** Consider a hybrid system $\mathcal{H} = (C, F, D, G)$ satisfying the Basic Assumptions. Let $A \subset U \subset \mathbb{R}^{m}$ be such that $A$ is compact, contained in the interior of $U$. If there exists a continuously differentiable function $V$ that is positive definite with respect to $A$ such that

$$
\langle \nabla V(\eta), F(\eta(0), 0) \rangle \leq 0, \forall \eta \in C \cap U, \quad (7a)
$$

then the set $A$ is locally stable. If furthermore, for each $\mu > 0$, no complete solution to $\mathcal{H}$ remains in $L_{V}(\mu) \cap U$, then the set $A$ is locally pre-asymptotically stable. In this case, the domain of attraction contains every sublevel set of $V$ which is a subset of $U$.

A consequence of Corollary 1 is that $A$ is globally asymptotically stable if $L_{V}(\mu) \subset U$ can be taken to be arbitrarily large.

**Proposition 1:** Consider a hybrid system $\mathcal{H} = (C, F, D, G)$ satisfying the Basic Assumptions. Let $L_{V}(s^{2}) \subset U \subset \mathbb{R}^{m}$ be such that $L_{V}(s^{2})$ is compact, contained in the interior of $U$. If there exists a continuously differentiable function $V$ that is positive definite with respect to $A$ such that

$$
\langle \nabla V(\eta), F(\eta(0), 0) \rangle \leq -\frac{1}{\gamma^{2}} s^{2} z^{T} + w^{T} w, \forall \eta \in C \cap U, \quad (8a)
$$

$$
V(\eta) - V(0) \leq 0, \forall \eta \in D \cap U, \quad (8b)
$$

then we have the following conclusion:

1) (Reachable region) If $\eta(0, 0) \in A$, $\eta(t, j) \in L_{V}(s^{2})$ for all $(t, j) \in \text{dom} \ A$ and $\|z\|_{2} \leq s$.

2) (Nonlinear $L_{2}$ gain) If $\eta(0, 0) \in A$ and $\|w\|_{2} \leq s$, $\|z\|_{2} \leq \gamma \|w\|_{2}$.

The finite $L_{2}$ gain $\gamma$ of the hybrid system $\mathcal{H}$ is global if $L_{V}(s^{2}) \subset U$ can be taken arbitrarily large as $s \to \infty$.

### IV. Main Results

**A. An explicit construction of the Lyapunov function**

To establish asymptotic stability of the compact set $A := \{ \eta : x = 0, \tau \in [0, h] \}$ and certain performance for the hybrid system $\mathcal{H} = (C, F, D, G)$ in (6), we consider a Lyapunov function:

$$
V(\eta) = V((\tau, x)) := x^{T} P(\tau), \quad (9)
$$

where $P : [0, h] \to \mathbb{R}_{+}^{n} \times \mathbb{R}^{n}$. Denote a set of symmetric, positive definite matrices.

Furthermore, the function (9) satisfies

$$
\gamma x^{T} x \leq \langle x, \gamma \rangle \leq \sigma_{\pi}(P),
$$

with $\gamma := \min_{\pi} \sigma_{\pi}(P(\tau))$, $\sigma_{\pi}(P)$ indicates the eigenvalue of $P$.

The function $P(\tau)$ is explicitly chosen to satisfy

$$
\nabla P(\tau) = -A^{T}P(\tau) - P(\tau)A - C_{\tau}^{T}C_{\tau} \gamma^{2} - C_{\tau}^{T}D_{\tau}M(P(\tau)B + \gamma^{2}C_{\tau}^{T}D_{\tau}T, \quad (10)
$$

where $M = (I - \gamma^{2}D_{\tau}^{T}D_{\tau})^{-1}$. This choice of $V$, for all $\eta \in C$, it is straightforward to obtain

$$
\langle \nabla V(\eta), F(\eta(0), 0) \rangle + \gamma^{2} \leq \langle x, w \rangle \leq 0,
$$

which satisfies the condition (8a) in Proposition 1. The idea and construction of Lyapunov function $V$ is proposed in [8].

The solution to the matrix differential equation (10) can be written by forming the Hamiltonian matrix

$$
H = \begin{bmatrix}
A + \frac{1}{\gamma^{2}} BM^{T} D_{\tau} C_{\tau} & BM^{T} \\
-C_{\tau}^{T}LC_{\tau} & -A^{T} - \frac{1}{\gamma^{2}} C_{\tau}^{T} D_{\tau} MB^{T}
\end{bmatrix}
$$

(11)

where $L = (\gamma^{2}I - D_{\tau} D_{\tau}^{T})^{-1}$, $\gamma > \sigma_{\text{max}}(D_{\tau})$, and $M$ is defined below (10). Furthermore, forming the matrix exponential

$$
E(\tau) := \begin{bmatrix}
E_{A}(\tau) & E_{B}(\tau) \\
E_{C}(\tau) & E_{D}(\tau)
\end{bmatrix} = \exp(H\tau),
$$

(12)

with the same partition as $H$ in (11). Note that $E(\tau)$ is symmetric, i.e., $E(\tau)^{T} X E(\tau) = X$. The typical choice of $X$ is $X = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$. In order to save space for the presentation of the main results, we define

$$
E := E(h) = \begin{bmatrix}
E_{A} & E_{B} \\
E_{C} & E_{D}
\end{bmatrix}
$$

(13)

so as to neglect the script $h$. 

1799
When Assumption 1 holds, the matrices derived the following equation by applying the fact written as

\[ P(\tau) := (E_C(\tau) + E_D(\tau)P_0)(E_A(\tau) + E_B(\tau)P_0)^{-1}, \]  

(14)

where \( P_0 := P(0) \). In a similar way, with (13) we also define

\[ P_h := P(h) = (E_C + E_D P_h)(E_A + E_B P_h)^{-1}, \]  

(15)

and \( P_0 \) can be solved in terms of \( P_h \) as

\[ P_0 = E_D^{-1}(I - P_h E_B E_D^{-1})^{-1} P_h E_D^{-T} - E_D^{-1} E_C \]  

(16)

by using the fact \( E(\tau) \) is symplectic.

It is important to discuss that \( P(\tau) \) is well-defined on \( \tau \in [0, h] \) if the following assumption holds:

*Assumption 1*: \( E_D(\tau) \) is invertible for all \( \tau \in [0, h] \). To check positive definiteness of the solution (14) on \([0, h] \), let \( G(\tau) \) be defined as \( G(\tau)G(\tau)^T = E_B(\tau)E_D^{-1}(\tau) \). We derive the following equation by applying the fact \( E(\tau) \) is symplectic. To save space, we omit the calculation:

\[
P(h - \tau) = E_D^{-1}(\tau) P_h G(\tau)(I - G(\tau)^T P_h G(\tau))^{-1} G(\tau)^T P_h E_D^{-T}(\tau) + E_D^{-1}(\tau) P_h E_D^{-T}(\tau) - E_D^{-1}(\tau) E_C(\tau).\]

When Assumption 1 holds, the matrices \( -E_D(\tau)^{-1} E_C(\tau) \) and \( E_B(\tau) E_D(\tau)^{-1} \) are positive semidefinite for all \( \tau \in [0, h] \), and monotonically increasing. Furthermore, if \( I - G(h)^T P_h G(h) \) is positive definite, then \( I - G(\tau)^T P_h G(\tau) \) is positive definite for \( \tau \in [0, h] \). Therefore, \( P \) is well-defined on \([0, h] \) when \( P_h \) and \( I - G(h)^T P_h G(h) \) are initialized to be positive definite. Both conditions are guaranteed by the LMIs in our analysis theory.

Assumption 1 is always satisfied for a small enough \( h \), which implies that our results apply for a small enough \( h \).

Furthermore, according to the matrix parameters below (5a), the Hamiltonian matrix \( H \) in (11) can be explicitly written as

\[
H = \begin{bmatrix}
H_{11} & 0_n & \tilde{B}_{pw} & M \tilde{B}_{pw}^T & 0_n \\
0_n & 0_n & 0_n & -H_{11} & 0_n \\
-C_{p\nu}^\top L_{p\nu} & 0_n & 0_n & -H_{11} & 0_n \\
0_n & 0_n & 0_n & -H_{11} & 0_n
\end{bmatrix}.
\]  

(17)

where \( H_{11} = \bar{A}_p + \gamma^{-2} \tilde{B}_{pw} M \tilde{B}_{pw}^T \bar{D}_{p\nu} C_{p\nu} \).

\( E(\tau) \) can be written in an explicit form with the same partition as \( H \) in (17):

\[
E(\tau) = \begin{bmatrix}
E_{A_1}(\tau) & 0_n & I & 0_n & E_{B_1}(\tau) & 0_n \\
0_n & 0_n & 0_n & 0_n & E_{C_1}(\tau) & 0_n \\
E_{A_1}(\tau) & 0_n & I & 0_n & E_{B_1}(\tau) & 0_n \\
0_n & 0_n & 0_n & 0_n & E_{C_1}(\tau) & 0_n \\
I & 0_n & 0_n & 0_n & I & 0_n
\end{bmatrix}.
\]  

(18)

where the structure of \( E_A(\tau), E_B(\tau), E_C(\tau) \) and \( E_D(\tau) \) is derived from the fact that \( E(\tau) \) is symplectic.

**B. Regional and global performance analysis**

In the following content, we start by asserting that Proposition 1 can be applied to formulate statements for the hybrid system defined in (6).

Before stating the analysis results, we define \( K \in \mathbb{R}^{2n \times 2n} \) to be any matrix satisfying

\[
K^T K = -E_D^{-1} E_C.
\]  

(19)

and \( G \in \mathbb{R}^{2n \times 2n} \) to be any matrix satisfying

\[
G G^T = E_B E_D^{-1}.
\]  

(20)

**Theorem 1**: Let Assumption 1 hold for a choice of \( h \) in the symplectic matrix \( E \) in (13). Consider the hybrid system \( \mathcal{H} = (\mathcal{C}, \mathcal{F}, D, \mathcal{G}) \) in (6) with \( \|u\|_2 \leq s \). Using \( K \) and \( G \) in (19), (20), suppose that there exist a positive definite matrix \( Q \in \mathbb{R}^{2n \times 2n} \), a diagonal matrix \( U > 0 \), \( U \in \mathbb{R}^{n_u \times n_u} \), and a matrix \( \Theta_Q \in \mathbb{R}^{n_u \times 2n} \), such that the following LMIs are feasible

\[
\begin{bmatrix}
-Q & \Theta_Q & Z_{22} \\
\Theta_Q^T & -I_n & * \\
-Z_{22}^T & * & -Q
\end{bmatrix} < 0
\]  

(21a)

\[
\begin{bmatrix}
\tilde{u}_i^2/s^2 & \Theta_Q & Q \\
\Theta_Q^T & -Q
\end{bmatrix} \geq 0, i = 1, 2, ..., n_u
\]  

(21b)

where \( Z_{22} = D_{u\eta} U + U D_{\nu \eta}^T - 2U \) and \( \Theta_Q \) indicates the \( i \)-th row of \( \Theta_Q \). Then a function \( V(\eta) \) in (9) can be constructed from (14) by using \( P_h = Q^{-1} \) and (16). Furthermore, we have

1) (Local asymptotic stability) If \( w = 0 \), then \( \mathcal{A} \) is locally asymptotically stable.

2) (Reachable region) If \( \eta(0, 0) \in \mathcal{A} \) and \( \|w\|_2 \leq s \), then \( \eta(t, j) \in L_\nu((s^2) \mathbb{R}^n) \) for all \( (t, j) \in \text{dom} \eta \).

3) (Regional \( L_2 \) gain) If \( \eta(0, 0) \in \mathcal{A} \) and \( \|w\|_2 \leq s \), then \( \|z\|_2 \leq \gamma \|w\|_2 \), i.e., the regional \( L_2 \) gain is bounded by \( \gamma \).

**Remark 1**: When \( s \to \infty \), (21b) enforces \( \Theta_Q \to 0 \), and the theorem guarantees global results: 1) If \( w = 0 \), then \( \mathcal{A} \) is globally asymptotically stable. 2) If \( \eta(0) \in \mathcal{A} \), then \( \|z\|_2 \leq \gamma \|w\|_2 \), i.e., the global \( L_2 \) gain is bounded by \( \gamma \).

**C. Connection to discrete-time control system**

The feasibility of the LMIs (21) can be identified with a synthesis problem for the discrete-time system

\[
x^+ = E_D^{-T} J x + E_D^{-T} F dz(\nu) + Gw \]

\[
\nu = C_\gamma x + D_{u\eta} dz(\nu) \]

\[
z = K J x + K F dz(\nu).
\]  

(22)

The synthesis problem corresponds to picking the parameters \( \Omega \) and \( \Delta \) in (5b) to ensure that the \( L_2 \) gain from the disturbance \( w \) to the output \( z \) is less than one. The calculation can be found in [12] by applying a Lyapunov function \( V(x) = x^T X x \) through the condition \( V(x^+) - V(x) \leq -\epsilon |x|^2 - |z|^2 + |w|^2 \) for some \( \epsilon > 0 \) and \( X = X^T > 0 \).

This connection is built due to the jump system framework that we choose to represent the sampled-data system, and also the explicit Lyapunov function that we construct. More specifically, we construct the Lyapunov function in such a way that (8a) is satisfied when the system follows the continuous-time dynamics in (5a). Hence, (21) is developed separately to guarantee that (8b) holds when a jump happens as in (5b), and that is the reason why the LMIs in (21) look similar to some results arising with some discrete-time control techniques, such as Model Predictive Control [3], [11].

**D. A LMI-based design**

From analysis results to synthesis is quite straightforward; this is similar to the continuous or discrete time \( H_{\infty} \) problems solved via LMIs, as in [7]. Also see e.g. [4], [9] for the synthesis problems solved for continuous-time systems with input saturation, and [12] for the discrete-time counterpart. With the knowledge of the explicit structure of
In (18), the matrix $G$ can be explicitly constructed as $G := \begin{bmatrix} G_A^T & 0_n \end{bmatrix}^T$ with $G_A$ being any matrix satisfying

$$G_A G_A^T = E_{B_1} E_{D_{11}}^{-1}. \quad (23)$$

In the same manner, $K$ is constructed as $K := \begin{bmatrix} K_A & 0_n \end{bmatrix}$ with $K_A$ being any matrix satisfying

$$K_A K_A^T = -E_{D_{11}}^{-1} E_{C_{11}}. \quad (24)$$

Before stating the synthesis results, we define

$$M_1 := -K_A \begin{bmatrix} 0_{n_p \times n_u} & I_{n_u} \end{bmatrix}, \quad M_2 := -E_{D_{11}}^{-1} \begin{bmatrix} 0_{n_p \times n_u} & I_{n_u} \end{bmatrix} \quad (25)$$

**Theorem 2:** Let Assumption 1 hold for a choice of $h$ in (13), and $N_{Y_3}$ be a basis of the null space of $[C_{pu} D_{p,yu}]$. Consider a hybrid system $\mathcal{H} = (C, \mathcal{F}, D, G)$ in (6) with $\|w\|_2 \leq s$. By using $G_A, K_A, M_1, M_2$ in (23), (24) and (25), suppose that there exist positive definite matrices $Q_{11}, P_{11} \in \mathbb{R}^{n \times n}$ and a matrix $\Theta_q \in \mathbb{R}^{n_u \times n_u}$, such that the following LMIs are feasible

$$\begin{bmatrix} -Q_{11} & * & * & * \\ K_A \tilde{J}_0 Q_{11} - M_1 \Theta_q & 0_n -I_n & * \\ E_{D_{11}}^{-1} \tilde{J}_0 Q_{11} - M_2 \Theta_q & G_A & 0_n -Q_{11} \end{bmatrix} < 0 \quad (26a)$$

$$\begin{bmatrix} Z_{11}^T P_{11} E_{D_{11}}^{-1} \tilde{J}_0 N_{Y_3} & G_A^T P_{11} G_A - I_n & * \\ K_A \tilde{J}_0 N_{Y_3} & 0_n -I_n \end{bmatrix} < 0 \quad (26b)$$

$$\begin{bmatrix} \bar{u}_i^2/s^2 & \Theta_q, Q_{11} \\ \Theta_q^T I_{P_{11}} \end{bmatrix} \geq 0, \quad i = 1, 2, \ldots, n_u \quad (26c)$$

$$\begin{bmatrix} Q_{11} & P_{11} \end{bmatrix} > 0, \quad (26d)$$

where $Z_{11} = \tilde{N}_{Y_3}^T (\tilde{J}_0 E_{D_{11}}^{-1} P_{11} E_{D_{11}} \tilde{J}_0 - P_{11}) \tilde{N}_{Y_3}$ and $\Theta_q$ indicates the $i$-th row of $\Theta_q$. Then a feasible solution of the parameters of the controller in (2) can be constructed to guarantee that $\mathcal{A}$ is locally asymptotically stable with a regional $L_2$ gain from $w$ to $z$ bounded by $s$.

**Remark 2:** When $s \to \infty$, (26d) enforces $\Theta_q \to 0$, and the theorem implies that a feasible solution of the parameters of the controller in (2) can be constructed to guarantee that $\mathcal{A}$ is globally asymptotically stable with a global $L_2$ gain from $w$ to $z$ bounded by $s$.

### E. Controller construction

In what follows, we provide a constructive algorithm for determining the matrices of the controller whose existence is established in Theorem 2.

**Procedure 1:** (Output feedback construction)

**Step 1.** Solve the feasibility LMIs. Find a solution $(Q_{11}, P_{11}, \Theta_q)$ to the feasibility LMI conditions (26).

**Step 2.** Construct the matrix $Q$. (See also [9].) Define the matrix $Q_{12} \in \mathbb{R}^{n \times n}$ as a solution to the following equation:

$$Q_{11} P_{11} Q_{11} - Q_{11} = Q_{12} Q_{12}^T. \quad (27)$$

Since $Q_{11}$ and $P_{11}$ are invertible and $Q_{11}^{-1} P_{11}$ by the feasibility conditions, then $Q_{11} P_{11} Q_{11} - Q_{11}$ is positive definite. Hence there always exists a matrix $Q_{12}$ satisfying equation (27). Define the matrix $Q_{22} \in \mathbb{R}^{n \times n}$ as

$$Q_{22} := I + Q_{12}^{-1} Q_{11} Q_{12}. \quad (28)$$

Finally, define the matrix $Q \in \mathbb{R}^{2n \times 2n}$ as

$$Q := \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix}. \quad (29)$$

**Step 3.** Controller synthesis LMI. Choose $\Theta = \begin{bmatrix} \Theta_1 & \Theta_2 \end{bmatrix} = \begin{bmatrix} 0_{n_u \times n_u} & 0_{n_u \times n_u} \end{bmatrix}$, with the same partition as $Q$, where $\Theta_1 = \Theta_q Q_{11}^{-1}$. Then construct the matrices $\Psi \in \mathbb{R}^{(2n+2n) \times (2n+2n)}$, $R \in \mathbb{R}^{(n+2n) \times (n+2n)}$, and $X \in \mathbb{R}^{(n+n+2n) \times (2n+2n)}$ using $P_{h} = Q^{-1}$ as follows:

$$\Psi := \begin{bmatrix} -P_h & \star & \star & \star \\ -\Theta & -2U & \star & \star \\ K_{J_0} & K_{F_0} & 0 \quad & 0 \quad \star \\ E_{D_{11}}^{-1} \bar{J}_0 \end{bmatrix} \begin{bmatrix} 0_{(2n \times 2n)} & 0_{(n \times 2n)} & -I_n & \star \\ K_{Y_1} & K_{Y_2} & 0 \quad & 0 \quad \star \\ E_{D_{11}}^{-1} \bar{Y}_1 \end{bmatrix} \begin{bmatrix} 0_{(2n \times 2n)} & 0_{(n \times 2n)} & 0_{((n+n) \times 4n)} & 0_{((n+4n) \times 4n)} \end{bmatrix}.$$
control input is constrained in the range of the D/A converter: [-5, +5] Volts. In [9] a dynamic anti-windup compensator is used to preserve the local LQG behavior and improve the response after saturation. All the synthesis discussed in [9] are in the continuous-time framework. Here, we apply the direct design results of the discrete-time controller in the sampled-data framework. We discuss this example with the disturbance happening at the time $t = 1s$ and the control input is also constrained to $\pm 5$.

Case 1: Choose $h = 0.2$ and $s \rightarrow \infty$. Theorem 2 gives feasible solutions when $\gamma = 182$. We apply the same tap as in [9] at $t = 1$. The thin solid curve in Figure 2 represents the response of the closed-loop system by using our output feedback "Controller I" with the sampling and hold period $h = 0.2s$. This controller is constructed by trial and error and selected from feasible solutions which give a desirable performance. The numerical strategy in simulation includes appropriately restricting parameters $P_{11}$ and $Q_{11}$ when computing LMIs in (26), and/or appropriately restricting $\Omega$ when solving (30).

Case 2: Choose $h = 1$ and $s \rightarrow \infty$. The achievable $L_2$ gain is $\gamma = 182$. We apply a larger tap at $t = 1$, which is modeled as $12N$ with duration 0.01. The dotted curve in Figure 2 represents the response of the closed-loop system by using our output feedback "Controller II" with the sampling and hold period $h = 1s$.

Case 3: Choose $h = 3$ and $s \rightarrow \infty$. The achievable $L_2$ gain is $\gamma = 182$. We apply a larger tap at $t = 1$, which is modeled as $12N$ with duration 0.01. The dotted curve in Figure 2 shows that the system is stabilized after about 22s by using our output feedback "Controller III" with the sampling and hold period $h = 3s$.

For completeness, we analyze the nonlinear $L_2$ gains achieved by the three controllers listed above by solving LMIs in Theorem 1 for increasing values of $s$. The corresponding curves are reported in the lower plot of Figure 1.

Fig. 1. Achievable (upper) and achieved (lower) nonlinear $L_2$ gains for the three cases of the cart-spring-pendulum system.

Fig. 2. The response of the cart-spring-pendulum system with different sampling and hold periods.

REFERENCES


