Uniting Two Output-Feedback Controllers with Different Objectives

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Abstract—The problem of robustly, globally stabilizing a point (or set) with two nonlinear output-feedback hybrid controllers is considered. These control laws may have different objectives, e.g., both closed-loop systems may have different attractors. We provide a control algorithm that combines the two hybrid controllers to accomplish the task. It consists of a hybrid supervisor that, based on the values of plant’s outputs and state estimates, selects the hybrid controller that should be applied to the plant. The accomplishment of the stabilization task relies on an output-to-state stability property induced by the controllers, which enables us to construct an estimator for the norm of the plant’s state.

I. INTRODUCTION

Many control applications cannot be solved by means of a single state-feedback controller. As a consequence, control algorithms combining more than one controller have been thoroughly investigated in the literature. Particular attention has been given to the problem of uniting local and global controllers, in which two control laws are used: one that is supposed to work only locally, perhaps guaranteeing good performance, and another that is capable of steering the system trajectories to a neighborhood of the operating point, where the local control law works; see, e.g., [20], [11], and [5]. More recently, these ideas have been extended in [15] to allow for the combination of more than two state-feedback laws as well as open-loop control laws. They have also been extended to the case when, rather than state-feedback, only output-feedback controllers are available [12].

The motivation of this paper is two fold. On the one hand, the impossibility of robustly, locally stabilizing an equilibrium point (or set) with smooth or discontinuous control laws (see, e.g., [13], [2]) precludes utilizing uniting controllers that combine smooth or discontinuous (non-hybrid) state-feedback laws. On the other hand, the typical limitation of measuring all of the plant variables for state-feedback control demands the use of output-feedback controllers. Building from the ideas in [12] on uniting output-feedback controllers and the supervisory control algorithms in [10], [9], [16], we propose a hybrid controller to solve the problem of uniting two output-feedback hybrid controllers with different objectives, one considered local and the other one global. Each of the output-feedback hybrid controllers is known to confer certain properties to each of the resulting closed-loop systems: the local controller renders, for the plant state, a target compact set locally asymptotically stable, while the global controller renders, for the plant state, a particular compact set globally attractive. The proposed supervisor controller for each of these hybrid control algorithms is shown to solve the uniting problem when the individual closed-loop systems are output-to-state stable (cf. [12], [4]). Our construction exploits the fact that, as established in [19] for continuous-time nonlinear systems and generalized to hybrid systems in [3], this property implies the existence of an estimator of the norm of the state. By combining a discrete state and a timer state, we design a robustly, globally stabilizing hybrid supervisor. We work on the hybrid systems framework of [7] (see also [6], [8]) and employ results on robust asymptotic stability in [8].

The remainder paper is organized as follows. After basic notation is introduced, Section II presents a short description of the framework used for analysis. The main results follow in Section III. It starts by introducing the problem to be solved, the proposed formulation of a solution, and the required assumptions. After presenting a detailed design procedure for the supervisor, it establishes the main robust stability properties of the closed-loop system. The design procedure is exercised in an example.

Notation

We use the following notation and definitions throughout the paper. $\mathbb{R}^n$ denotes $n$-dimensional Euclidean space. $\mathbb{R}_{\geq 0}$ denotes the nonnegative real numbers, i.e., $\mathbb{R}_{\geq 0} = [0, \infty)$. $\mathbb{N}$ denotes the natural numbers including 0, i.e., $\mathbb{N} = \{0, 1, \ldots\}$. $B$ denotes the open unit ball in Euclidean space. Given a set $S$, $\overline{S}$ denotes its closure. Given a set $S \subset \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$, $|x| := \inf_{y \in S} |x-y|$. Given a vector $x \in \mathbb{R}^n$, $|x|$ denotes the Euclidean vector norm. A function $\alpha : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is said to belong to the class $K_L$ if it is continuous, zero at zero, and strictly increasing. A function $\alpha : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is said to belong to the class $K无限$ if it is continuous, zero at zero, and strictly increasing. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is said to belong to the class $K_L$ if it is continuous, zero at zero, and strictly increasing. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is said to belong to the class $K_L$ if it is continuous, zero at zero, and strictly increasing.

II. PRELIMINARIES

In this paper, we consider hybrid systems as in [7] (see also [6], [8]) where solutions can evolve continuously (flow) and/or discretely (jump) depending on the continuous and discrete dynamics and the sets where those dynamics apply. In general, a hybrid system $\mathcal{H}$ is given by data $(F, C, G, D)$
and can be written in the compact form
\[ H \left\{ \begin{array}{ll}
\dot{x} & \in F(x) \quad x \in C \\
{\chi}^{+} & \in G(x) \quad x \in D,
\end{array} \right. \]
where \( \chi \in \mathbb{R}^{n} \) is the state taking values in a Euclidean space \( \mathbb{R}^{n} \), the set-valued map \( F \) defines the continuous dynamics on the set \( C \) and the set-valued map \( G \) defines the discrete dynamics on the set \( D \). The notation \( {\chi}^{+} \) indicates the value of the state \( \chi \) after a jump.

A set \( E \) is a hybrid time domain if for all \( (T, J) \in E, E \cap ([0, T] \times \{0, 1, \ldots, J\}) \) is a compact hybrid time domain, i.e., it can be written as \( \bigcup_{j=0}^{J} ([t_{j}, t_{j+1}], j) \) for some finite sequence of times \( 0 = t_{0} \leq t_{1} \ldots \leq t_{J} \). A solution \( \chi \) to \( H \) is a hybrid arc \( \chi \) consisting of a hybrid time domain \( \text{dom} \chi \) and a function \( \chi : \text{dom} \chi \rightarrow \mathbb{R}^{n} \) such that \( \chi(t, j) \) is locally absolutely continuous on \( I_{j} := \{ t \in (t, j) : \dot{x} \in \text{dom} \chi \} \) for each \( j \in \mathbb{N} \) and satisfies:

(S1) for each \( j \in \mathbb{N} \) such that \( I_{j} \) has nonempty interior
\[ \chi(t, j) \in C \quad \text{for all} \quad t \in [\min I_{j}, \sup I_{j}] \]
\[ \chi(t, j) \in F(\chi(t, j)) \quad \text{for almost all} \quad t \in I_{j}, \] (1)

(S2) for each \( (t, j) \in \text{dom} \chi \) such that \( (t, j + 1) \in \text{dom} \chi \),
\[ \chi(t, j) \in D, \quad \chi(t, j + 1) \in G(\chi(t, j)). \] (2)

Then, the state trajectory \( \chi \) is parameterized by \( (t, j) \), where \( t \) is the ordinary time and \( j \) is an independent variable that corresponds to the number of jumps of the solution.

A solution \( \chi \) to \( H \) is said to be complete if \( \text{dom} \chi \) is unbounded, Zeno if it is complete but the projection of \( \text{dom} \chi \) onto \( \mathbb{R}_{\geq 0} \) is bounded, and maximal if there does not exist another hybrid arc \( \chi' \) such that \( \chi \) is a truncation of \( \chi' \) to some proper subset of \( \text{dom} \chi' \). For more details about this hybrid systems framework, we refer the reader to [7].

When the data \( (F, C, G, D) \) of \( H \) satisfies certain conditions, which we refer to as hybrid basic conditions, hybrid systems are well posed in the sense that they inherit several good structural properties of their solution sets. These include sequential compactness of the solution set, closedness of perturbed and unperturbed solutions, etc. We refer the reader to [8] (see also [6]) and [18] for details on and consequences of these conditions.

**Definition 2.1:** (Well-posed hybrid systems) The hybrid system \( H \) with data \( (F, C, G, D) \) is said to be well posed if it satisfies the following hybrid basic conditions: the sets \( C \) and \( D \) are closed, the mappings \( F : C \rightrightarrows \mathbb{R}^{n} \) and \( G : D \rightrightarrows \mathbb{R}^{n} \) are outer semicontinuous and locally bounded, \( F(x) \) is nonempty and convex for all \( x \in C \), and \( G(x) \) is nonempty for all \( x \in D \).

For a hybrid system \( H = (C, F, D, G) \), the compact set \( A \subset \mathbb{R}^{n} \) is said to be

- Stable if for each \( \epsilon > 0 \) there exists \( \delta > 0 \) such that each solution \( \chi \) to \( H \) with \( |\chi(0, 0)|_{A} \leq \delta \) satisfies \( |\chi(t, j)|_{A} \leq \epsilon \) for all \( (t, j) \in \text{dom} \chi \);
- Locally attractive if there exists \( \mu > 0 \) so that every solution \( \chi \) to \( H \) with \( |\chi(0, 0)|_{A} \leq \mu \) is complete and satisfies \( \lim_{t 
rightarrow +\infty} |\chi(t, j)|_{A} = 0 \), and globally attractive if every solution \( \chi \) to \( H \) is complete and satisfies \( \lim_{t 
rightarrow +\infty} |\chi(t, j)|_{A} = 0 \);
- Locally asymptotically stable if it is both stable and locally attractive, and globally asymptotically stable if it is both stable and globally attractive.

The *basis of attraction* of an asymptotically stable set \( A \) is the set of points from which every solution is complete and converges to \( A \). Note that, under the hybrid basic conditions, points not in \( C \cup D \) are automatically in the basin of attraction since there is nothing to be checked. For results about asymptotically stable compact sets for hybrid systems, see Section VI in [8] and Section VI and VII in [17].

The following output-to-state stability definition for hybrid systems \( H \) with an output was introduced in [3]: a hybrid system \( H_{y} \) with data \( (h, C, F, D, G) \) and given by
\[ H_{y} \left\{ \begin{array}{ll}
\dot{x} & \in F(x) \quad x \in C, \\
{\chi}^{+} & \in G(x) \quad x \in D, \\
y & = h(x) \end{array} \right. \]
\[ \text{is output-to-state stable (OSS) with respect to a set} \ A \subset \mathbb{R}^{n} \text{if there exist a class-K-Lyapunov function} \ \beta \text{and a class-K function} \ \rho \text{such that each maximal solution} \ \chi \text{to} \ H_{y} \text{ satisfies, for all} \ (t, j) \in \text{dom} \chi, \]
\[ |\chi(t, j)|_{A} \leq \max \left\{ \beta(|\chi(0, 0)|_{A}, t, j), \rho(\|y\|(t, j)) \right\}, \]
where, for each \( (t, j) \in \text{dom} y \),
\[ \|y\|(t, j) := \sup_{(t', j') \in \text{dom} y} |y(t', j')|. \] (3)

From the definition of \( y \), we have \( y(t, j) = h(\chi(t, j)) \).

### III. UNIFYING TWO OUTPUT FEEDBACK CONTROLLERS

#### A. Problem statement and assumptions

We consider robust, global stabilization of a compact set for nonlinear control systems of the form
\[ \mathcal{P} : \quad \dot{\xi} = f_{p}(\xi, u_{p}) \quad \xi \in \mathbb{R}^{m_{p}}, \quad u_{p} \in \mathbb{R}^{m_{p}}, \] (4)
with only measurements of two outputs given by functions of the state \( h_{0} \) and \( h_{1} \). That is, we are interested in solving the following problem:

\( \ast \) Given a compact set \( A \subset \mathbb{R}^{m_{p}} \) and continuous functions \( h_{0}, h_{1} \) defining outputs \( h_{0}(\xi), h_{1}(\xi) \) of (4), design an output feedback controller \( K_{e} \) that renders \( A \) robustly globally asymptotically stable.

For starters, we assume there exist two hybrid controllers, denoted \( K_{0} \) and \( K_{1} \), with “local” and “global” stabilizing capabilities, respectively, which are properties that will be made precise later. The controllers have state \( \zeta_{0} \) and \( \zeta_{1} \), both
in $\mathbb{R}^{n_r}$, respectively. For each $i = 0, 1$, the hybrid controller $K_i$ takes the form $K_i = (\kappa_i, \xi_i, C_i, \eta_i, f_i, D_i, g_i)$, where $
abla_c$ is the controller's state, whose solutions are defined on hybrid time domains.

$$\begin{align*}
\forall i = 0, 1, \kappa_i &= \frac{\partial}{\partial t} x_i(t, j) + \frac{\partial}{\partial j} x_i(t, j) + \frac{\partial}{\partial x_i(t, j)} + g_i x_i(t, j),
\forall i = 0, 1, \xi_i &= \frac{\partial}{\partial t} \xi_i(t, j) + \frac{\partial}{\partial j} \xi_i(t, j) + \frac{\partial}{\partial \xi_i(t, j)} + f_i \xi_i(t, j),
\end{align*}$$

(5)

where $\xi_i(t, j)$ are subsets of $\mathbb{R}^{n_r}$ for the plant output $y_p$.

By construction and well posedness of the hybrid system $(P, K_0)$, there exist a class $\mathcal{K}$ function $\beta$ and a class $\mathcal{L}$ function $\rho$ such that for each solution $(\xi, \zeta)$ to $(P, K_0)$, $(\xi(0), \zeta(0,0))$ exists in $\mathbb{R}^{n_r}$.

$$\begin{align*}
\|\xi(t, j)\| &\leq \max \left\{ \beta\left(\|\xi(0,0), \zeta(0,0)\| \right) , |A_t|, t, j, \rho \right\},
\end{align*}$$

(8)

for all $(t, j) \in \text{dom}(\xi, \zeta)$. Then, under the stated assumptions, as established in [3, Theorem 1], the output-to-state stability property in (8) implies that there exists an exponential-decay OSS-Lyapunov function with respect to $A_t \times \Phi_t$ for $(P, K_0)$. As defined in [3, Definition 5], this smooth function $V_t : R^n \times \mathbb{R}^2$ is such that there exist class $\mathcal{K}$ functions $\alpha_{i,1}, \alpha_{i,2}, (i, \zeta) \in \text{dom}(\zeta, \zeta)$, satisfying for all $(\xi, \zeta) \in \mathbb{R}^{n_r}$,

$$\begin{align*}
\alpha_{i,1}(\|\xi, \zeta\|) \leq V_t(\xi, \zeta) \leq \alpha_{i,2}(\|\xi, \zeta\|),
\end{align*}$$

(9)

for all $(\xi, \zeta) \in E_1$, such that:

1) A well-posed hybrid controller $K_0 = (\kappa_{c,0}, C_{c,0}, f_{c,0}, D_{c,0}, g_{c,0})$ for the plant output $y_p = h_0(\xi)$ inducing the following properties exists:

2The case where the hybrid controllers have a dynamical state $\xi_0$ (respectively $\zeta_0$) in a set $\mathbb{R}^{n_{c,0}}$ (resp. $\mathbb{R}^{n_{c,1}}$) of different dimension $n_{c,0} \neq n_{c,1}$ can be treated by embedding both sets into the set of larger dimension $\mathbb{R}^{n_{c,1}}$ with $n_{c,1} = \max\{n_{c,0}, n_{c,1}\}$. 

$$a) \text{Stability: For each } \varepsilon_0 > 0 \text{ there exists } \delta_0 > 0 \text{ with } ||(\xi, 0)(0)| \leq \delta_0 \text{ satisfies } ||(\xi(t, j), 0)(t, j)| \leq \varepsilon_0 \text{ for all } (t, j) \in \text{dom}(\xi, \zeta) \text{.}$$

$$b) \text{Local attractiveness: There exists } \mu > 0 \text{ such that every solution } (\xi, \zeta) \text{ to } (P, K_0) \text{ with } ||(\xi(0,0), \zeta(0,0))| \leq \mu \text{ is complete and satisfies}$$

$$\lim_{t \to \infty} [(\xi(t, j), \zeta(t, j))]|_{A_t \times \Phi_t} = 0;$$

$$c) \text{Output-to-state stability: The hybrid system } (P, K_0) \text{ with output } y_{p,0} = h_0(\xi) \text{ is output-to-state stable with respect to } A_t \times \Phi_t;$$

and have the property that $A_t$ and the basin of attraction of the hybrid controller $K_0$, denoted $B_0$, satisfy, for some $\mu > 0$,

$$\begin{align*}
\mathcal{A}_1 + \mu \mathbb{B} &\subseteq \{ \xi \in \mathbb{R}^{n_r} : (\xi, \zeta) \in \mathcal{B}_1 \},
\end{align*}$$

(7)

Remark 3.2: By construction and well posedness of the hybrid controller $K_i$, the hybrid system $(P, K_i)$ is well posed for each $i = 0, 1$. Assumption 3.1.1.c and Assumption 3.1.2.b imply that, for each $i = 0, 1$, there exist a class $\mathcal{K}$ function $\beta_i$ and a class-$\mathcal{L}$ function $\rho_i$ such that for each solution $(\xi, \zeta)$ to $(P, K_i)$, $(\xi(0), \zeta(0,0))$ exists in $\mathbb{R}^{n_r}$.

$$\begin{align*}
\|\xi(t, j)\| &\leq \max \left\{ \beta_i\left(\|\xi(0,0), \zeta(0,0)\| \right) , |A_t|, t, j, \rho_i\left(\|y_p, i\| \right) \right\},
\end{align*}$$

(10)

for all $(\xi, \zeta) \in D_i$.

$$\begin{align*}
\text{max } V_{t}(g) - V_{i}(\xi, \zeta) &\leq \text{max } \left\{ \beta_i\left(\|\xi(0,0), \zeta(0,0)\| \right) , |A_t|, t, j, \rho_i\left(\|y_p, i\| \right) \right\},
\end{align*}$$

(11)

The plant state $\xi$ is parameterized by $(t, j)$ since it is a component of the closed-loop hybrid system's state $x$, whose solutions are defined on hybrid time domains.
As stated in [3, Proposition 2], a norm estimator for the state $(\xi, \zeta_i)$ (and, hence, for $\xi$) exists. A particular construction is

$$
\dot{z}_i = -\varepsilon_i z_i + \gamma_i |h_i(\xi)| \\
\quad (\xi, \zeta_i) \in C_i, \\
z_i^+ = (1 - \varepsilon_i) z_i + \gamma_i |h_i(\xi)| \\
\quad (\xi, \zeta_i) \in D_i. 
$$

(12)

In fact, given a solution $(\xi, \zeta_i)$ to $(P, K_i)$, with (10) and (11), we obtain, using the upperbound in (9), for all $(t, j) \in \text{dom}(\xi, \zeta_i)$,

$$
V_i(\xi(t, j), \zeta_i(t, j)) \leq z_i(t, j) + \exp(-\varepsilon_i t)(1 - \varepsilon_i)j \\
\alpha_{i,2}(|(\xi(0,0), \zeta_i(0,0))|, \Phi_i + |z_i(0,0)|, t, j). 
$$

Assuming, without loss of generality, that $\alpha_2(s) \geq s \forall s \geq 0$ and defining $\beta_i(s, t, j) := 2 \exp(-\varepsilon_i t)(1 - \varepsilon_i)j \alpha_{i,2}(s)$ gives

$$
V_i(\xi(t, j), \zeta_i(t, j)) \leq z_i(t, j) + \beta_i(|(\xi(0,0), \zeta_i(0,0))|, \Phi_i + |z_i(0,0)|, t, j). 
$$

(13)

We impose mild regularity conditions on the nominal model of the plant $P$ in (4).

**Assumption 3.3:** The functions $f_p : \mathbb{R}^{n_p} \times \mathbb{R}^{m_p} \rightarrow \mathbb{R}^{n_p}$, $h_0 : \mathbb{R}^{n_p} \rightarrow \mathbb{R}^{m_0}$, and $h_1 : \mathbb{R}^{n_p} \rightarrow \mathbb{R}^{m_1}$ are continuous.

For analysis of robustness, the following model of the plant with perturbations is considered

$$
\dot{\xi} = f_p(\xi, u + d_1) + d_2, 
$$

(14)

with outputs $y_p, 0 = h_0(\xi) + d_3$ and $y_p, 1 = h_1(\xi) + d_4$, where $d_1$ corresponds to actuator error, $d_2$ models unmodeled dynamics, and $d_3, d_4$ represent measurement noise.

In the next section, we provide a solution to problem (*) that consists of a hybrid controller coordinating, using control logic and norm observers, the two (well-posed) output-feedback hybrid controllers $K_0$ and $K_1$.

**B. Controller design**

We propose a hybrid controller to supervise $K_0, K_1$. This hybrid controller, referred to as the hybrid supervisor, is denoted $K_s$ and designed to perform the uniting task as follows:

A) Apply the hybrid ("global") controller $K_1$ when the estimate of $|\xi|_{A_1}$ is away from the origin.

B) Permit estimate of $|\xi|_{A_1}$ to converge.

C) Apply $K_0$ when the estimate of $|\xi|_{A_1}$ are close enough to zero.

To accomplish these tasks, the supervisor has a discrete state $q \in Q := \{0, 1\}$ and a timer state $\tau \in \mathbb{R}$ with parameter $\tau^* > 0$. The dynamics of the state $q$ are designed to guarantee that the controller $K_q$ is connected to the plant. We now describe the control mechanisms in the hybrid supervisor.

1) Global Controller ($q = 1$): Assumption 3.1.2.a implies that for every solution $(\xi, \zeta_i)$ to $(P, K_1)$, we have

$$
\lim_{t \rightarrow +\infty} \gamma_1(|h_i(\xi(t, j))|) = 0.
$$

Using (12) for $i = 1$, it follows that $z_1$ also approaches zero, and that, eventually, when $t$ and $j$ are large enough, $|\xi|_{A_1}$ is small enough. This suggests that the supervisor should apply $K_1$ until, eventually, $z_1$ is small enough. This can be implemented as follows:

- Flow according to

$$
\dot{\xi} = f_p(\xi, \kappa_{c,1}(h_1(\xi), \zeta_i)) \\
\dot{\zeta}_i = f_{c,1}(h_1(\xi), \zeta_i) \\
\dot{z}_i = -\varepsilon_i z_i + \gamma_i |h_i(\xi)| \\
\dot{q} = 0 
$$

(15)

when, for a design parameter $\varepsilon_{1a} > 0$,

$$
z_0 = 0 \quad \text{and} \quad z_1 \geq \varepsilon_{1a} \quad \text{and} \quad q = 1, \quad \text{or}
$$

$$
z_0 = 0 \quad \text{and} \quad z_1 \geq 0 \quad \text{and} \quad \tau \leq \tau^* \quad \text{and} \quad q = 1. \quad \text{(17)}
$$

- Jump according to

$$
\zeta_0^+ \in \Phi_0, \quad \zeta_1^+ \in \Phi_1, \quad z_0^+ = 0, \quad z_1^+ = 0, \quad q^+ = 0 \quad \text{(18)}
$$

when

$$
z_0 = 0 \quad \text{and} \quad 0 \leq z_1 \leq \varepsilon_{1a} \quad \text{and} \quad \tau \geq \tau^* \quad \text{and} \quad q = 1. \quad \text{(19)}
$$

The flows defined in (15) enforce, in particular, that $q$ remains constant and that the estimate of $|\xi|_{A_1}$ converges. Condition (16) allows jumps when the estimate of $|\xi|_{A_1}$ is not small enough, while, when condition (19) holds, the state $q$ is set to 0 so that $K_0$ is applied. The state $\zeta_0$ of the "local" controller is updated to a point in $\Phi_0$ and the estimator state $z_0$ is reset to zero. These selections are to properly initialize $K_0$. Note that the values to which $z_1$ and $\zeta_1$ are updated are not important since the dynamics of $K_0$ do not depend on them. Due to the impossibility of measuring $\xi$, it is not possible to ensure that $\xi$ is such that $(\xi, \zeta_0)$ is in the basin of attraction $B_0$ after jumps from $q = 1$ to $q = 0$ occur. Hence, it could be the case that there are jumps from $q = 0$ back to $q = 1$. Condition (17) enforces that, perhaps after a few jumps to $q = 0$ and back to $q = 1$, $\xi$ eventually is so that $(\xi, \zeta_0)$ is in the said basin of attraction by allowing the estimate $|\xi|_{A_1}$ to converge. The conditions $z_0 = 0$ in (16), (17), and (19) force $z_0$ to remain nonnegative along solutions.

2) Local Controller ($q = 0$): The local asymptotic stability property assured by Assumption 3.1.1.a and b guarantees that there exists $\varepsilon_{0b} > 0$ such that

$$
\{(\xi, \zeta_0) : V_0(\xi, \zeta_0) \leq \varepsilon_{0b}\} \quad \text{(20)}
$$

is a subset of the basin of attraction $B_0$ for the asymptotic stabilization of $\mathcal{A}$ with $K_0$. Moreover, from condition (7) in Assumption 3.1, it follows that there exists $\varepsilon_{0a} > 0$ and $\varepsilon_{1b} > 0$ such that, for each solution $(\xi, \zeta_0)$ to $(P, K_0)$ from

$$
\{(\xi, \zeta_0) : V_1(\xi, \zeta_0) \leq \varepsilon_{1b}, \quad \zeta_1 \in \Phi_1 \times \Phi_0\} \quad \text{(21)}
$$

we have $\gamma_0(|h_0(\xi(t, j))|) < \varepsilon_{0a} \forall (t, j) \in \text{dom}(\xi, \zeta_0)$. Note that from (12) for $i = 0$ it follows that $z_0$ approaches $\gamma_0(|h_0(\xi(t, j))|)$ along solutions. Furthermore, when $z_0 \leq \varepsilon_{0a}$, $\zeta_0 \in \Phi_0$, and $t$ and $j$ are large enough, it follows from (13) for $i = 0$ that after jumps to $q = 0$, $(\xi, \zeta_0)$ will be in the set (20). Then, the supervisor is designed to apply $K_0$ as long as $z_0$ is smaller or equal than $\varepsilon_{0a}$, and when is larger or equal to that parameter, a jump to $q = 1$ is triggered. Note that the logic for $q = 1$ eventually forces flows for at least $\tau^*$
units of time, which allows \( t \) and \( j \) to become large enough, and with that, guarantee that \((\xi, \zeta_0)\) is in the set \((20)\). This mechanism is implemented as follows:

- **Flow according to**
  \[
  \begin{align*}
  \dot{\xi} &= f_p(\xi, \kappa_{c,0}(h_0(\xi), \zeta_0)) \\
  \dot{\zeta}_0 &= f_{c,0}(h_0(\xi), \zeta_0) \\
  \dot{z}_0 &= -\varepsilon_0 z_0 + \gamma_0(h_0(\xi)) \\
  \dot{q} &= 0 
  \end{align*}
  \]  
  \((22)\)

  when \( 0 \leq z_0 \leq \varepsilon_{0a} \) and \( z_1 \geq 0 \) and \( q = 0 \).  

- **Jump according to**
  \[
  \begin{align*}
  \zeta^+_0 \in \Phi_0, \ z^+_1 \in \Phi_1, \ z^+_0 = 0, \ z^+_1 = 0, \ q^+ = 1. 
  \end{align*}
  \]  
  \((24)\)

  As \((15)\), the flows defined in \((22)\) enforce, in particular, that \( g \) remains constant and that the estimate of \( |\xi|_{A_6} \) converges. While optimal choices might be possible to maximize performance, the values to which \( \zeta_0, z_0, z_1, \) and \( z_1 \) are updated to at jumps are not crucial since the controller \( \mathcal{K}_1 \) enforces a global attractivity property. The condition \( z_1 \geq 0 \) in \((23)\) and \((25)\) forces \( z_1 \) to be nonnegative along solutions.

3) **Closed-loop system:** We are now ready to write the resulting closed loop as a hybrid system. Let \( \varepsilon_{0a}, \varepsilon_{1a}, \) and \( \tau^* \) be design parameters selected as suggested in Sections III-B.1 and III-B.2. The closed-loop hybrid system has state \( \chi = (\xi, \zeta_0, \zeta_1, z_0, z_1, q, \tau) \) taking values in \( \chi := \mathbb{R}^n_x \times \mathbb{R}^{n_c} \times \mathbb{R}^{n_\nu} \times \mathbb{R}^n_v \times \mathbb{R}^n_{\zeta} \times \mathbb{R}^\tau \). Collecting the definitions in Sections III-B.1 and III-B.2, the resulting closed-loop system, denoted as \( \mathcal{H}_{cl} \), has dynamics given as follows:

\[
\begin{align*}
\dot{\chi} &= \begin{bmatrix}
  f_p(\xi, \kappa_{c,0}(h_0(\xi), \zeta_0)) \\
  (1-q)f_{c,0}(h_0(\xi), \zeta_0) \\
  g f_{c,1}(h_1(\xi), \zeta_1) \\
  (1-q)(-\varepsilon_0 z_0 + \gamma_0(h_0(\xi))) \\
  q(-\varepsilon_1 z_1 + \gamma_1(h_1(\xi))) \\
  0 \\
  q
\end{bmatrix} =: F(\chi),
\end{align*}
\]

\( \chi \in \bar{C} \),

\[
\chi^+ \in \mathcal{G}_0(\chi) \cup \mathcal{G}_1(\chi) \cup \mathcal{G}_s(\chi) =: \mathcal{G}(\chi), \quad \chi \in \bar{D},
\]

where: for each \( q = 0, (\xi, \zeta_0) \in D_q

\[
\mathcal{G}_0(\chi) = \begin{bmatrix}
  \xi \\
  g_{c,0}(h_0(\xi), \zeta_0) \\
  \zeta_1 \\
  (1-\varepsilon_0)z_0 + \gamma_0(h_0(\xi)) \\
  z_1 \\
  q \\
  \tau
\end{bmatrix},
\]

\[
\mathcal{G}_1(\chi) = \begin{bmatrix}
  \xi \\
  \zeta_0 \\
  g_{c,1}(h_1(\xi), \zeta_1) \\
  (1-\varepsilon_1)z_1 + \gamma_1(h_1(\xi)) \\
  q \\
  \tau
\end{bmatrix},
\]

\( \mathcal{G}_s(\chi) = \emptyset \) otherwise; for each \( q = 1, (\xi, \zeta_0) \in D_q

\[
\mathcal{G}_0(\chi) = \emptyset \) otherwise; for each \( q = 1, (\xi, \zeta_0) \in D_q

The flow map \( F \) is defined in terms of the discrete state \( q \) to “select” the appropriate flow dynamics when \( K_0 \) and \( K_1 \) are applied. In particular, when \( q = 0, \zeta_1 \) and \( z_1 \) remain constant during flows while, when \( q = 1, \zeta_0 \) and \( z_0 \) remain constant. The flow set \( \bar{C} \) allows flow when both \( (\xi, \zeta_0) \) is in the flow set \( C_q \) and the conditions for flow imposed by the hybrid supervisor are satisfied. The latter are given in \((23), (17), \) and \((16)\), which are captured in the sets \( C_{s,a}, C_{s,c}, \) and \( C_{s,b}, \) respectively. The jump maps \( \mathcal{G}_0, \mathcal{G}_1, \) and \( \mathcal{G}_s \) above are defined to execute the jumps of the individual hybrid controllers when their state jumps due to \( \zeta_q \in D_{c,q} \) or when reset of the appropriate states is required by the supervisor jump sets \( D_{s,a} \) and \( D_{s,b} \), which are given in \((25)\) and \((19)\), respectively. Note that since \( g_{c,q} \) is only defined on \( D_{c,q} \), the set-valued maps \( \mathcal{G}_0 \) and \( \mathcal{G}_1 \) are nonempty at points \( \chi \) with \( \zeta_q \) components in \( D_{c,q} \).

\subsection{Closed-loop system properties}

\textbf{Theorem 3.4: (Nominal global asymptotic stability)}

Suppose Assumptions 3.1 and 3.3 hold. Then, there exist a hybrid controller \( \mathcal{K}_s \) that provides a solution to the uniting problem (*). Moreover, the hybrid controller \( \mathcal{K}_s \) is such that, for the hybrid system \( \mathcal{H}_{cl} \), every solution is bounded and complete, and the compact set

\[
\mathcal{A}_s := \mathcal{A} \times \Phi_0 \times \Phi_1 \times \{0\} \times \{0\} \times \mathcal{Q} \times [0, \tau^*]
\]

is globally asymptotically stable.

With the plant in \((14)\) under the presence of perturbations, the resulting closed-loop system \( \mathcal{H}_{cl} \) results in a perturbed hybrid system, denote as \( \mathcal{H}_{cl} \), which can be written as

\[
\begin{align*}
\dot{\chi} &= F(\chi + \tilde{d}_1) + \tilde{d}_2, \quad \chi + \tilde{d}_1 \in \bar{C} \\
\chi^+ &\in \mathcal{G}(\chi + \tilde{d}_1) + \tilde{d}_2, \quad \chi + \tilde{d}_1 \in \bar{D}.
\end{align*}
\]
The following result asserts that the closed-loop system is robust to a class of perturbations. It follows from the asymptotic stability property established in Theorem 3.4 and the fact that the construction of the hybrid supervisor is such that the hybrid basic conditions of Definition 2.1 hold for the closed-loop system $\mathcal{H}_{cl}$, that is, $K_a$ is well posed.

**Theorem 3.5: (Nominal robustness)** Suppose Assumptions 3.1 and 3.3 hold. Then, for the hybrid system $\mathcal{H}_{cl}$, there exists $\beta \in K\mathcal{L}\mathcal{L}$, for each $\epsilon > 0$ and each compact set $K \subset \mathbb{R}^n$ there exists $\delta > 0$ such that for each measurable $d_1, d_2 : \mathbb{R} \rightarrow \delta \mathbb{B}$ every solution $\chi$ to $\mathcal{H}_{cl}$ from compact subsets of $X$ with $x(0,0) \in K$ satisfies

$$|x(t,j)|_{A} \leq \beta(|x(0,0)|_{A}, t, j) + \epsilon \quad \forall (t,j) \in \text{dom} \chi.$$  

**D. Illustrative Example**

Consider the global stabilization of the origin of $^4$

$$\dot{\xi} = f_p(\xi, u_p) := \left[ \begin{array}{c} -\xi_1 + (u_1 - \xi_2)\xi_2^2 \\ -\xi_2 + \xi_1^2 + \alpha + u_2 \end{array} \right],$$

(26)

where $\xi \in \mathbb{R}^2$ is the state, $u_p = [u_1 \ u_2]^T$ is the control input constrained to $u_1 u_2 = 0$, and $\alpha \in \left[ -\frac{1}{2}, \frac{1}{2} \right]$. Measurements of $\xi_1$ and $\xi_2$ are available but not simultaneously. The proposed hybrid controller $K_a$ is used to coordinate two controllers, $K_0$ and $K_1$. The “local” controller $K_0$ measures $h_0(\xi) = \xi_1$ and stabilizes $\xi$ to $A_0 = (0,0)$. It is given as in (5) with $\kappa_{c,0}(\xi) := [0, -\alpha]^T$ on the set $C_{c,0} = \mathbb{R}$, $n_c = 0$, and empty set $D_{c,0}$. For $V_0(\xi) = \frac{1}{2}\xi^2$, it follows that

$$\langle \nabla V_0(\xi), f_p(\xi, \kappa_{c,0}(\xi)) \rangle = -\xi_1^2 - \xi_2^2 - \xi_1 \xi_2 + \xi_1 \xi_2^2 \leq -V_0(\xi) + \xi_1^2 (1 + \xi_2^2)$$

(27)

and thus, by letting $\gamma_0(s) = s^4(1 + s^2)$ for all $s \geq 0$, a norm observer for $[\xi]_{A_0}$ is given by $\dot{z}_0 = -z_0 + \gamma_0(h_0(\xi))$, where $z_0 = \xi_1$. It can be verified that $\{\xi : V_0(\xi) = \frac{1}{2}\xi^2 \} \subset B_0$. A “global” controller $K_1$ can be designed to steer the solutions to $A_1 := (0, 0, \alpha)$. From (27), it follows that $\{0, \alpha\}$ belongs to the interior of $B_0$. Let $h_1(\xi) := \xi_2 - \alpha$. The controller $K_1$ is given as in (5) with $\kappa_{c,1}(\xi) := [h_1(\xi) + \alpha, 0]^T$ on the set $C_{c,1} = \mathbb{R}$, $n_c = 0$, and empty set $D_{c,1}$. With this controller, the function $V_1(\xi) = \frac{1}{2}\xi_1^4 + \frac{1}{2}(\xi_2 - \alpha)^2$ satisfies

$$\langle \nabla V_1(\xi), f_p(\xi, \kappa_{c,1}(\xi)) \rangle \leq -V_1(\xi),$$

from where a norm observer for $[\xi]_{A_1}$ follows. Then, Assumptions 3.1 and 3.3 hold and the hybrid supervisor $K_a$ robustly, globally asymptotically stabilizes the origin of (26).

Figure 1 shows a trajectory to the closed-loop system when $\alpha = \frac{1}{4}$, $\varepsilon_{d_0} = \varepsilon_{d_1} = 0.01$, and $\tau^* = 1$.

**REFERENCES**


