Decentralized Robust Control Invariance for a Network of Integrators

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Abstract—Robust control of networked storage devices is considered. Each network node is a storage element modeled as a single-state, discrete time integrator with bounded control input and subject to additive bounded disturbance. Nodes exchange matter through links of limited capacity. We characterize the maximal robust control invariant set and consider a decentralized solution to the robust feasibility problem. We show how to compute the set of link capacities which ensure the feasibility of the proposed decentralized design by using convex optimization. The results allow to guarantee persistent feasibility when decentralized control strategies are used to robustly control the network flow while satisfying nodes and links capacity constraints.

I. INTRODUCTION

Networks of integrators are commonly used models to characterize a variety of large scale systems. These include networked devices which can store a certain amount of matter (such as water, energy, goods), produce it and transfer it to its peers. Some problems in production–distribution systems, communication networks, water supply systems, cells of batteries and transportation networks can be modeled within such framework.

In this paper we adopt a model where each node of the network is a discrete time single-state integrator and its state corresponds to the amount of matter stored within the node. The amount of matter is regulated by a control input associated to the node. Additionally, each node can transfer an amount of its stored matter to other nodes, and equally, receive some amount of matter from other nodes through directed links. A subset of the nodes is subject to an unknown bounded disturbance in the form of matter demand or supply. In such setting the control problem consists of finding feasible productions for each node and flows between the nodes in order to satisfy the uncertain demand and keep the state of each node within the admissible bounds and according to some optimality criterion.

The closely related problem of flows in static networks has been treated extensively, starting from early classical works [1], [2] to more recent monographs [3]–[5]. There the nodes are not dynamical systems but represent merely topological connections. Robust control of dynamic networks subject to uncertain demands is treated in a series of papers by Blanchini and co–authors [6]–[10], both for the discrete–time as well as for continuous–time case, where the disturbances are modeled as non–stochastic and of bounded magnitude. In [6]–[8], [11] the common approach is to characterize the maximal robust control invariant (RCI) set, i.e. the set of network states for which there exists a network flow that guarantees the demand satisfaction at all time instants. In [6] the conditions for the existence of the RCI set are used for the purpose of optimal network design of production–distribution system in order to achieve feasibility of flows and levels of stored goods in the network for all possible realizations of the demand, while minimizing the cost associated to the network operation. Cost minimization is also addressed in [11], with the focus on existence of the control law that drives the amount of stored goods to the least feasible storage levels for all demands and in the presence of failures. An extension of the earlier results to networks featuring delays in the flows (the control inputs) is provided in [7]. Specific combinatorial structure of the maximal RCI sets was hinted on in [6] using the formalism of sub-modular functions and base polyhedra (cf. [5]). Using the projection algorithm the explicit expressions for RCI and positive robust invariant set for a particular control policy are derived in [12] for a specific buffer network structure.

In this paper we do not assume any specific graph structure and consider flow capacities and state constraints more general than those considered in [6], [7], [11]. In the first part of the paper we characterize the maximal robust control invariant set for the defined scenario. In the second part, motivated by the combinatorial structure of the RCI set and computational difficulties that may arise in their applications for large networks, we consider a decentralized design, where the nodes or groups of nodes act as decision makers which perceive the flow from adjacent nodes as an additional bounded disturbance [13]. We propose a design of robust control invariant sets parametrized with respect to network link capacities. Then we show that a convex problem can provide the set of network link capacities which ensure that the corresponding decentralized robust control invariant set is non–empty. This result can be used to compute terminal sets to ensure persistent feasibility when a decentralized constrained control strategy, for instance model predictive control, is used to control the network flow while satisfying input and state constraints.

The structure of the paper is the following. The problem statement and necessary preliminaries are given in Section II. We discuss the characterization of the RCI set for the integrator network in Section III. Section IV contains the details related to decentralized design. Numerical example is reported in Section V, while Section VI gives concluding remarks. The proofs for all the results in the paper, due to the space limitation, are given in [14].
Notation: The set of real numbers is denoted by \( \mathbb{R} \) and the set of non–negative (greater or equal to 0) real numbers as \( \mathbb{R}^+ \). \( \mathbb{N} \) is the set of natural numbers, i.e. \( \mathbb{N} := \{1, 2, \ldots \} \). \( \mathbb{N}_{[q_1, q_2]} \) stands for \( \{q_1, \ldots, q_2\} \), where \( q_1, q_2 \in \mathbb{N} \) and \( q_1 \leq q_2 \). Empty set is denoted as \( \emptyset \). Given a finite set \( \mathcal{N} \), for any \( \mathcal{S} \subseteq \mathcal{N} \) we use notation \( \mathcal{S}' := \mathcal{N} \setminus \mathcal{S} \), i.e. the set of all \( i \in \mathcal{N} \) such that \( i \notin \mathcal{S} \). Given any set \( \mathcal{S} \), \( 2^\mathcal{S} \) denotes power set of \( \mathcal{S} \), i.e. the set of all subsets of \( \mathcal{S} \). Cardinality of a set \( \mathcal{S} \) is denoted as \( |\mathcal{S}| \). For a vector \( x \in \mathbb{R}^n \) the ith component is denoted as \( x_i \) and \( x(S) := \sum_{i \in S} x_i \) for some \( S \subseteq \mathbb{N}_{[1,n]} \), with convention \( x(\emptyset) = 0 \). Binary operation \( \ominus \) denotes Minkowski sum of two non–empty sets \( \mathcal{X} \) and \( \mathcal{Y} \): \( \mathcal{X} \oplus \mathcal{Y} := \{ x + y : x \in \mathcal{X}, y \in \mathcal{Y} \} \). Binary operation \( \odot \) denotes Minkowski (Pontryagin) difference between two non–empty sets \( \mathcal{X} \) and \( \mathcal{Y} \): \( \mathcal{X} \ominus \mathcal{Y} := \{ z : z \neq y \land z + y \in \mathcal{X} \} \). Set \( \mathcal{P} \subseteq \mathbb{R}^n \) is a polyhedron if it can be represented as: \( \mathcal{P} = \{ x \in \mathbb{R}^n : h_i^T x \leq h_k, \ i \in [1,q] \} \), bounded polyhedron is referred to as polytope. Given a polyhedron \( \mathcal{P} \), for all \( x \in \mathbb{R}^n \), an inequality \( h_i^T x \leq k_i \) is redundant for \( \mathcal{P} \) if \( \mathcal{P} = \{ x \in \mathbb{R}^n : h_i^T x \leq k_i, \ i \in [1,q], i \neq j \} \).

II. Problem Statement and Preliminaries

Consider a directed graph \( G = (\mathcal{N}, \mathcal{E}) \) where \( \mathcal{N} = \mathbb{N}_{[1,n]} \) is the set of nodes and \( \mathcal{E} \subseteq \mathcal{N} \times \mathcal{N} \) is a set of ordered pairs, edges, whose order indicates the direction of links between two nodes. We will use the following notation for adjacent nodes of the node \( i \in \mathcal{N} \):

\[
\mathcal{A}_i^- = \{ j \in \mathcal{N} : (i, j) \in \mathcal{E} \},
\]

\[
\mathcal{A}_i^+ = \{ j \in \mathcal{N} : (j, i) \in \mathcal{E} \},
\]

where \( \mathcal{A}_i^- \) is the set of adjacent nodes leaving node \( i \) and \( \mathcal{A}_i^+ \) is the set of adjacent nodes entering node \( i \). Each node \( i \) in the graph \( G \) is associated with a discrete–time dynamical system with the state update equation:

\[
x_i^+ = x_i + \phi_i - d_i,
\]

where \( x_i \) denotes the state of node \( i \) at the next time step. The state \( x_i \) represents the current amount of matter stored in the node \( i \), whose level is controlled by the total net flow \( \phi_i \) into the node \( i \) which is given as:

\[
\phi_i = u_i + \phi_i^+ - \phi_i^-,
\]

where \( u_i \) represents matter which can be added to or subtracted from the node \( i \) from an external source or sink and \( \phi_i^+ \) and \( \phi_i^- \) denote the amount of matter supplied to taken from the node \( i \) by other nodes in the graph, respectively. The dynamics of each storage element is additionally affected by an unknown disturbance \( d_i \) which can be thought of as uncontrolled demand or supply associated to the node \( i \). If we denote the transfer of matter between the node \( i \) and \( j \) as \( u_{ij} \), with the convention that \( u_{ij} = 0 \) for \( i, j \notin \mathcal{E} \), the variables \( \phi_i^+ \) and \( \phi_i^- \) in (2) are then given as:

\[
\phi_i^+ = \sum_{j \in \mathcal{A}_i^+} u_{ij}, \quad \phi_i^- = \sum_{j \in \mathcal{A}_i^+} u_{ij}.
\]

We further introduce the net flow from the node \( i \) to the node \( j \) as a difference between the amount of matter transferred from \( i \) to \( j \) and the amount transferred from \( j \) to \( i \):

\[
f_{ij} := u_{ij} - u_{ji},
\]

Note that using these conventions we have:

\[
f_{ij} + f_{ji} = 0 \quad \text{and} \quad f_{ii} = 0,
\]

\[
\phi_i = u_i + \sum_{j \in A_i} f_{ij}.
\]

We use the following notation for the net flow between the two disjoint sets \( \mathcal{S}, \mathcal{T} \subseteq \mathcal{N} \):

\[
f_{\mathcal{S}, \mathcal{T}} := \sum_{i \in \mathcal{S}, j \in \mathcal{T}} f_{ij}.
\]

The capacity function \( \tau : (\mathcal{S}, \mathcal{T}) \to \mathbb{R}^+ \), defined for all disjoint subsets \( \mathcal{S} \) and \( \mathcal{T} \) of the set \( \mathcal{N} \), bounds the flow \( f_{\mathcal{S}, \mathcal{T}} \) between two disjoint sets of nodes \( \mathcal{S} \) and \( \mathcal{T} \), with:

\[
\tau(\emptyset, \cdot) = \tau(\cdot, \emptyset) = 0.
\]

The dynamics of the whole network consisting of dynamical systems (1) can be compactly written as:

\[
x^+ = x + \phi - d,
\]

where \( x = [x_1 \ldots x_n]^T \), \( \phi = [\phi_1 \ldots \phi_n]^T \) and \( d = [d_1 \ldots d_n]^T \).

The state \( x \), the flow vector \( \phi \) and the disturbance \( d \) are assumed to be bounded. In particular we consider the following bounding functions: \( x, \tau, u, d, \bar{d} : 2^\mathcal{N} \to \mathbb{R} \), where \( \tau(\cdot, \cdot), u(\cdot), \bar{d}(\cdot) \) are defined for all \( \mathcal{S} \subseteq \mathcal{N} \) and satisfy the relations:

\[
\tau(S) \leq \tau(S), \quad u(S) \leq u(S) \quad \text{for all} \ S \subseteq \mathcal{N},
\]

with the convention: \( \tau(\emptyset) = \tau(\emptyset) = u(\emptyset) = \tau(\emptyset) = \bar{d}(\emptyset) = \bar{d}(\emptyset) = 0 \). The constraints on variables \( x, \phi, d \) and \( u = [u_1 \ldots u_n]^T \) are compactly represented as follows:

\[
x \in \mathcal{X} := \{ x : \tau(S) \leq x(S) \leq \tau(S), \forall S \subseteq \mathcal{N}, \}
\]

\[
d \in D := \{ d : d(S) \leq d(S) \leq d(S), \forall S \subseteq \mathcal{N}, \}
\]

\[
u \in U := \{ u : u(S) \leq u(S) \leq u(S), \forall S \subseteq \mathcal{N}, \}
\]

\[
\phi \in \mathcal{F} := \{ \phi : \phi \in U, \mathcal{f}_{S,T} \leq \tau(S, S'), \}
\]

We remark that the states, inputs and flow constraints defined in (6) represent a wide class of network constraints. As an example, they include the box–type constraints where each variable is individually bounded by some minimal and maximal value. In this case the user would specify only the functions \( \tau(i), \bar{d}(i), u(i), \bar{u}(i) \) for each node \( i \in \mathcal{N} \) and \( \tau(i,j) \) for all edges \( i, j \in \mathcal{E} \) while the remaining combinatorial set of constraints in (6) would be implicitly defined as \( \tau(S) = \sum_{i \in S} \tau(i) \), for all \( S \subseteq \mathcal{N} \), and so on.

The most general form given by (6) is convenient to define aggregated bounds on variables, such as total amount of matter stored in a subset of graph, or the upper bound on the
total supply or demand within the network. The sets \( \mathcal{X}, \mathcal{U}, \mathcal{D} \) and \( \mathcal{F} \) specified by (6) are defined as a finite intersection of closed half-spaces and are, therefore, closed polyhedral sets. Furthermore, due to (5), the sets \( \mathcal{X}, \mathcal{U}, \mathcal{D} \) and \( \mathcal{F} \) are, in fact, polytopes. We further introduce, without loss of generality, the following assumption on the constraints in (6).

Assumption 2.1: For each non-empty \( S \subseteq \mathcal{N} \) there exist \( x_*, x^* \in \mathcal{X}, d_*, d^* \in \mathcal{D} \) and \( \phi_*, \phi^* \in \mathcal{F} \) such that:

\[
x_*(S) = x(S), \quad x^*(S) = \pi(S), \\
d_*(S) = \delta(S), \quad d^*(S) = \delta(S), \\
\phi_*(S) = \omega(S) - \pi(S, S'), \quad \phi^*(S) = \pi(S) + \pi(S', S).
\]

All non-redundant constraints clearly satisfy the above assumption, while all redundant constraints can be modified to comply to Assumption 2.1 since all corresponding sets defined by (6) are closed. Redundant constraints that satisfy Assumption 2.1 will be hereafter referred to as weakly redundant.

Next, we recall some basic terms and results related to robust control invariance. For more in-depth treatment of this rather mature field, the reader is referred to the review [15] and the recent monograph [16].

Definition 2.1 (RCI set): A set \( \mathcal{R} \) is a robust control invariant (RCI) set for the dynamical system (4) subject to constraints (6) if \( \mathcal{R} \subseteq \mathcal{X} \) and for any \( x \in \mathcal{R} \) there exists a control (a flow vector) \( \phi \in \mathcal{F} \) such that \( x + \phi - d \in \mathcal{R} \) for all \( d \in \mathcal{D} \).

Characterization of an RCI sets for constrained, uncertain system (4) represents the fundamental part of a constrained, robust control synthesis, as for, instance, model predictive control (MPC) (cf. [16], [17]). It is usually of interest to identify a maximal RCI set, which we denote as \( \mathcal{R}_\infty \), i.e. the RCI set that is a superset for all RCI sets for a given system and the constraints.

Consider the mapping \( \rho : 2^\mathcal{X} \rightarrow 2^\mathcal{X} \) defined, for a nonempty \( S \subseteq \mathcal{X} \), as follows:

\[
\rho(S) := \{ x \in S : \exists \phi \in \mathcal{F} \text{ s.t. } x + \phi - d \in S, \forall d \in \mathcal{D} \}.
\]

The mapping \( \rho(\cdot) \) gives the set of all states belonging to the argument set that can be “robustly steered” into the same set. From the Definition 2.1 and the expression for \( \rho(\cdot) \) it is evident that a non-empty RCI set \( \mathcal{R} \subseteq \mathcal{X} \) is a fixed point of the mapping \( \rho(\cdot) \), i.e. that \( \rho(\mathcal{R}) = \mathcal{R} \). Direct transilation of the quantifiers in the definition (7) yields the following convenient expression for \( \rho(S) \), for some \( S \neq \emptyset \), in terms of Minkowski sum and difference:

\[
\rho(S) = \{ S \ominus (\mathcal{N}^{\mathcal{D}}) \ominus (\mathcal{F}) \} \cap S.
\]

In the rest of the paper we will first focus on the characterization of the maximal RCI set \( \mathcal{R}_\infty \) for the dynamical system (4) and the constraints (6). We then proceed to consider a decentralized scenario in which each dynamical system (1), or a collection of systems, implements its own robust control law while guaranteeing robust feasibility for all systems, leading to a scheme of decentralized robust control invariance.

**III. Characterization of RCI Sets for Dynamic Networks**

Simple dynamics of the system (7) allows us to characterize the corresponding maximal RCI set \( \mathcal{R}_\infty \) by a simple expression. We assume the following:

Assumption 3.1:

1. \( \mathcal{D} \subseteq \mathcal{F} \).
2. \( \mathcal{X} \ominus (\mathcal{N}^{\mathcal{D}}) \neq \emptyset \).

The following theorem provides the characterization of the maximal RCI set \( \mathcal{R}_\infty \) for the system (4) in terms of the constraints (6):

**Theorem 3.1:** The maximal robust control invariant set \( \mathcal{R}_\infty \) for the system (4) subject to constraints (6) is given as:

\[
\mathcal{R}_\infty = \rho(\mathcal{X}),
\]

where \( \rho(\cdot) \) is defined by (7)–(8). Furthermore, \( \mathcal{R}_\infty \) is non-empty if and only if the Assumption 3.1 holds.

Theorem 3.1 is originally proved in [6], while in [14] we provide an alternative compact proof. Simple form of the maximal RCI set given by (9) allows for the straightforward, optimization based, verification whether a given \( x \in \mathbb{R}^n \) belongs to the set \( \mathcal{R}_\infty \). Namely, for a given \( x \in \mathbb{R}^n \) the condition \( x \in \mathcal{R}_\infty \) is equivalent to feasibility of the set of convex (linear) constraints:

\[
\begin{align*}
x &\in \mathcal{X}, \quad x = y + z, \quad (10a) \\
y &\in \tilde{\mathcal{X}}, \quad z \in \mathcal{N}^{\mathcal{F}}. \quad (10b)
\end{align*}
\]

where \( \tilde{\mathcal{X}} := \mathcal{X} \ominus (\mathcal{D}) \). Similarly, given a state \( x \), one can compute a control vector \( \phi \) that ensures feasibility of the successor state for all admissible realizations of the disturbance \( d \) by finding a flow \( \phi \) that satisfies polyhedral constraints:

\[
x + \phi \in \mathcal{R}_\infty \ominus (\mathcal{D}), \quad \phi \in \mathcal{F}. \quad (11)
\]

The particular structure of the set \( \mathcal{F} \) postulated by (6d) is motivated by the fact that the set of feasible flows indeed assumes such a form, which is immediate consequence of the Gale’s theorem on feasible flows in networks [1]. At this point, we stress that the number of constraints that define the set \( \mathcal{F} \) in general grows exponentially with the number of nodes in the graph.

**IV. Decentralized Solution**

The implicit characterization of the maximal RCI set for the system (4) and the constraints (6) allows for a simple computation of a feasible flow vector \( \phi \in \mathcal{F} \) which, for a given \( x \in \mathcal{R}_\infty \), ensures that \( x + \phi - d \in \mathcal{R}_\infty \) for all possible realizations of \( d \). However, the number of constraints that determine the set \( \mathcal{F} \) in constraints (11), as previously mentioned, in general may grow as \( 2^{\mathcal{N}^{\mathcal{D}}} \) and the centralized robust control of the system (4) may become computationally impractical for a large number of nodes. This motivates our next step, namely to consider a decomposition of the problem into sub-problems through a decentralization.
A. Problem definition

The basic idea behind the decentralized design for the control of dynamical systems (4) subject to particularly structured constraints (6) is the decomposition of the network of nodes with the dynamics (1) into sub–networks in which the flow from other sub–networks is considered as an unknown disturbance of known bounds, determined by the capacities of the flows between the sub–networks. In such setting, a robust controller, deciding on the values of the inflows $u$ and outflows $\delta^-$, is implemented for each sub–network individually, using only the information on the current states of the corresponding collection of nodes.

More formally, assume that the set of nodes $N$ can be decomposed into disjoint subsets $S_1, \ldots, S_q$, $\bigcup_{i=1}^q S_i = N$, such that the constraint sets $\mathcal{X}, \mathcal{D}$ and $\mathcal{F}$ are given as a Cartesian products:

$$
\mathcal{X} = \prod_{i=1}^q \mathcal{X}_{S_i}, \quad \mathcal{D} = \prod_{i=1}^q \mathcal{D}_{S_i}, \quad \mathcal{F} = \prod_{i=1}^q \mathcal{F}_{S_i},
$$

(12)

where $\mathcal{X}_{S_i}, \mathcal{D}_{S_i}$ and $\mathcal{F}_{S_i}$ denote, respectively, the orthogonal projection of the set $\mathcal{X}, \mathcal{D}$ and $\mathcal{F}$ to the subspace $\mathbb{R}^{S_i}$. The collection of such sets $\Delta = \{S_1, \ldots, S_q\}$ for which the Cartesian decomposition (12) exists will be termed network decomposition. The dynamics (4) can now be written as:

$$
x_{S_i}^+ = x_{S_i} + u_{S_i} - \phi_{S_i}^- + \phi_{S_i}^+, \quad i = 1, \ldots, q,
$$

(13)

where $x_{S_i} \in \mathcal{X}_{S_i}$, $u_{S_i} - \phi_{S_i}^- + \phi_{S_i}^+ \in \mathcal{F}_{S_i}$, and $d_{S_i} \in \mathcal{D}_{S_i}$. For each of the $q$ sub–systems in (13), the implemented control strategy is based on the following decision model:

**Interpretation 1:** At each time instant the controller for the $i$th subsystem in (13) has only the information on the current state vector $x_{S_i}$ and decides on the values of $u_{S_i}$ and $\phi_{S_i}^-$. According to Interpretation 1, the information on magnitudes of the inflow $\phi_{S_i}^+$ and the demand $d_{S_i}$ are not available to the controller implemented for each of the $q$ subsystems and, therefore, $\phi_{S_i}^-$ and the demand $d_{S_i}$ represent bounded uncertainties in the dynamics. In order to precisely define the robust control problem for the uncertain systems (13) in the view of the decision model given by Interpretation 1, we first need to determine the constraints on the new control and disturbance variables for each of the $q$ controllers. In previous sections the bounds on the transfers between the individual nodes $u_{ij}$ and $u_{ji}$, which certainly exist in any practical instance of the problem, were not explicitly introduced, as they were not needed in order to define the constraints on the flow $\phi$ with a compact and convex set $\mathcal{F}$. We now assume that the transfers between the individual nodes $u_{ij}, (i, j) \in \mathcal{E}$, are subject to the bounds:

$$
u_{ij} \leq u_{ij} \leq \nu_{ij}, \quad (i, j) \in \mathcal{E}.
$$

(14)

The dynamics of the sub–systems in (13) can be compactly written as:

$$
x_{S_i}^+ = x_{S_i} + \phi_{S_i}^- - d_{S_i}, \quad i = 1, \ldots, q,
$$

(15)

where $\phi_{S_i} := u_{S_i} - \phi_{S_i}$ and $d_{S_i} := d_{S} - \phi_{S_i}^+$. Given $S \subseteq S_i \subseteq N$, the following notation will be used:

$$
\pi(S) := \sum_{j \in S} \pi_{kj} - \sum_{j \notin S} \pi_{jk},
$$

(16a)

$$
\sigma(S) := \sum_{j \in S} \sigma_{jk} - \sum_{j \notin S} \sigma_{kj},
$$

(16b)

$$
\bar{\delta}(S) := \sum_{j \in S} \bar{\sigma}_{kj}, \quad \text{and} \quad \bar{\delta}(S) := \sum_{j \notin S} \bar{\sigma}_{kj},
$$

(16c)

Introduce the sets:

$$
\mathcal{F}^S_i = \left\{ \phi \in \mathbb{R}^{|S_i|} \mid \phi(S) \leq \pi(S) + \bar{\sigma}(S), \forall S \subseteq S_i \right\},
$$

(17a)

$$
\mathcal{D}^S_i = \left\{ d \in \mathbb{R}^{|S_i|} \mid d(S) \leq \bar{\delta}(S) - \bar{\delta}(S), \forall S \subseteq S_i \right\}.
$$

(17b)

The constraints on the variables in (13) are given as:

$$
x_{S_i} \in \mathcal{X}_{S_i}, \quad \phi_{S_i}^- \in \mathcal{F}^S_{S_i} \quad \text{and} \quad d_{S_i} \in \mathcal{D}^S_{S_i}, \quad i \in \mathbb{N}_{1,q}.
$$

We can now formally define decentralized robust control invariance.

**Definition 4.1:** Given a network decomposition $\Delta = \{S_1, \ldots, S_q\}$, let $\mathcal{R}^S_{\infty}$ be a maximal RCI set for the ith sub–system in (13), subject to constraints (16)–(17). Then, the decentralized robust control invariant set for the system (4), the constraints (6) and the network decomposition $\Delta = \{S_1, \ldots, S_q\}$ is given as $\mathcal{R}^\Delta = \prod_{i=1}^q \mathcal{R}^S_i$. It is straightforward to show that the set $\mathcal{R}^\Delta$ for any decomposition $\Delta$ is itself an RCI set for the system (4) and the constraints (6), and as such $\mathcal{R}^\Delta \subseteq \mathcal{R}_\infty$ for any network decomposition $\Delta$. In fact, since for the ith subsystem defined by a set $S_i \in N$ a part of control authority is effectively taken away and added to the uncertainty, it may happen, for given constraints (6) and bounds (14), that the set $\mathcal{R}^\Delta$ is empty for a network decomposition $\Delta$. This motivates our next step, namely, the design of the network parameters that redefine the constraints (6) such that the decentralized design is possible for a given network decomposition $\Delta$.

B. Design for decentralized robust control invariance

If constraints (6) is suitable for the decentralization scheme described in the previous subsection, the decentralized solution can be an attractive alternative to the centralized design due to difficulties that may arise in the centralized implementation of a robust controller for the system (4). In such cases there may even exist an incentive to modify the parameters of the network, i.e., the functions $u(\cdot), \pi(\cdot)$ and the bounds $\nu_{ij}, \nu_{ji}$ for $(i, j) \in \mathcal{E}$, with the goal to make the decentralized design feasible, i.e., the decentralized RCI set $\mathcal{R}^\Delta$ non–empty, for a given network decomposition $\Delta$. In this section we outline important aspects of one possibility of a network parameter design for decentralization.
Consider a network decomposition \( \Delta = \{ S_1, \ldots, S_q \} \). Introduce the vectors of parameters \( \mu^{S_i}, \pi^{S_i} \in \mathbb{R}^{2|S_i|} \) for each \( S_i \in \Delta \) and the parameter vectors:

\[
\begin{align*}
\mathcal{U} &= [\mathcal{U}_{i,j}, \ldots, \mathcal{U}_{m,j,n}]^T, \quad (i,k,j) \in \mathcal{E}, \quad k \in \mathbb{N}_{[1,|\mathcal{E}|]} \\
\mathcal{V} &= [\mathcal{V}_{i,j}, \ldots, \mathcal{V}_{m,j,n}]^T, \quad (i,k,j) \in \mathcal{E}, \quad k \in \mathbb{N}_{[1,|\mathcal{E}|]}.
\end{align*}
\]

Denote the vector of all parameters as:

\[
\xi := [(\mu^{S_1})^T \ldots (\mu^{S_q})^T (\pi^{S_1})^T \ldots (\pi^{S_q})^T \mu^T \pi^T]^T
\]

and consider the following parametrized sets:

\[
\tilde{F}^{S_i}(\xi) = \left\{ \phi \in \mathbb{R}^{2|S_i|}: \begin{array}{l}
\phi(S) \leq \pi^{S_i}(S) + \pi^{S_i}(S) \\
\phi(S) \geq \mu^{S_i}(S) + \mu^{S_i}(S)
\end{array}, \forall S \subseteq S_i \right\}
\]

\[
\tilde{D}^{\delta_i}(\xi) = \left\{ d \in \mathbb{R}^{2|S_i|}: \begin{array}{l}
d(S) \leq \tilde{d}^{S_i}(S) + \delta^{S_i}(S) \\
d(S) \geq d(S) + \delta^{S_i}(S)
\end{array}, \forall S \subseteq S_i \right\}
\]

where \( \pi^{S_i}(S), \delta^{S_i}(S), \tilde{\delta}^{S_i}(S) \) and \( \delta^{S_i}(S) \) represent the parametrized versions of the expressions in (16) obtained by replacing \( \mathcal{U}_{ij} \) and \( \mathcal{V}_{ij} \) by \( \tilde{\mathcal{U}}_{ij} \) and \( \tilde{\mathcal{V}}_{ij} \), respectively. Clearly, the parameters stacked in the vector \( \xi \) defined by (19) represent the bounds on the control inputs \( u \) and the transfers between the individual nodes \( u_{ij}, (i,j) \in \mathcal{E} \). The parameters \( \xi \) that are of interest for the decentralized design are those for which the decentralized RCI set \( \mathcal{R}^{\Delta} \) is non-empty. From the Definition 4.1 (the decentralized RCI set \( \mathcal{R}^{\Delta} \)) and the Theorem 3.1, it is immediate that the set of such parameters is given as:

\[
\Pi := \left\{ \xi: \tilde{F}^{S_i}(\xi) \supseteq \tilde{D}^{\delta_i}(\xi) \right\}
\]

We make use of the following simple lemma:

\textit{Lemma 4.1:} Given a finite \( N = [1, n] \), let \( \mathcal{X} \subset \mathbb{R}^n \) and \( \mathcal{Y} \subset \mathbb{R}^n \) be non-empty polytopes of the form:

\[
\mathcal{X} = \left\{ x: \varphi(S) \leq x(S) \leq \psi(S), \forall S \subseteq N \right\}, \quad \mathcal{Y} = \left\{ y: \varphi(S) \leq y(S) \leq \psi(S), \forall S \subseteq N \right\}
\]

where \( \varphi, \psi, \pi, \gamma: 2^N \to \mathbb{R} \) are defined for all \( S \subseteq N \), with \( \varphi(\emptyset) = \psi(\emptyset) = \pi(\emptyset) = \gamma(\emptyset) = 0 \). Furthermore, assume that all redundant inequalities in (22) are \textit{weakly redundant}, i.e. that for each \( S \subseteq N \) there exist some \( x_1, x_2 \in \mathcal{X} \) and \( y_1, y_2 \in \mathcal{Y} \) such that \( x_1(S) = \varphi(S), x_2(S) = \psi(S), y_1(S) = \psi(S) \) and \( y_2(S) = \varphi(S) \). Then:

(i) \( \mathcal{Y} \subseteq \mathcal{X} \) if and only if \( \varphi(S) \leq \pi(S) \) and \( \psi(S) \geq \gamma(S) \), for all \( S \subseteq N \).

(ii) \( \mathcal{X} \cap \mathcal{Y} \neq \emptyset \) if and only if \( \pi(S) - \gamma(S) \geq \varphi(S) - \psi(S) \), for all \( S \subseteq N \).

Finally, in the view of Lemma 4.1, we can state the following computationally relevant observation on the form of the set of parameters \( \Pi \).

\textit{Proposition 4.1:} The set of parameters \( \Pi \) given by (21) is a polyhedral set (possibly empty).

Note that in general, for generic type of polytopes (or polyhedra) \( \mathcal{F} \) and \( \mathcal{D}^{\delta_i} \) that do not have the particular structure as defined by (17), finding the parameters \( \xi \) such that inclusion \( \mathcal{F}^{\delta_i} \supseteq \mathcal{D}^{\delta_i}(\xi) \) cannot be formulated by means of convex (linear) constraints.

The computation of the appropriate vector of parameters \( \xi \) for which the decentralized RCI set is non-empty can be carried out by minimizing some cost function \( f(\xi) \), subject to the constraint \( \xi \in \Pi \). Additionally, one can add upper/lower bound on the parameters \( \xi \) that reflect the admissible range of link capacities and supply rates in a particular application. If the selected parameter cost function \( f(\xi) \) is convex, then the problem of synthesis of a decentralized RCI set, according to the Proposition 4.1, can be casted as a convex optimization problem.

The first relevant issue is the complexity of such optimization problem. In the definition (19) of the parameter vector \( \xi \), the dimensions of the vectors \( \mu^{S_i} \) and \( \pi^{S_i} \) are declared as exponential in the cardinality of the sets \( S_i \). Similarly, the number of constraints that define the set \( \Pi \) grows exponentially in the number of constraints in the sets \( X_{S_i} \) and the sets in (17). Note however, that the numbers \( 2^{|S_i|} \) for \( |S_i| \) much smaller than \( |S| \) may result in a reasonable number of small optimization problems of manageable size compared to the centralized solution, both from the perspective of the decentralized design and the robust control synthesis.

We considered the scenario in which the design variables are only the parameters that determine the flow between the nodes. In such case, as indicated in Proposition 4.1, for a given set \( D \) of admissible demands, the set of parameters \( \Pi \) for which there exists a decentralized RCI set may be empty. To resolve this, one may introduce additional parameters for lower and upper bounds in the constraints on the state vector \( x \), thus permitting the variation of the storage capacities in order to cope with the given demands \( d \in D \). This extension is direct and the resulting extended set of parameters is again, using observations in Lemma 4.1, a polyhedral set.

V. AN EXAMPLE

Consider a network of integrators defined by the graph shown in Figure 1. The upper bounds on the states \( \mathcal{U}(i) \), inputs \( \mathcal{U}(i) \) and the disturbances \( \mathcal{V}(i) \) are given in Table I. Lower bounds on \( x_i \) and \( d_i \) and \( u_i \) are 0 for all \( i \in [1,11] \).

<table>
<thead>
<tr>
<th>Node</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathcal{U}(i) )</td>
<td>1</td>
<td>1</td>
<td>1.5</td>
<td>1.5</td>
<td>1.5</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( d(i) )</td>
<td>2</td>
<td>2</td>
<td>2.5</td>
<td>2</td>
<td>2.5</td>
<td>1.5</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( \mathcal{V}(i) )</td>
<td>3</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
</tr>
</tbody>
</table>

Additional constraints are imposed on the states:

\[
x_1 + x_3 + x_6 \leq 10, \quad x_2 + x_4 + x_5 \leq 10, \quad x_7 + x_8 + x_9 \geq 2.
\]
Considered network decomposition, as indicated in Figure 1, is $\Delta = \{S_1, S_2, S_3, S_4\}$, where $S_1 = \{1, 3, 6\}$, $S_2 = \{2, 4, 5\}$, $S_3 = \{7, 8, 9\}$ and $S_4 = \{10, 11\}$. The goal is to find a set of edge capacities $u_{ij}$ and $\pi_{ij}$ connecting the nodes such that the decentralized robust control invariant set exists for the network. The upper and lower bounds $u(i)$ and $\pi(i)$ are kept fixed, while the edge capacities $u_{ij}$ and $\pi_{ij}$, $(i, j) \in \mathcal{E}$, are optimized by minimizing the cost: $\|u_{ij}\|_1 + \|\pi_{ij}\|_1$ subject to feasibility constraints defined by (21) and the bounds: $-2 \leq u_{ij} \leq \pi_{ij} \leq 2$, for $(i, j) \in \mathcal{E}$. The values obtained through optimization, that guarantee the existence of a decentralized RCI set, are given in Table II.

**TABLE II**

**Bounds on the flow between the nodes that guarantee existence of a decentralized RCI set (obtained through optimization).**

<table>
<thead>
<tr>
<th>Edges $(i, j)$</th>
<th>$(1,3)$</th>
<th>$(2,1)$</th>
<th>$(3,6)$</th>
<th>$(3,8)$</th>
<th>$(5,2)$</th>
<th>$(5,4)$</th>
<th>$(6,1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u_{ij}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\pi_{ij}$</td>
<td>1</td>
<td>1</td>
<td>0.5</td>
<td>1.5</td>
<td>1.3</td>
<td>0.5</td>
<td>2</td>
</tr>
<tr>
<td>Edges $(i, j)$</td>
<td>$(6,9)$</td>
<td>$(7,5)$</td>
<td>$(8,7)$</td>
<td>$(9,10)$</td>
<td>$(11,6)$</td>
<td>$(11,10)$</td>
<td></td>
</tr>
<tr>
<td>$u_{ij}$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>-0.5</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>$\pi_{ij}$</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>

It is interesting that in this decentralized design some lower bounds on link capacities become negative. Under given constraints and the values of $\pi(i)$ and $u(i)$ fixed for all $i \in \mathcal{N}$ to the values in Table I, there is no parameter vector $\xi \in \Pi$ (given by (21)) such that $u_{ij} \geq 0$ for all nodes $(i, j) \in \mathcal{E}$. Thus in the resulting implementation of a decentralized controller node 9 has “right to demand” flow of 0.5 from the node 10 at any time, and such demand would always be feasible.

Computational benefits in robust control synthesis are visible already in this small example. The set of feasible flows $\mathcal{F}$ for the centralized design for the given network and the parameters in Tables I and II is 11-dimensional polyhedron defined by 360 (non-redundant) inequalities. On the other hand, the evaluation of the robust control law in the proposed decentralized scenario requires solving 3 optimization problems with 3 decision variables and 1 optimization problem with 2 decision variables of the form (11), where the local sets of feasible flows $\mathcal{F}^{S_1}$, $\mathcal{F}^{S_2}$, $\mathcal{F}^{S_3}$ and $\mathcal{F}^{S_4}$ are defined by 14, 12, 12 and 6 inequalities, respectively.

**VI. Conclusion**

We considered a robust control problem of a network of integrators with a different aim compared to previous work on this topic. In particular, we proposed an approach how to split the problem into small robust control sub–problems in such a way that the robust feasibility for the whole network is satisfied. Conditions for the capacities of the network links and the supplies are derived, which guarantee the existence of a decentralized robust control invariant set.

Future work on the topic may include validation of the proposed scheme on a relevant and computationally interesting case study. Further research may be related to more general network design which includes the possibility of adding/removing the links in the network (according to some optimality criterion) with the decentralization as a goal.

**References**