Equivalent Conditions for Uniform Asymptotic Consensus Among Distributed Agents

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Abstract—A set of conditions equivalent to uniform asymptotic consensus for distributed agents involving switched stability theory, linear matrix inequalities, and graph-theoretic notions has been established. These conditions are presented for both leaderless and leader-follower situations and extend previous results to wider classes of multi-agent systems. In particular, the uniformity requirement imposed on the convergence rate of mixed matrix products not only plays a crucial role in theoretical developments, but it also meets the practical needs of reaching consensus robustly against disturbances.

I. INTRODUCTION

The study of multi-agent asymptotic consensus problems has received continued attention in recent years due to their applications in varied fields such as distributed formation control, study of flocking theory, synchronization of coupled oscillators, etc [1]. Based on linear time-varying state-space models of multi-agent systems, sufficient conditions for the asymptotic convergence of the states of the agents to a common value was established in [2]–[4] under undirected connectivity graphs, and in [5] under directed connectivity graphs. Moreover, an extension of these results to a class of nonlinear time-varying state-space models was presented in [6]. A common sufficient condition for asymptotic consensus is that the union of the connectivity graphs over every time interval of a certain length must form a jointly connected graph in some sense.

A connection between the asymptotic consensus problems for multi-agent systems, where the connectivity graph keeps jumping from one to another over time, and the asymptotic stability analysis problems for switched linear systems, where the system dynamics keeps switching from one to another over time, was first observed in [3]. Then, based on recent advances in the stability analysis of switched linear systems, an exact characterization of uniform asymptotic consensus of multi-agent systems has been obtained in [7]. The uniformity requirement strengthens the standard notion of asymptotic consensus by requiring the existence of a single rate of convergence that is valid over all initial times; this requirement in fact allows the agents to remain near consensus even under persistent disturbances, which is not guaranteed under mere asymptotic consensus. It was found in [7] that the joint connectedness condition in [3] is in fact an exact condition for uniform asymptotic consensus under undirected graphs.

Asymptotic stability analysis of switched linear systems gives us further characterizations of uniform asymptotic consensus. Such a characterization can be given in terms of the feasibility of a certain set of linear matrix inequalities. Another characterization can be made by the study of joint spectral radius, which generalizes the notion of spectral radius to multiple matrices. These results were implicit in [7] under undirected connectivity graphs in leaderless situations.

In this paper, we extend all the above-mentioned equivalent characterizations of leaderless uniform asymptotic consensus to directed graphs. Moreover, we consider the leader-follower situations [3], [5] as well, and extend our equivalence result to a wider class of matrices. In particular, while in the previous studies of this kind the state matrices were constrained to have strictly positive diagonal entries, we allow the state matrices to have possibly zero diagonal entries. As a numerical example shows, this enables us to analyze asymptotic consensus under a realistic scenario, which previous results do not cover.

The organization of the paper is as follows. In Section 2, we present basic definitions and lemmas required to develop the main result in the next section. Section 3 gives several different characterizations of uniform asymptotic consensus in both leaderless and leader-follower situations. This result is illustrated by examples in Section 4, and its proof is sketched in Section 5. Lastly, concluding remarks are made in Section 6.

Notation. The set of real numbers is denoted by \( \mathbb{R} \) and the set of nonnegative integers by \( \mathbb{N}_0 \). For a vector \( x \in \mathbb{R}^n \), the Euclidean vector norm of \( x \), denoted by \( ||x|| \), equals \( \sqrt{x^T x} \). For a matrix \( A \in \mathbb{R}^{m \times n} \), the spectral norm of \( A \) is given by \( ||A|| = \max_{||x|| = 1} ||Ax|| \). Denoted by \( 1 \) is a vector of appropriate dimension with all entries equal to 1. The identity matrix of appropriate dimension is denoted \( I \); similarly, the zero matrix, with its dimension understood, is denoted \( 0 \). If \( X, Y \in \mathbb{R}^{n \times n} \) are symmetric (i.e., \( X = X^T \) and \( Y = Y^T \)), then we write \( X < Y \) to mean that \( X - Y \) is negative definite as well as symmetric.

II. DEFINITIONS AND BASIC LEMMAS

A. Asymptotic Consensus

Let \( S \subset \mathbb{R}^{n \times n} \) be the set of all stochastic matrices; that is, for any \( F = (f_{ij}) \in S \), we have \( f_{ij} \geq 0 \) for all \( i, j \) and \( \sum_{j=1}^{n} f_{ij} = 1 \) for all \( i \). Let the set

\[ \mathcal{F} = \{ F_1, \ldots, F_N \} \subset S \]  

(1)

define a discrete linear inclusion (i.e., a discrete-time switched linear system) whose state-space representation is given by

\[ x(t+1) = F_{\theta(t)} x(t) \]  

(2)
for each switching sequence $\theta = (\theta(0), \theta(1), \ldots) \in \{1, \ldots, N\}^\infty$. For a discrete linear inclusion $F$ we have the following definitions.

**Definition 1:** Let $F$ be as in (1). A switching sequence $\theta \in \{1, \ldots, N\}^\infty$ is said to achieve asymptotic consensus for $F$ if there exists an $x_f : \mathbb{R}^n \to \mathbb{R}$ such that (2) satisfies

$$\lim_{t \to \infty} x(t) = x_f(x_0)1$$

whenever $x(0) = x_0$.

**Definition 2:** With $F$ as in (1), a switching sequence $\theta \in \{1, \ldots, N\}^\infty$ is said to achieve uniform asymptotic consensus for $F$ if there exist $c > 0$, $\lambda \in (0, 1)$, and $x_f : \mathbb{N}_0 \times \mathbb{R}^n \to \mathbb{R}$ such that, whenever $t_0 \in \mathbb{N}_0$ and $x(t_0) = x_0$, equation (2) satisfies

$$\|x(t) - x_f(t_0, x_0)1\| \leq c\lambda^{t-t_0}\|x_0 - x_f(t_0, x_0)1\|$$

for all $t \in \mathbb{N}_0$ with $t \geq t_0$.

It is known that a switching sequence $\theta \in \{1, \ldots, N\}^\infty$ achieves asymptotic consensus for $F$ if and only if the infinite product $F_{\theta(t-1)} \cdots F_{\theta(0)}$, with $t_0 = 0$, converges to a matrix whose rows are all equal to each other as $t \to \infty$. Moreover, uniform asymptotic consensus is achieved if a single rate of convergence is valid regardless of the initial time $t_0 \in \mathbb{N}_0$. These characterizations are summarized in the following lemma. (Use the convention that the product $F_{\emptyset} = I$ if $t = t_0$.)

**Lemma 3:** Let $F$ be as in (1). We have the following:

(a) A switching sequence $\theta \in \{1, \ldots, N\}^\infty$ achieves asymptotic consensus for $F$ if and only if there exists an $f \in \mathbb{R}^n$ such that

$$\lim_{t \to \infty} F_{\theta(t-1)} \cdots F_{\theta(0)} = f^T.$$

(b) A switching sequence $\theta \in \{1, \ldots, N\}^\infty$ achieves uniform asymptotic consensus for $F$ if and only if there exist $c > 0$, $\lambda \in (0, 1)$, and $f : \mathbb{N}_0 \times \mathbb{R}^n \to \mathbb{R}$ such that

$$\|F_{\theta(t-1)} \cdots F_{\theta(0)} - 1f(t_0)^T\| \leq c\lambda^{t-t_0}$$

for all $t, t_0 \in \mathbb{N}_0$ with $t \geq t_0$.

**Proof:** Part (a) is proved in, e.g., [5, Lemma 2.1]. The proof of part (b) is omitted due to space constraints.

**B. Asymptotic Stability**

As noted in [3], associated with each $F_i \in F$, where $F$ is as in (1), is an $A_i \in \mathbb{R}^{(n-1) \times (n-1)}$ such that

$$[I \quad -1]F_i = A_i [I \quad -1].$$

The set

$$A = \{A_1, \ldots, A_N\}$$

defines the discrete linear inclusion with state space representation

$$y(t+1) = A_{\theta(t)}y(t)$$

for all $\theta \in \{1, \ldots, N\}^\infty$. While Definitions 1 and 2 are about asymptotic consensus with respect to $F$, the following definitions pertain to asymptotic stability with respect to $A$.

**Definition 4:** Let $A$ be as in (4). A switching sequence $\theta \in \{1, \ldots, N\}^\infty$ is said to be asymptotically stabilizing for $A$ if (5) satisfies

$$\lim_{t \to \infty} y(t) = 0$$

for all $y(0) \in \mathbb{R}^{n-1}$.

**Definition 5:** With $A$ as in (4), a switching sequence $\theta \in \{1, \ldots, N\}^\infty$ is said to be uniformly asymptotically stabilizing for $A$ if there exist $c > 0$ and $\lambda \in (0, 1)$ such that (5) satisfies

$$\|y(t)\| \leq c\lambda^{t-t_0}\|y(t_0)\|$$

for all $t_0, t \in \mathbb{N}_0$ with $t_0 \leq t$, and for all $y(t_0) \in \mathbb{R}^{n-1}$.

As the following lemma shows, the discrete linear inclusions $F$ and $A$ are closely related, and in fact (uniform) asymptotic consensus with respect to $F$ is equivalent to (uniform) asymptotic stability with respect to $A$.

**Lemma 6:** Let $F$ and $A$ be as in (1) and (4), respectively. We have the following:

(a) A switching sequence $\theta \in \{1, \ldots, N\}^\infty$ achieves asymptotic consensus for $F$ if and only if it is asymptotically stabilizing for $A$.

(b) A switching sequence $\theta \in \{1, \ldots, N\}^\infty$ achieves uniform asymptotic consensus for $F$ if and only if it is uniformly asymptotically stabilizing for $A$.

**Proof:** Part (a) is proved in [3, Page 992]. Part (b) can be proved applying a similarity transformation with

$$P = P^{-1} = \begin{bmatrix} I & -1 \\ 0 & -1 \end{bmatrix} \in \mathbb{R}^{n \times n}$$

to $F$. Sufficiency can be shown using a procedure similar to the one used in proving [9, Theorem 4.2]. Necessity can then be proved following some elementary operations. The details are omitted due to space constraints.

Recent advances in the stability analysis of switched systems have given us a Lyapunov inequality–based convex condition for a switching sequence to be uniformly asymptotically stabilizing for a discrete linear inclusion. Given a switching sequence $\theta \in \{1, \ldots, N\}^\infty$, define $N_L(\theta)$ as the largest set of switching paths of length $L$ such that the following holds: for each $(t_0, \ldots, i_L) \in N_L(\theta)$, there exist an integer $M > L$, a switching path $(t_{L+1}, \ldots, t_M) \in \{1, \ldots, N\}^{M-L}$, and a set of time instants $\{t_0, \ldots, t_{M-L}\} \subset \mathbb{N}_0$ such that $(i_j, \ldots, i_{j+L}) = (\theta(t_j), \ldots, \theta(t_j + L))$ for all $j \in \{0, \ldots, M - L\}$ and such that $(i_0, \ldots, i_L) = (M-L, \ldots, i_L)$. For each $L \in \mathbb{N}_0$, switching paths of length $L$ that do not belong to $N_L(\theta)$ are irrelevant to uniform asymptotic stability. Using the convention that $\{1, \ldots, N\}^0 = \{0\}$, we have the following lemma:

**Lemma 7:** Let $A$ be as in (4). A switching sequence $\theta \in \{1, \ldots, N\}^\infty$ is uniformly asymptotically stabilizing for $A$ if and only if there exist $L \in \mathbb{N}_0$ and $X_{(j_1, \ldots, j_L)} > 0$, $(j_1, \ldots, j_L) \in \{1, \ldots, N\}^L$, satisfying the Lyapunov inequalities

$$A_{t_L}^T X_{(i_1, \ldots, i_L)} A_{i_L} - X_{(i_0, \ldots, i_{L-1})} < 0$$

for each switching sequence $\theta = (\theta(0), \theta(1), \ldots) \in \{1, \ldots, N\}^\infty$. For a discrete linear inclusion $F$ we have the following definitions.
for all switching paths \((i_0, \ldots, i_L) \in \mathcal{N}_L(\theta)\).

Proof: This is a special case of [8, Corollary 3.4].

Along with the notions already described, there are various other notions of stability for discrete linear inclusions. One such notion is that of asymptotic stability under arbitrary switching.

Definition 8: Let \(A\) be as in (4). The discrete linear inclusion \(A\) is said to be asymptotically stable if the state equation (5) satisfies (6) for all \(y(0) \in \mathbb{R}^{n-1}\) and for all \(\theta \in \{1, \ldots, N\}^\infty\). In particular, \(A\) is said to be uniformly asymptotically stable if there exist \(c > 0\) and \(\lambda \in (0, 1)\) such that (5) satisfies (7) for all \(t_0, t \in \mathbb{N}_0\) with \(t_0 \leq t\), for all \(y(t_0) \in \mathbb{R}^{n-1}\), and for all \(\theta \in \{1, \ldots, N\}^\infty\).

The following lemma states that a discrete linear inclusion is asymptotically stable if and only if it is uniformly asymptotically stable, and characterizes the asymptotic stability of \(A\) in terms of joint spectral radius and Lyapunov inequalities. The joint spectral radius of \(A\) is defined by

\[
\rho(A) = \limsup_{L \to \infty} \max_{(j_1, \ldots, j_L) \in \{1, \ldots, N\}^L} \|A_{j_L} \cdots A_{j_1}\|^{1/L}.
\]

Lemma 9: With \(A\) as in (4), the following are equivalent:

(a) The discrete linear inclusion \(A\) is asymptotically stable;
(b) The discrete linear inclusion \(A\) is uniformly asymptotically stable;
(c) The joint spectral radius \(\rho(A) < 1\);
(d) There exist \(L \in \mathbb{N}_0\) and \(X_{(j_1, \ldots, j_L)} > 0\), \((j_1, \ldots, j_L) \in \{1, \ldots, N\}^L\), such that (8) holds for all \((i_0, \ldots, i_L) \in \{1, \ldots, N\}^{L+1}\).

Proof: The equivalence of conditions (a) and (c) is established in [9, Theorem 4.1], [10]. Conditions (a) and (b) are equivalent by [9, Corollary 4.1a], [10]; see also [11, Proposition 8]. Conditions (b) and (d) are equivalent due to [11, Theorem 9], or [8, Corollary 3.4], specialized to all switching sequences.

III. EQUIVALENT CONDITIONS FOR UNIFORM ASYMPTOTIC CONSENSUS

Uniform asymptotic consensus with respect to \(F\), and hence uniform asymptotic stability with respect to \(A\), can also be characterized using graph theoretic notions. A directed graph \(G\) is defined by the pair \((V, E)\) of the set \(V = \{1, \ldots, n\}\) of the nodes in the graph and a set \(E \subset V \times V\) of the directed edges in the graph, so that \((i, j) \in E\) if and only if there is a directed edge from node \(i\) to node \(j\). In graph \(G\), a node \(i\) is said to be connected to node \(j\) if there is a path \((i_0, \ldots, i_L) \in V^{L+1}\) such that \(i_0 = i, i_{L+1} = j\), and \((i_k, i_{k+1}) \in E\) for all \(k = 0, \ldots, L - 1\).

In this section, we present several equivalent conditions for uniform asymptotic consensus under two different graph theoretic assumptions.

A. Joint Connectedness for Leaderless Case

The standard assumption for asymptotic consensus under leaderless situations is the following:

Assumption 10: In (1), all the diagonal entries of each \(F \in F\) are positive; that is, if \(F = (f_{ij}) \in F\), then \(f_{ii} > 0\) for all \(i \in \{1, \ldots, n\}\).

In this case a directed graph \(G = (V, E)\) associated with \(F \in F\) is called connected if there is a node which is connected to all the other nodes, or equivalently, if there exists an \(E \subset E\) such that \((V, E)\) is a directed spanning tree. Associated with each matrix \(F_i = (f_{jk}) \in \mathbb{R}\) is a directed graph \(G_i = (V, E_i)\) such that \((k, j) \in E_i\) if and only if \(f_{jk} \neq 0\).

Definition 11: An indexed family of graphs \(\{G_j = (V_i, E_j)\}_{j \in J}\) is said to be jointly connected if \((V, \bigcup_{j \in J} E_j)\) is connected.

A switching sequence \(\theta \in \{1, \ldots, N\}^\infty\) leads to a sequence \(\{G_{\theta(j)}(0), G_{\theta(j)}(1), \ldots\}\) of directed graphs with \(G_{\theta(j)}(t) = (V, E_{\theta(j)(t)})\) for all \(t \in \mathbb{N}_0\). For such a sequence of directed graphs, the following result is well-known.

Lemma 12: With \(F\) as in (1), let \(G_i\) be the directed graph of \(F_i\) for all \(i \in \{1, \ldots, N\}\). Suppose Assumption 10 holds true. A switching sequence \(\theta \in \{1, \ldots, N\}^\infty\) achieves asymptotic consensus for \(F\) if there exists a \(T \in \mathbb{N}_0\) such that \(G_{\theta(j(t))} = G_{\theta(j(t)+T)}\) is jointly connected for all \(t \in \mathbb{N}_0\).

Proof: The result is immediate from, e.g., [5, Theorem 3.10].

If the graphs \(G_i\) are undirected (i.e., every edge is bidirectional for all \(i\)), then Lemma 12 reduces to [3, Theorem 2]. That is, the joint connectedness condition in [5], which reverts to that in [3] for undirected graphs, is in fact equivalent to the condition in Lemma 12. The former requires that \(\theta\) leads to a sequence of directed graphs which are jointly connected over nonempty, nonoverlapping time intervals such that the lengths of these time intervals are bounded above by some \(\tau \geq 0\) and such that the distance between any two consecutive such time intervals is also bounded above by some \(\eta \geq 0\); letting \(T = 2\tau + \eta\), it is evident that \(\{G_{\theta(j(t))}, \ldots, G_{\theta(t(t)+T)}\}\) is jointly connected for all \(t \in \mathbb{N}_0\). However, as shown in Example 1, these conditions are only sufficient, and not necessary, for mere asymptotic consensus.

B. Joint Connectedness for Leader-Follower Case

In this subsection, we present a new graph theoretic condition applicable to asymptotic consensus under “leader-follower” situations. These situations have been discussed earlier in, e.g., [3], [5]. All these previous discussions, however, impose Assumption 10. We replace this assumption with the following one, which allows some of the matrices to have possibly zero diagonal entries.

Assumption 13: In (1), there exists an \(i^* \in \{1, \ldots, n\}\) such that each \(F = (f_{ij}) \in F\) satisfies the following:

1. \(f_{i^*i^*} = 1\).
2. For \(i \neq i^*\), either \(f_{ii^*} > 0\) or \(f_{i^*i} = 1\).

In this assumption, index \(i^*\) denotes the “leader” node which never updates its state. To take into account the presence of a leader, the notion of joint connectedness in Definition 11 needs to be modified.

Definition 14: Given an \(i^* \in \{1, \ldots, n\}\), a directed graph \(G = (V, E)\) is said to be connected from \(i^*\) if node \(i^*\) is connected to all other nodes. Similarly, an indexed family of
graphs \( \{G_j = (V, E_j) : j \in J \} \) is said to be jointly connected from \( i^* \) if \((V, \bigcup_{j \in J} E_j)\) is connected from \( i^* \).

As in the leaderless case, we are able to relate the joint connectedness of associated directed graphs to asymptotic consensus of \( \mathcal{F} \). Though the structure of our proof remains similar to that used for proving [12, Lemma 5.2.1], these two differ on several accounts. This is because our result only needs \( \{G_{\theta(t)} \cdots G_{\theta(t+T)}\} \) to be jointly connected for any \( t \) and the connected graph is allowed to be time-varying, whereas in [12] the sets \( \{G_{\theta(t)} , \cdots , G_{\theta(t+T)}\} \) are assumed to yield the same jointly connected graph for all \( t \).

C. Main Result

In this subsection, we state our main result which presents an equivalence of many different conditions for uniform asymptotic consensus.

**Theorem 15:** Let \( \mathcal{F} \) be as in (1), and let \( A \) be as in (4). Let \( G_i \) be the graph of \( F_i \) for all \( i \in \{1, \ldots , N\} \). Suppose Assumption 10 holds. Then, for a given switching sequence \( \theta \in \{1, \ldots , N\}^\infty \), the following are equivalent:

1. The switching sequence \( \theta \) achieves uniform asymptotic consensus for \( \mathcal{F} \).
2. The switching sequence \( \theta \) is uniformly asymptotically stabilizing for \( A \).
3. There exists a \( T \in \mathbb{N}_0 \) such that \( \{G_{\theta(t)} \cdots G_{\theta(t+T)}\} \) is jointly connected for all \( t \in \mathbb{N}_0 \).
4. There exists an \( M \in \mathbb{N}_0 \) such that the joint spectral radius \( \rho(A_M) < 1 \), where
   \[ A_M = \{A_{\theta(t+M)} \cdots A_{\theta(t)} : t \in \mathbb{N}_0\} \]
5. There exist an \( L \in \mathbb{N}_0 \) and symmetric positive definite matrices \( X_{(j_1, \ldots , j_L)} \in \mathbb{R}^{(n-1) \times (n-1)} \), \( (j_1, \ldots , j_L) \in \{1, \ldots , N\}^L \), such that the Lyapunov inequalities (8) hold for all \( (i_0, \ldots , i_L) \in \mathbb{N}_L \). (c')

On the other hand, suppose Assumption 13 holds. Then, for a given switching sequence \( \theta \in \{1, \ldots , N\}^\infty \), conditions (a)–(e) above, with (c) replaced by (c') below, are equivalent:

(c') There exist a \( T \in \mathbb{N}_0 \) such that \( \{G_{\theta(t)} \cdots G_{\theta(t+T)}\} \) is jointly connected from \( i^* \) for all \( t \in \mathbb{N}_0 \).

**Proof:** The proof is sketched in Section V. 

In Theorem 15, the period \( T \) in conditions (c) and (c') can be taken to be equal to the path length \( M \) in condition (d); on the other hand, the path length \( L \) in condition (e) is often much smaller than the period \( T \) in conditions (c) and (c')—see, e.g., the numerical examples in [8], [11].

It follows from the results in [2]–[4] that the joint connectedness condition (c) is sufficient for asymptotic consensus under undirected connectivity graphs satisfying Assumption 10. The generalization of these results to directed graphs satisfying the same assumption is presented in [5]. Theorem 15 shows that the joint connectedness condition (c) is both sufficient and necessary for uniform asymptotic consensus involving general directed graphs. The uniformity requirement guarantees that as long as condition (c) is satisfied, the agents’ states will remain near consensus even under persistent exogenous disturbances. Theorem 15 also establishes a connection among a graph-theoretic condition, a joint spectral radius condition, and a linear matrix inequality feasibility condition; this connection is a generalization of what was implicitly made in [7] under undirected graphs satisfying Assumption 10.

Theorem 15 not only gives a more precise definition of uniform asymptotic consensus than that in [7], but also relates it to various other conditions for directed graphs. Hence, [7, Theorem 4] can be thought of as a special case of Theorem 15. The equivalence of a uniform global attractiveness condition and the joint connectedness condition for a class of discrete nonlinear inclusions satisfying a condition similar to Assumption 10 was established by Moreau [6]. The equivalence of conditions (a) and (c) makes it explicit that this uniform global attractiveness condition reverts precisely to the uniform asymptotic consensus condition for discrete linear inclusions. Moreover, Theorem 15 extends this result to discrete linear inclusions under Assumption 13 as well.

As stated earlier, the leader-follower cases satisfying Assumption 13 are different from the ones discussed in, e.g., [3], [5]. It is because the state matrices can have possibly zero diagonal entries under this assumption. The leader-follower model discussed in [2] is very similar to ours but requires that the connectivity graphs form the “same strongly connected” directed graph for every time interval of length \( T \). Theorem 15 does not rely on such a strong assumption. An application for this framework is, for example, the formation control of multi-vehicle systems [1]. As noted in [5], we may have unidirectional flow of information giving rise to directed graphs, and moreover, if only one vehicle is equipped with a transmitter, it may act as a leader giving rise to the leader-follower scenario. Getting each of these vehicles to follow the leader even under persistent disturbances may be the objective of the entire network.

IV. ILLUSTRATIVE EXAMPLES

**Example 1.** This example deals with the study of uniform asymptotic consensus for discrete linear inclusions satisfying Assumption 10. The agents follow the nearest neighbor rule for updating their states (i.e., headings) as in [3]. We illustrate how asymptotic consensus can be lost under the presence of a disturbance, whereas uniform asymptotic consensus guarantees convergence in spite of such a disturbance. The state matrices \( F_i \in \mathcal{F} \) are as follows:

\[
F_1 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix},
F_2 = \begin{bmatrix}
1 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 \\
0 & 0 & 1
\end{bmatrix},
F_3 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & \frac{1}{2} & \frac{1}{2}
\end{bmatrix},
F_4 = \begin{bmatrix}
1 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 \\
0 & 0 & 1
\end{bmatrix},
F_5 = \begin{bmatrix}
1 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 \\
0 & \frac{1}{2} & \frac{1}{2}
\end{bmatrix},
F_6 = \begin{bmatrix}
1 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 \\
0 & \frac{1}{2} & \frac{1}{2}
\end{bmatrix},
F_7 = \begin{bmatrix}
1 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 \\
0 & \frac{1}{2} & \frac{1}{2}
\end{bmatrix},
F_8 = \begin{bmatrix}
1 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 \\
0 & \frac{1}{2} & \frac{1}{2}
\end{bmatrix}.
\]
A three-agent model was simulated using initial \((x, y)\)-
position and heading vectors as
\[
\begin{align*}
    x(0) &= \begin{bmatrix} 0.3594 \\ 0.0669 \\ 0.9659 \end{bmatrix}, \\
    y(0) &= \begin{bmatrix} 0.6495 \\ 0.4206 \\ 0.5964 \end{bmatrix}, \\
    \phi(0) &= \begin{bmatrix} 0.9612 \\ 0.1510 \\ 2.8015 \end{bmatrix},
\end{align*}
\]
respectively, with a common speed \(v = 0.5\) and a common
communication radius \(r = 0.8\). The simulation results as
shown in Fig. 1 depict that the connectivity graph becomes
complete at time \(t = 2\); i.e., \(F_{\theta(2)} = F_8\), which allows the
agents to reach consensus at time \(t = 2\) even though the
agents remain disconnected thereafter. But the application of
a single disturbance at \(t = 15\) drives them out of consensus,
and they can never reach consensus again. This shows,
the necessity of a condition stronger than mere asymptotic
consensus. Consider another three-agent model with
\[
\begin{align*}
    x(0) &= \begin{bmatrix} 0.9695 \\ 0.1838 \\ 0.2999 \end{bmatrix}, \\
    y(0) &= \begin{bmatrix} 0.4112 \\ 0.2365 \\ 0.1951 \end{bmatrix}, \\
    \phi(0) &= \begin{bmatrix} 0.1253 \\ 0.7290 \\ 2.0287 \end{bmatrix},
\end{align*}
\]
a common speed \(v = 0.5\), and a common communication
radius \(r = 0.8\). As Fig. 2 shows, since the graphs are jointly
connected for any interval of length \(T = 4\) in this case, we
have uniform asymptotic consensus, which is robust against
disturbances. This example illustrates the importance of the
uniformity requirement in asymptotic consensus problems.

**Example 2.** We will show the uniform asymptotic consen-
sus of a discrete linear inclusion satisfying Assumption 13
in this example. Here, we consider a discrete linear inclusion
\(\mathcal{F} = \{F_1, \ldots, F_5\}\), where
\[
\begin{align*}
    F_1 &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \\
    F_2 &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \\
    F_3 &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \\
    F_4 &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \\
    F_5 &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.
\end{align*}
\]
As Fig. 3 shows, a switching sequence \(\theta = (1, 3, 2, 3, 1, 4,
5, 2, 3, 1, 3, 2, 4, 2, 5, 1, 2, 3, 4, 2, 4, 5, 2, 1, 4, 4, \ldots)\) achieves
uniform asymptotic consensus for \(\mathcal{F}\), since it leads to graphs
that are jointly connected from the leader over any interval of
length \(T = 3\). Application of a disturbance at \(t = 35\)
disturbs the consensus momentarily, but it is again restored
since we have uniform asymptotic consensus here rather than
mere asymptotic consensus.

Fig. 4 shows explicitly the difference between the graphs
in this example and similar ones used previously in, e.g.,
[3], [5], [6]. The graph \(\mathcal{G}_{\theta(5)} = (V, E_4)\) for our example
is shown in part (a) and what we would have instead under
the settings of [3], [5], [6] is shown in part (b). While the
graph in part (b) reflects the implicit requirement of previous
results that every node has a self-loop, our graph in part (a)
shows that we do not impose such a requirement. This arises
from the fact that, in the leader-follower case, we are able to replace Assumption 10 with Assumption 13. This example clearly shows that our results apply to some situations that previous results do not cover.

V. SKETCH OF PROOF OF THEOREM 15

In Theorem 15, conditions (a) and (b) are equivalent by Lemma 6(b), and conditions (b) and (e) are equivalent by Lemma 7. We will sketch how one can show that (e) ⇒ (c) ⇒ (d) ⇒ (b) under Assumption 10, and that (e) ⇒ (c') ⇒ (d) ⇒ (b) under Assumption 13.

A. Leaderless Case (under Assumption 10)

The proof of (e) ⇒ (c) is somewhat involved and requires adoption of additional notions and results such as Lemma 1 and Theorem 2 from [13]; otherwise, it is essentially the same as that of the necessity part of [7, Theorem 4], except that we consider directed graphs here.

Similarly, except for the fact that we consider directed graphs here, the proof of (c) ⇒ (d) is essentially contained in that of the sufficiency part of [7, Theorem 4].

To show (d) ⇒ (b), suppose (d) holds, so that there exists an \( M \in \mathbb{N}_0 \) such that \( \rho(A_M) < 1 \). Then Lemma 9 implies that there exist \( c > 0 \) and \( \lambda \in (0, 1) \) such that, in particular, equation (5) satisfies

\[
\|y(k(M + 1))\| \leq z\lambda^{k-k_0}\|y(k_0(M + 1))\|
\]

for all \( k, k_0 \in \mathbb{N}_0 \) with \( k \geq k_0 \) and for all \( y(k_0(M + 1)) \in \mathbb{R}^{n-1} \). A standard technique reveals that this implies (b).

B. Leader-Follower Case (under Assumption 13)

The proof remains identical to the leaderless case except that Lemma 17 below is used in place of Lemma 12 for showing the implication (c') ⇒ (d).

Lemma 16: Let \( F \) be as in (1). Suppose that there exist an \( \bar{\alpha} > 0 \) and an \( i^* \in \{1, \ldots, n\} \) such that each \( F = (f_{ij}) \in F \) satisfies \( f_{ii} \geq \bar{\alpha} \) for all \( i \in \{1, \ldots, n\} \). Then, for any switching sequence \( \theta = (\theta(0), \theta(1), \ldots) \in \{1, \ldots, N\}^\infty \), there exists an \( f^* : \mathbb{N}_0 \to \mathbb{R}^n \) such that

\[
\lim_{t \to \infty} f_{\theta(t-1)} \cdots f_{\theta(t_0)} = 1 f(t_0)^T
\]

for all \( t_0 \in \mathbb{N}_0 \).

Proof: See [12, Lemma A.1].

Lemma 17: Let \( F \) be as in (1); let \( G_i \) be the directed graph of \( F_i \) for all \( i \in \{1, \ldots, N\} \). Suppose Assumption 13 holds. A switching sequence \( \theta \in \{1, \ldots, N\}^\infty \) achieves asymptotic consensus for \( F \) if there exists a \( T \in \mathbb{N}_0 \) such that \( \{G_{\theta(t)}, \ldots, G_{\theta(t+T)}\} \) is jointly connected from \( i^* \) for all \( t \in \mathbb{N}_0 \).

Proof: We present only a brief sketch of the proof and omit the details due to space constraints. Assume without loss of generality that \( T > 0 \). Let \( \alpha \in (0, 1] \) be the minimum of all nonzero elements of the entries of \( F \), define the discrete-time state transition matrix as \( \Phi(k_0, k_0) = 1 \) for \( k_0 \in \mathbb{N}_0 \), and \( \Phi(k, k_0) = (\phi_{ij}(k, k_0)) = F_{\theta(k-1)} \cdots F_{\theta(k_0)} \) for \( k, k_0 \in \mathbb{N}_0 \) with \( k > k_0 \). Following an approach similar to the one used in proving [12, Lemma 5.2.1], it can be shown that putting \( \bar{\alpha} = \alpha(2^{n-1}nT) \) yields that \( \bar{\alpha} \in (0, 1] \) and \( \phi_{ii}(k_0 + nT, k_0) \geq \bar{\alpha} \) for all \( i \in \{1, \ldots, n\} \) and \( k_0 \in \mathbb{N}_0 \).

For each \( k \in \mathbb{N}_0 \), the matrix product \( F_\theta(k(nT+1)) \cdots F_\theta(k(nT+1)) \) is associated with a directed graph which is connected from node \( i^* \); that is, all the entries of column \( i^* \) of this matrix product are positive and bounded below by \( \bar{\alpha} \). Thus it follows from Lemma 16 and part (a) of Lemma 3 that, regardless of \( y(0) \in \mathbb{R}^{n-1} \), equation (5) satisfies \( y(k(nT + 1)) \to 0 \) as \( k \to \infty \). A standard argument then reveals that this implies \( \theta \) is asymptotically stabilizing for \( A \). By part (a) of Lemma 6, this gives the desired result.

VI. CONCLUSIONS

A complete characterization of leaderless uniform asymptotic consensus for multi-agent systems and its equivalence with various other notions related to switched systems theory and discrete linear inclusions was presented. Then, the result was extended to a particular type of leader-follower problems with state matrices having possibly zero diagonal entries. Possible extensions of this work include analyzing the transient behavior of these systems, studying uniform asymptotic consensus for higher-order consensus algorithms [14], and examining cases where agents can possibly form factions.

REFERENCES


