Non-Fragile Control for Trajectory Tracking of Nonholonomic Mobile Robots

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Abstract—This paper is devoted to the non-fragile controller design for the trajectory tracking of nonholonomic mobile robots. Using non-linear state feedback and proper coordinate transformation, the model of nonholonomic mobile robots is exactly linearized. Based on which, the non-fragile controller is designed by employing the linear matrix inequality (LMI) approach. Simulation examples are included to illustrate the effectiveness of the proposed controller.

I. INTRODUCTION

Recently, nonholonomic systems, which can be modeled with constraints concerning velocity or acceleration as well as coordinates and position angle, have become a hot research topic of the mechanical systems. As a class of typical nonholonomic systems, the mobile robots have caused the extensive concern. Nonholonomic mobile robots have good flexibility, since they could realize autonomous movement in the case of nobody involving. The motion tasks of the mobile robots can be divided as point-to-point motion, path following and trajectory tracking [1]. Specifically,

1) Point-to-point motion: The robot must reach a desired goal configuration starting from a given initial configuration.

2) Path following: The robot must reach and follow a geometric path in the Cartesian space starting from a given initial configuration (on or off the path).

3) Trajectory tracking: The robot must reach and follow a trajectory in the Cartesian space (i.e., a geometric path with an associated timing law) starting from a given initial configuration (on or off the trajectory).

The three tasks are sketched in Fig. 1, with reference to the mobile robots. Generally, the last study is more difficult and has more practical significance than the former two. The trajectory tracking control of nonholonomic mobile robots has been given enough attention by researcher in the world. Kanayama et al. proposed a control rule based on the linear approximation of the tracking system error model [2]. Ma et al. showed that the nonholonomic chained systems could be completely linearized by dynamic feedback linearization if the proper output variables were chosen and certain conditions were satisfied [3]. A trajectory tracking control method based on the terminal sliding mode technique was proposed for a kinematic model of two degree of freedom mobile robots in [4]. In [5], a nonlinear state feedback control law was designed for the tracking error system using a Lyapunov function. Li solved the tracking problem of the reference model whose angular velocity and linear velocity both tended to zero [6]. An adaptive controller for trajectory tracking was developed based on the learning ability of wavelet network (WN) in [7]. In [8], a global asymptotically stable control law was designed via Lyapunov direct method using the idea of integral back stepping method. And the fuzzy control solution was introduced to resolve the robustness of mobile robots trajectory tracking problem in [9].

Fig. 1. Motion tasks: (a) Point-to-point motion (b) Path following (c) Trajectory tracking

However, we know that the mobile robots are driven by the DC motor. The designed control strategies need to be digitized in order to realize control goal. In this process, the small change of the control parameters exists and may cause control failure, or even destroy the system. Keel et
al. indicated that the traditional controller design method like optimal control and robust control only led to fragile controller [10]. It means that small offset of controller gain coefficient will be likely to damage the stability of the closed-loop system and degrade the performance. This requires that the designed controller gain coefficient should have sufficient adjustable redundancy or non-fragility in order to meet different performance requirements.

In this paper, the non-fragile control for global asymptotic stability is applied to the nonholonomic mobile robots in order to improve the practicability. The paper is organized as follows. In Section 2, the equation of a nonholonomic mobile robot is linearized via state feedback. The non-fragile trajectory tracking controller is designed in Section 3. Finally, the effectiveness of designed controller is verified through comparing with the normal state-feedback controller by the simulation in Section 4.

II. PROBLEM FORMULATION AND PRELIMINARIES

The model of a mobile robot with two independently drivable wheels considered in this paper is shown in Fig. 2, where \( XOY \) is the world coordinate system, \( X_aO_aY_a \) is the coordinate system fixed the mobile robot body, \( X_a \) is the center of the axle of two driving wheels, \((x, y)\) indicates the coordinate of the robot in world coordinate system, \( \theta \) is the angle of moving direction (right angle to the wheel axis), \( v \) is the linear velocity of the robot and \( \omega \) is its angular velocity.

![Fig. 2. The planar graph of a mobile robot](image)

Although the model is the simplest one which has constrained by velocity, it has inherent difficulty of the nonholonomic system. Suppose that the wheels of the robot rotate without slipping. Thus, the constraint of the mobile robot motion is denoted by

\[
\dot{x} \sin \theta - \dot{y} \cos \theta = 0. \tag{1}
\]

And the model of a nonholonomic mobile robot can be obtained

\[
\begin{bmatrix}
\dot{x} \\
\dot{y} \\
\dot{\theta}
\end{bmatrix} =
\begin{bmatrix}
\cos \theta & 0 \\
\sin \theta & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
v \\
\omega
\end{bmatrix}. \tag{2}
\]

We can see that system (2) can’t be exact linearization via input-state feedback.

Consider now the following nonlinear system:

\[
\begin{align*}
\dot{x} &= f(x) + g(x)u \\
y &= h(x)
\end{align*} \tag{3}
\]

where

\[
g(x) = \begin{bmatrix} g_1(x) & \ldots & g_m(x) \end{bmatrix},
\]

\[
h(x) = \text{col}(h_1(x), \ldots, h_m(x))
\]

are respectively an \( n \times m \)-matrix and an \( m \)-vector, \( f(x), g_1(x), \ldots, g_m(x) \) are smooth vector fields, and \( h_1(x), \ldots, h_m(x) \) are smooth function defined on an open set of \( \mathbb{R}^n \).

**Definition 1:** (relative degree) [11] The nonlinear system of the form (3) has a (vector) relative degree \( \{r_1, r_2, \ldots, r_m\} \) at point \( x^0 \) if

1) For all \( 1 \leq j \leq m, 1 \leq i \leq m, k \leq r_i - 1 \), and for all \( x \) in a neighborhood of \( x^0 \),

\[
L_{g_j} L_f^{r_j - r_i} h_i(x) = 0;
\]

2) The \( m \times m \) matrix

\[
\begin{bmatrix}
L_{g_1} L_f^{r_1 - 1} h_1(x) & \ldots & L_{g_m} L_f^{r_m - 1} h_1(x) \\
\vdots & \ddots & \vdots \\
L_{g_1} L_f^{r_1 - 1} h_m(x) & \ldots & L_{g_m} L_f^{r_m - 1} h_m(x)
\end{bmatrix}
\]

is nonsingular at \( x = x^0 \).

**Lemma 1:** [11] Suppose the matrix \( g(x^0) \) has rank \( m \). Then, the exact feedback linearization problem is solvable if and only if there exist a neighbourhood \( U \) of \( x^0 \) and \( m \) real-valued functions \( h_1(x), \ldots, h_m(x) \) defined on \( U \), such that the system (3) has (vector) relative degree \( \{r_1, r_2, \ldots, r_m\} \) at \( x^0 \) and \( \sum_{i=1}^{m} r_i = n \).

From Definition 1 and Lemma 1, the system (2) can’t be exact feedback linearization through choosing outputs \( Y = [x \ y]^T \), because it has no relative degree.

To solve this problem, an auxiliary variable

\[
\psi = a \tag{4}
\]

is introduced. Combine (4) with (2), and set

\[
X = [x \ y \ \theta \ \psi]^T,
\]

\[
Y = [x \ y]^T,
\]

\[
u = [a \ \omega]^T.
\]

Then, the system (2) is transformed to the following form

\[
\begin{align*}
\dot{X} &= f(X) + g(X)u \\
Y &= h(X)
\end{align*} \tag{5}
\]

where

\[
f(X) = \begin{bmatrix}
\cos \theta \\
\sin \theta \\
0
\end{bmatrix} v, 
\]

\[
g(X) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},
\]

\[
h(X) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} X.
\]

**Lemma 2:** For \( v \neq 0 \), the system (5) can be exact feedback linearization if the output vector is chosen as \( Y = [y_1 \ y_2]^T = [x \ y]^T \).
Proof: The derivative of $Y$ is given by

\[
\begin{align*}
\dot{y}_1 &= x = v \cos \theta \\
\ddot{y}_1 &= a \cos \theta - \omega v \sin \theta \\
\dot{y}_2 &= y = v \sin \theta \\
\ddot{y}_2 &= a \sin \theta + \omega v \cos \theta.
\end{align*}
\]  

Then we can obtain the decoupling matrix by the equation (7) and (9)

\[
D(X) = \begin{bmatrix}
\cos \theta & -v \sin \theta \\
\sin \theta & v \cos \theta
\end{bmatrix}
\]

and we have

\[
\det(D(X)) = v \neq 0.
\]

We can see that if we choose output vectors as $y_1 = x$ and $y_2 = y$ when $v \neq 0$, the system has relative degree $(r_1, r_2) = (2, 2)$ and $r_1 + r_2 = 4$. Then, by Lemma 1, the system (5) can be exact feedback linearization.

Let

\[
\begin{align*}
\xi(t) &= [y_1 y_1 y_2 y_2]^T = [x v \cos \theta y v \sin \theta]^T, \\
Y(t) &= [y_1 y_2]^T, \\
u(t) &= [a \cos \theta - \omega v \sin \theta \quad a \sin \theta + \omega v \cos \theta].
\end{align*}
\]

The system (5) becomes:

\[
\begin{align*}
\dot{\xi}(t) &= \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} \xi(t) + 
\begin{bmatrix}
1 & 0 \quad 0 & 0
\end{bmatrix} \begin{bmatrix}
u(t) \\
0 & 1
\end{bmatrix} \\
= & \ddot{A} \xi(t) + \ddot{B} u(t)
\end{align*}
\]

\[
Y(t) = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix} \xi(t) = \ddot{C} \xi(t).
\]

Consider the non-fragile controller

\[
u(t) = (K + \Delta K) \xi(t)
\]

where $K$ is the state feedback controller; $\Delta K = \Delta(t) F$ is the additive gain perturbation of the controller with $E, F$ the real matrices of known appropriate dimensions, $\Delta(t)$ Lebesgue measurable matrix and $\Delta'(t) \Delta(t) \leq I$.

Combining the system (10) and controller (11) yields the following closed-loop system:

\[
\begin{align*}
\dot{\xi}(t) &= [\ddot{A} + \ddot{B}(K + \Delta K)] \xi(t) \\
Y(t) &= \dddot{C} \xi(t).
\end{align*}
\]

The objective of this paper is to design a non-fragile controller (11) such that the closed-loop system (12) is asymptotically stable.

Lemma 3: [12] Let $E, F$ be real matrices of appropriate dimensions, and $\Delta(t)$ be time-varying matrix satisfying $\Delta'(t) \Delta(t) \leq I$. Then, for any scalar $\varepsilon > 0$,

\[
E \Delta(t) F + F^T \Delta'(t) E^T \leq \varepsilon EE^T + \varepsilon^{-1} F^T F.
\]

Lemma 4: [13] Let $M > 0, N > 0, L$ be matrices of appropriate dimensions. Then,

\[
M + L^T N^{-1} L < 0
\]

if and only if

\[
\begin{bmatrix}
M & L^T \\
L & -N
\end{bmatrix} < 0.
\]

III. NON-FRAGILE CONTROLLER DESIGN

The following theorem gives the sufficient condition for existence of non-fragile state feedback controller for system (10) when the controller has additive gain perturbation.

Theorem 1: The system (12) is asymptotically stable if there exist a scalar $\varepsilon > 0$ and matrices $Q > 0, P$ such that the following LMI holds

\[
\begin{bmatrix}
\dot{A} Q + Q \dot{A}^T + \dot{B} P + (\dot{B} P)^T & \varepsilon \dot{B} E & (F Q)^T \\
\varepsilon (\dot{B} E)^T & -\varepsilon I & 0 \\
F Q & 0 & -\varepsilon I
\end{bmatrix} < 0.
\]

And further, the non-fragile controller is taken as $K = PQ^{-1}$.

Proof: For system (12), define Lyapunov function candidate

\[
V(t) = \xi^T(t) R \xi(t) > 0
\]

where $R = Q^{-1}$ and $Q$ is a solution of the LMI (13). The derivative of $V$ is given by

\[
\dot{V}(\xi(t)) = \dot{\xi}^T(t) [R(\dot{A} + \dot{B}(K + \Delta K)) + (\dot{A} + \dot{B}(K + \Delta K))^T R \xi(t)].
\]

Let $Z = R(\dot{A} + \dot{B}(K + \Delta K)) + (\dot{A} + \dot{B}(K + \Delta K))^T R$. If $Z < 0$, then system (12) is asymptotically stable. Pre- and post-multiplying $Z$ by matrix $Q$, and setting $P = KQ$, we can obtain

\[
\begin{align*}
QZ Q &= \dot{A} Q + \dot{Q} \dot{A}^T + \dot{B} Q K + Q K^T \dot{B}^T + \dot{B} E \Delta(t) F Q \\
&+ Q F \Delta(t) E^T \dot{B}^T \\
&= \ddot{A} Q + \dot{Q} \dot{A}^T + \dot{B} P + P^T \dot{B}^T + \dot{B} E \Delta(t) F Q \\
&+ Q F \Delta(t) E^T \dot{B}^T.
\end{align*}
\]

From the Lemma 2, if there exists a scalar $\varepsilon > 0$, we have

\[
\dot{\xi}^T(t) [R(\dot{A} + \dot{B}(K + \Delta K)) + (\dot{A} + \dot{B}(K + \Delta K))^T R \xi(t)] \\
\leq \varepsilon [\dot{B} E]^T (F Q)^T (F Q).
\]

It follows from (16) and (17) that

\[
Q Z Q \\
\leq \ddot{A} Q + \dot{Q} \dot{A}^T + \dot{B} P + P^T \dot{B}^T + \varepsilon [\dot{B} E]^T (F Q)^T (F Q) \\
= \ddot{A} Q + \dot{Q} \dot{A}^T + \dot{B} P + P^T \dot{B}^T + \varepsilon^{-1} (F Q)^T (F Q) \\
\leq \varepsilon [\dot{B} E]^T (F Q)^T (F Q) \\
< 0
\]

where $U = \text{diag}(-\varepsilon I, -\varepsilon I)$.

By Lemma 3, the upper formula can be expressed by the following

\[
\begin{bmatrix}
\dot{A} Q + \dot{B} P + (\dot{B} P)^T & \varepsilon \dot{B} E & (F Q)^T \\
\varepsilon (\dot{B} E)^T & -\varepsilon I & 0 \\
F Q & 0 & -\varepsilon I
\end{bmatrix} < 0.
\]
That is to say, (13) can guarantee \( \dot{V}(t) < 0 \). Since the defined Lyapunov functional \( V(t) > 0 \), it shows that the system (12) is asymptotically stable. And the the non-fragile controller is \( K = PR = PQ^{-1} \). This completes the proof.

Obviously, if \( \Delta(t) = 0 \), Theorem 1 degenerates to the normal state feedback control. Now applying the designed controller \( K \) in Theorem 1 to the nonholonomic mobile robots (10), we obtain

\[
u(t) = K \xi = \begin{bmatrix} a \cos \theta - \omega v \sin \theta \\ a \sin \theta + \omega v \cos \theta \end{bmatrix}.
\]

IV. SIMULATION EXAMPLES

In this section, we confirm the validity of the proposed method and compare the control effect between the non-fragile controller and the normal state feedback controller through the simulation examples.

Consider the systems (12) with

\[
E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad F = \begin{bmatrix} 0.5 & 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0.5 & 0.5 \end{bmatrix}, \\
\Delta_1 = -I = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \Delta_2 = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\]

From Theorem 1, we get the non-fragile controller by employing the LMI Toolbox in MATLAB

\[
K_{nf} = \begin{bmatrix} -2.2500 & -2.0000 & -0.9375 & -1.0625 \\ -0.9375 & -1.0625 & -2.2500 & -2.0000 \end{bmatrix},
\]

and the normal state feedback controller is

\[
K_{nor} = \begin{bmatrix} -1.3125 & -0.9375 & 0 & 0 \\ 0 & 0 & -1.3125 & -0.9375 \end{bmatrix}.
\]

When the two controllers resist the perturbation \( \Delta_1 \), they all can make the system stable (see Fig.3 and Fig.4). Using the normal control with the perturbation \( \Delta_2 \), the system becomes unstable (see Fig.5), while the non-fragile controller with the perturbation \( \Delta_2 \) still makes the system stable (see Fig.6).

We require the mobile robot to make the circular motion:

\[
x_d(t) = 0.5 \cos(0.01t) , \quad y_d(t) = 0.5 \sin(0.01t) , \quad \theta_d(t) = t.
\]

Fig.7 and Fig.8 are the state \( x \) and \( y \) tracking trajectories respectively. Fig.9 is the mobile robot motion trajectory using the non-fragile controller on the \( x-y \) plane. The red dash lines express the mobile robot motion trajectory and the blue lines describe the desired trajectory. We can see that the trajectory can converge to the expected trajectory quickly.
V. CONCLUSIONS AND FUTURE WORKS

In this paper, the non-fragile control has been introduced to nonholonomic mobile robots for the first time. After exactly linearizing the model of nonholonomic mobile robots via state feedback, the non-fragile controller with the additive gain perturbation is designed in order to make the mobile robots asymptotically stable. Finally, the proposed non-fragile controller is used to the mobile robots trajectory tracking control. Simulation results show that the designed non-fragile controller has strong robustness to controller gain perturbations, which guarantee the fast response and superior control effect to the normal controller. In our future works, the non-fragile control will be applied to the multi-agent systems.

REFERENCES


