Minimum second moment state for the existence of average optimal stationary policies in linear stochastic systems

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Abstract—This note considers the long-run average cost control problem for a class of discrete-time stochastic systems. The stochastic system is assumed to be linear with respect to the state but the controls possess a general structure, possibly a nonlinear one. The main contribution of this paper is to show that the existence of a minimal second moment system state implies the existence of an optimal stationary policy for the long-run average cost problem. A numerical example illustrates the derived result.

I. INTRODUCTION

Consider a discrete-time stochastic linear system defined in a filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_k\}, P)\) as follows:

\[
x_{k+1} = A(g_k)x_k + Ew_k, \quad g_k \in \mathcal{G}, \quad \forall k = 0, 1, \ldots, \quad x_0 \in \mathbb{R}^n,
\]

where \(x_k\) and \(w_k, k = 0, 1, \ldots\) are processes taking values respectively, in \(\mathbb{R}^n\) and \(\mathbb{R}^q\), which represent the system state, and additive noisy input, respectively. The noisy input \(\{w_k\}\) forms an iid process with zero mean and covariance matrix equal to the identity for each \(k \geq 0\). The matrix \(E\), of dimension \(n \times q\), is given. The variable \(g_k\), at the \(k\)-th stage, represents the control action and belongs to a prescribed set \(\mathcal{G}\). We assume that \(A\) is a given operator, possibly nonlinear, that maps \(\mathcal{G}\) to the space of real matrices of dimension \(n \times n\).

The long-run average cost is defined as

\[
J = \limsup_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} E\left[Q(g_k)x_k\right],
\]

(2)

where \(E[\cdot] = E[\cdot|\mathcal{F}_0]\) denotes the mathematical expectation and \(Q\) is a given operator that maps \(\mathcal{G}\) to the space of nonnegative symmetric matrices of dimension \(n \times n\).

At this point we characterize precisely the sequence of control actions \(g_k, k = 0, 1, \ldots\). Indeed, we assume that the control has no access to the filtration \(\{\mathcal{F}_k\}\), so that each control action \(g_k\) is based on the evolving probability distribution only. In particular, this situation arises naturally when \(g_k\) is a gain matrix to be designed.

For instance, in the output feedback control problem one seeks a matrix sequence \(\{g_k\}\) so that the average cost in (2) is minimized with \(A(g_k) := A + Bg_kC\) and \(Q(g_k) := Q + C'g_kRg_kC\), where \(A, B, C, Q, R\) are matrices of appropriate dimensions \([1]\). It will be seen in the sequel that the control action \(g_k \in \mathcal{G}\) is heavily dependent on a suitable function \(f_k\). As a matter of fact, the value of \(g_k\) depends on both a set of functions \((f_0, \ldots, f_k)\) and \(x_0 \in \mathbb{R}^n\).

The infinite set of functions \(f = \{f_0, \ldots, f_k, \ldots\}\) is referred to as a policy.

Following these motivations, we now provide the precise setup and objectives of this paper. From the assumption on the process \(\{w_k\}, k \geq 0\), we have that the second moment of the system state \(x_k\) in (1), represented by

\[
X_k = E[x_kx_k'], \quad \forall k \geq 0,
\]

satisfies a deterministic matrix recurrence (see (8)), and there holds the identity

\[
E[x_k'Q(g_k)x_k] = (X_k, Q(g_k)), \quad g_k \in \mathcal{G}, \quad \forall k \geq 0,
\]

where \(\langle \cdot, \cdot \rangle\) denotes the Frobenius inner product. Hence the long-run average cost (2) is identical to

\[
J = \limsup_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} \langle X_k, Q(g_k) \rangle.
\]

(4)

The deterministic expression of (4) allows us to choose a control action in the form \(g_k = f_k(X_k)\) at the \(k\)-th stage, thus it is associated with a policy \(f = \{f_0, \ldots, f_k, \ldots\}\). Let \(\mathcal{F}\) be the set of all feasible policies \(f\), and let \(\mathcal{F}_s \subset \mathcal{F}\) be the set of all stationary policies, in such a way that if \(f \in \mathcal{F}_s\), then \(f = \{f, f, \ldots\}\). Let \(J(f, X)\) be the cost \(J\) in (4) corresponding to a given policy \(f\) and \(X_0 = X\). The average control problem is to find a policy \(f^*\) such that

\[
J(f^*, X) = \inf_{f \in \mathcal{F}} J(f, X), \quad \text{for each } X.
\]

Clearly

\[
\inf_{f \in \mathcal{F}} J(f, X) \leq \inf_{f \in \mathcal{F}_s} J(f, X), \quad \text{for each } X,
\]

(5)

and the above inequality incites the following question.

(Q) Does the stationary class \(\mathcal{F}_s\) contain the optimal policy \(f^*\)?

The main contribution of this paper is to provide conditions to answer affirmatively the question in (Q). In fact, the question in (Q) was previously investigated in [2, 3], and [4]. In [2] and [3], the authors require the existence of an exponential decay for the optimal discounted problem, whereas in [4] they introduce a controllability to
the origin condition linked with a condition based on the optimal discounted solution. The approach in this paper does not require any condition on the optimal discounted solution, so that it can be seen as an improvement on the ones in [2], [3], and [4].

The working paper [5] is also devoted to study the system (1), but that paper focuses on a technique to evaluate the optimal long-run average cost by approximating it by solutions of the finite horizon problem.

The paper is organized as follows. Section II presents the necessary notation, definitions, assumptions, and the main result. This section also presents an example to illustrate the derived theory. Finally, some concluding remarks are presented in Section III.

II. PRELIMINARIES, NOTATIONS, AND MAIN RESULTS

The real and natural numbers are denoted by $\mathbb{R}$ and $\mathbb{N}$, respectively. The set of nonnegative real numbers is denoted by $\mathbb{R}_+$, and $\mathbb{R}_{0,m}$ is used to represent the space of all $n \times m$ real matrices. The superscript $'$ indicates the transpose of a matrix. Let $\mathbb{S}^d_0$ be the closed convex cone \{ $U \in \mathbb{R}^{d,m} : U = U' \geq 0$ \}; $\langle \cdot, \cdot \rangle$ will stand for the inner product in $\mathbb{S}^d_0$, and $\| \cdot \|$ will denote either the standard Euclidean norm in $\mathbb{R}^n$ or the Frobenius norm for matrices. We say that a matrix sequence $\{ U_k; k \geq 0 \}$ is bounded if $\sup_{k \in \mathbb{N}} \| U_k \| < \infty$.

The following definitions and conventions will apply throughout this paper.

(i) $\mathcal{X}$ and $\mathcal{G}$ are given sets referred to as state space and control space, respectively. In particular, we assume that $\mathcal{X} \subseteq \mathbb{S}^d_0$ and $\mathcal{G}$ are Borel spaces.

(ii) For each $X \in \mathcal{X}$, there is given a nonempty measurable subset $\mathcal{G}(X)$ of $\mathcal{G}$. The set $\mathcal{G}(X)$ represents the set of feasible controls or actions when the system is in state $X \in \mathcal{X}$, and with the property that the graph $\text{Gr} := \{ (X, g) | X \in \mathcal{X}, g \in \mathcal{G}(X) \}$ of feasible state-actions pairs is measurable.

(iii) (inf-compactness assumption). Let $Q : \mathcal{G} \to \mathbb{S}^d_0$ be a lower semi-continuous function. The one-stage cost functional $\mathcal{E} : \text{Gr} \to \mathbb{R}_+$ is defined as follows:

$$\mathcal{E}(X, g) = \langle Q(g), X \rangle, \quad \forall (X, g) \in \text{Gr}. \quad (7)$$

Moreover, for each $X \in \mathcal{X}$ and $r \in \mathbb{R}_+$, the set $\{ g \in \mathcal{G}(X) | \mathcal{E}(X, g) \leq r \}$ is compact (see the inf-compact definition in [6, p.28], [7, p.13]).

(iv) A policy $\mathbf{f} = \{ f_0, f_1, \ldots \}$ is a sequence of measurable functions $f_k, k \geq 0$, where $f_k(X) \in \mathcal{G}(X)$ for each $X \in \mathcal{X}$. The set of all policies is denoted by $\mathbf{F}$, and elements of $\mathbf{F}$ in the form $\mathbf{f} = \{ f_k \} \in \mathbf{F}$, the second moment matrix $X_k \in \mathcal{X}$ in (3) corresponding to $\mathbf{f}$ satisfies the recurrence (c.f. [8, Ch.2])

$$X_{k+1} = A(g_k)X_kA(g_k)' + \Sigma, \quad \forall k \geq 0, \quad X_0 = X \in \mathcal{X}, \quad (8)$$

where the control action applied at the $k$-th stage obeys the rule $g_k = f_k(X_k)$. Sometimes we will use the notation $X_k(g_{k-1}, \ldots, g_0, X)$ for each $k \geq 1$ to denote the solution in (8).

The long-run average cost can be defined as follows:

$$J(\mathbf{f}, X) := \lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} \langle Q(g_k), X_k \rangle, \quad (9)$$

for each policy $\mathbf{f} \in \mathbf{F}$ and each initial state $X_0 = X \in \mathcal{X}$. The average cost control problem is to find a policy $\mathbf{f}^*$ such that

$$J(\mathbf{f}^*, X) = \inf_{\mathbf{f} \in \mathbf{F}} J(\mathbf{f}, X) =: J^*(X), \quad \forall X \in \mathcal{X}. \quad (10)$$

The policy $\mathbf{f}^*$ satisfying (10) is referred to as average cost optimal, and the main contribution of this paper is to provide conditions under which $\mathbf{f}^*$ is stationary.

A. The main result

Let us consider the next assumption.

Assumption 2.1: (Minimum second moment state). For some $N \in \mathbb{N}$, there exist a finite sequence of control actions $\{ \bar{g}_0, \ldots, \bar{g}_N \}$ and a fixed matrix $Z \in \mathbb{S}^d_0$ such that

$$Z = X_{N+1}(\bar{g}_N, \ldots, \bar{g}_0, X) \leq X_{N+1}(\bar{g}_N, \ldots, g_0, X), \quad \forall g_i \in \mathcal{G}, \quad i = 0, \ldots, N, \quad \forall X \in \mathcal{X}. \quad (11)$$

Now we can state the main result of this paper.

Theorem 2.1: If Assumption 2.1 is satisfied, then there exists a stationary policy $\mathbf{f}^* = \{ f^*, f'^*, \ldots \} \in \mathbf{F}$ which is average cost optimal, and there is a constant $\rho \geq 0$ so that

$$J^*(X) = J(\mathbf{f}^*, X) = \rho, \quad \forall X \in \mathcal{X}. \quad (12)$$

The proof of Theorem 2.1 is postponed to the next section.

Remark 2.1: Assumption 2.1 implies the controllability to the origin, i.e., it assures the existence of a control sequence $\{ \bar{g}_0, \ldots, \bar{g}_N \}$ such that $A(\bar{g}_N) \cdots A(\bar{g}_0) = 0$. Indeed, from (8) we can write

$$X_{N+1}(\bar{g}_N, \ldots, \bar{g}_0, X) = \phi(N, 0)X \phi(N, 0)' + \Sigma + \sum_{j=1}^{N} \phi(N, j) \Sigma \phi(N, j)',$$

where $\phi(N, j) := A(\bar{g}_N) \cdots A(\bar{g}_j)$ for each $j = 0, \ldots, N$. Since the value of $Z$ in Assumption 2.1 is invariant with respect to $X \in \mathcal{X}$, one can then see that it must be $\phi(N, 0) = 0$. Thus controllability to the origin is a necessary condition to Assumption 2.1.

Example 2.1: In this example we consider the output feedback control problem as follows. Let us define

$$A = \begin{bmatrix} 0.75 & -1.25 \\ 2.25 & -2.75 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix},$$

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and \( A(g) := A + BgC \) for each \( g \) in \( \mathcal{S} := \mathbb{R} \). Let the matrix \( E \) be such that
\[
\Sigma = EE' = \begin{bmatrix} 0.226 & 0.1918 \\ 0.1918 & 0.5754 \end{bmatrix}.
\]

We claim that Assumption 2.1 holds. Indeed, one can readily verify that if \( \bar{g}_0 = \bar{g}_1 = 1 \) then
\[
A(\bar{g}_1)A(\bar{g}_0) = 0.
\]

Moreover, by representing \( \lambda_{\text{max}}(g) \) (\( \lambda_{\text{min}}(g) \)) as the maximum (minimum) eigenvalue of the matrix \( A(g)\Sigma A(g)' - A(\bar{g}_1)\Sigma A(\bar{g}_1)' \), we have that \( \lambda_{\text{max}}(g) \geq \lambda_{\text{min}}(g) \geq 0 \) for each \( g \in \mathcal{S} \) (see the curves in Fig. 1). It follows that
\[
A(g)\Sigma A(g)' - A(\bar{g}_1)\Sigma A(\bar{g}_1)' \leq 0, \quad \forall g \in \mathcal{S},
\]

which combined with (12) implies that
\[
Z := X_2(g^*_0, g^*_0, X) = A(g^*_1)\Sigma A(g^*_1)' + \Sigma \leq A(g)\Sigma A(g)' + \Sigma, \quad \forall g \in \mathcal{R}.
\]

Hence \( Z = X_2(g^*_1, g^*_0, X) \leq X_2(g_1, g_0, X) \) for each \( g_1, g_0 \in \mathcal{S} \) and \( X \in \mathcal{X} \), so that Assumption 2.1 is verified.

Now, consider an arbitrary (perhaps nonlinear) continuous function \( Q: \mathcal{S} \to \mathbb{R}_+ \). We then have from Theorem 2.1 that there exist a stationary policy \( f^* = \{f^*, f^*, \ldots\} \) and a constant \( \rho^* \geq 0 \) such that, by setting \( g^*_k = f^*(X^*_k), k \geq 0 \), we get
\[
\rho^* = J^*(X) = \lim_{N \to \infty} \sup_{N} \frac{1}{N} \sum_{k=0}^{N-1} (X^*_k, Q(g^*_k))
\]

\[
\leq J(f, X), \quad \forall f \in \mathcal{F}, \quad \forall X \in \mathcal{X},
\]

where \( X^*_k+1 = A(g^*_k)X^*_k + \Sigma \) and \( X_0 = X \in \mathcal{X} \).

For the numerical evaluation of \( \rho^* \), let \( C: \mathcal{S} \to \mathbb{R}^{2 \times 2} \) be the Vinogradov function \( [9] \), i.e.,
\[
C(g) = \begin{bmatrix} c_{11}(g) & c_{12}(g) \\ c_{21}(g) & c_{22}(g) \end{bmatrix}, \quad \forall g \in \mathcal{S},
\]

with
\[
c_{11}(g) = -1 - 9 \cos^2(6g) + 12 \sin(6g) \cos(6g),
\]
\[
c_{12}(g) = 12 \cos^2(6g) + 9 \sin(6g) \cos(6g),
\]
\[
c_{21}(g) = -12 \sin^2(6g) + 9 \sin(6g) \cos(6g),
\]
\[
c_{22}(g) = -1 - 9 \sin^2(6g) - 12 \sin(6g) \cos(6g),
\]

and define \( Q(g) = (C(g)')^{-1}C(g)^{-1} + 10^{-5}I, \quad \forall g \in \mathcal{S} \).

Suppose now that the limit \( g^* = \lim_{k \to \infty} g^*_k \) exists. In this case, with the aid of a numerical evaluation we obtain \( (\rho^*, g^*) = (0.3654, 0.8388) \), Fig. 2. The existence of the limit \( g^* = \lim_{k \to \infty} g^*_k \) is an open problem and conditions to assure it is under development.

**B. Proof of Theorem 2.1**

Before introducing the main argument to prove Theorem 2.1, we need some preliminary notation.

For each \( \alpha \in (0, 1) \), the discount criterion is defined as

\[
V_\alpha(f, X) := \sum_{k=0}^{\infty} \alpha^k Q(g_k, X_k), \quad \forall f \in \mathcal{F}, \quad \forall X_0 = X \in \mathcal{X},
\]

where \( \alpha \) denotes the discount factor, and \( X_k \) and \( g_k \) are defined as in (8) and they correspond to the policy \( f \in \mathcal{F} \) and initial state \( X_0 = X \in \mathcal{X} \). The associated control problem is

\[
V^*_\alpha(X) := \inf_{f \in \mathcal{F}} V_\alpha(f, X), \quad \forall X_0 = X \in \mathcal{X}.
\]

**Proposition 2.1:** \( [10, \text{Prop.5.11, p.87}, [6, \text{Th.4.2.3, p.46}]) \) Under the inf-compact assumption (see item (iii), p. 2), there exists a policy \( f^*_\alpha \in \mathcal{F} \) such that

\[
V^*_\alpha(X) = V_\alpha(f^*_\alpha, X), \quad \forall X_0 = X \in \mathcal{X}, \forall \alpha \in (0, 1).
\]

The next result establishes the monotonicity of the function \( V^*_\alpha(\cdot) \).

**Proposition 2.2:** \( ([2, [4, \text{Lem.2.4}])] \). If \( X, Y \in \mathcal{X} \), then

\[
X \geq Y \Rightarrow V^*_\alpha(X) \geq V^*_\alpha(Y), \quad \forall \alpha \in (0, 1).
\]

The next lemma will be useful in the sequel.
Lemma 2.1: Suppose that Assumption 2.1 holds. If $N \geq 1$ is a finite natural number, then there is a constant $c = c(N) > 0$ such that

$$(1 - \alpha^N) V_\alpha^*(X) \leq c \|X\|, \quad \forall \alpha \in (0, 1), \forall X \in \mathcal{X}.$$  

Proof: From Assumption 2.1 one can set

$$g_{N+i} = \tilde{g}_i, \quad \forall k \geq 0, \quad i = 0, \ldots, N,$$

which in turn implies (see (8))

$$X_{N+i+1} = g_{N+i+1}(g_{N+i}, \ldots, g_0, X) = X_{i+1}(\tilde{g}_i, \ldots, \tilde{g}_0, X).$$

for each $k \geq 0$ and $i = 0, \ldots, N$. From this and (8) one can show that there is a constant $c > 0$ such that

$$\sup_{k \geq 0} \|X_k\| \leq c \|X\|. \quad (15)$$

Using an optimality argument, we can write

$$V_\alpha^*(X) \leq \sum_{k=0}^{\infty} \alpha^k \langle Q(g_k), X_k \rangle \leq \frac{\|X\| \cdot \max_{i=0, \ldots, N} \|Q(\tilde{g}_i)\|}{1 - \alpha}.$$  

Multiplying the above inequality by $(1 - \alpha^N)$, we get that

$$(1 - \alpha^N) V_\alpha^*(X) \leq (1 - \alpha^N) \frac{\|X\| \cdot \max_{i=0, \ldots, N} \|Q(\tilde{g}_i)\|}{1 - \alpha}.$$  

Using a simple induction argument, one can show the validity of the identity

$$1 - \alpha^{N+1} = (1 - \alpha^N) (1 + \alpha (1 + \alpha^2) \cdots (1 + \alpha^{N-1})) \quad \forall k \geq 1.$$  

Now, take $N_1 > 0$ such that $2^{N_1} \geq N$, and note that

$$1 - \alpha^{N_1} \geq 1 - \alpha^N.$$  

From this fact, the above identity, and (16), we have

$$(1 - \alpha^N) V_\alpha^*(X) \leq (1 - \alpha^{N_1}) \frac{c \|X\| \cdot \max_{i=0, \ldots, N} \|Q(\tilde{g}_i)\|}{1 - \alpha} \leq (1 + \alpha (1 + \alpha^2) \cdots (1 + \alpha^{N-1})) \frac{c \|X\| \cdot \max_{i=0, \ldots, N} \|Q(\tilde{g}_i)\|}{1 - \alpha} \leq 2^{N_1-1} c \|X\| \cdot \max_{i=0, \ldots, N} \|Q(\tilde{g}_i)\|,$$

which shows the desired result.

The next result plays a key role in the proof of Theorem 2.1.

Proposition 2.3: ([6, Th. 5.4.3, p. 88], [11, Th. 3.8]). Let us consider the next two assumptions:

(H_1) For some $\alpha_0 \in (0, 1)$, there exist a constant $c > 0$ and a matrix $Z \in \mathbb{S}^{n^0}$ such that

$$(1 - \alpha) V_\alpha^*(Z) \leq c, \quad \forall \alpha \in [\alpha_0, 1).$$

(H_2) There exist a measurable function $b: \mathcal{X} \to \mathbb{R}_+$ and a constant $c_2 > 0$ such that

$$c_2 \leq h_\alpha(X) \leq b(X), \quad \forall X \in \mathcal{X}, \quad \forall \alpha \in [\alpha_0, 1),$$

where $h_\alpha(X) = V_\alpha^*(X) - V_\alpha^*(Z)$. If both (H_1) and (H_2) hold, then all the conclusions of Theorem 2.1 are valid.

The next step of the proof is to show that both (H_1) and (H_2) in Proposition 2.3 are verified.

Proof of Theorem 2.1 continued: Let us now consider the next claim.

CLAIM 1: Condition (H_1) holds true. Indeed, the result follows by taking $X = Z$ in Lemma 2.1.

CLAIM 2: Condition (H_2) holds true. Indeed, take $\alpha \in [\alpha_0, 1)$ and $X \in \mathcal{X}$, and let $X_k^*$ and $g_k^*$ be as in (8) according to the policy $f_\alpha^* \in \mathcal{F}$. Hence

$$V_\alpha^*(X) = V_\alpha(f_\alpha^*, X) = \sum_{k=0}^{\infty} \alpha^k \langle Q(g_k^*), X_k^* \rangle.$$  

Note that

$$V_\alpha^*(X) = \sum_{k=0}^{N} \alpha^k \langle Q(g_k^*), X_k^* \rangle + \alpha^{N+1} V_\alpha^*(X_{N+1}^*) \geq \sum_{k=0}^{N} \alpha^k \langle Q(g_k^*), X_k^* \rangle + \alpha^{N+1} V_\alpha^*(X_{N+1}^*). \quad (18)$$

Now, let $(\tilde{g}_0, \ldots, \tilde{g}_N)$ be the control sequence that satisfies Assumption 2.1, and set

$$\tilde{X}_i = X_i(\tilde{g}_i, \ldots, \tilde{g}_0, X), \quad i = 0, \ldots, N.$$  

It follows from (11) that

$$Z = \tilde{X}_{N+1} \leq X_N^* \quad \text{for some fixed } Z \in \mathbb{S}^{n^0},$$  

so that from Proposition 2.2 we have $V_\alpha^*(\tilde{X}_{N+1}) \leq V_\alpha^*(X_N^*)$. This and (18) imply that

$$V_\alpha^*(X) \geq \sum_{k=0}^{N} \alpha^k \langle Q(g_k^*), X_k^* \rangle + \alpha^{N+1} V_\alpha^*(\tilde{X}_{N+1}) \geq \alpha^{N+1} V_\alpha^*(\tilde{X}_{N+1}). \quad (20)$$

On the other hand, using an optimality argument we get that

$$V_\alpha^*(X) \leq \sum_{i=0}^{\infty} \alpha^i \langle Q(g_i^*), X_i^* \rangle + \alpha^{N+1} V_\alpha^*(\tilde{X}_{N+1}) \leq \sum_{i=0}^{\infty} \alpha^i \langle Q(g_i^*), X_i^* \rangle + V_\alpha^*(\tilde{X}_{N+1}) \leq c \|X\| \max_{i=0, \ldots, N} \|Q(\tilde{g}_i)\| + V_\alpha^*(\tilde{X}_{N+1}). \quad (21)$$

Combining (19), (20), and (21), we obtain

$$-(1 - \alpha^{N+1}) V_\alpha^*(Z) \leq V_\alpha^*(X) - V_\alpha^*(Z) \leq b(X). \quad (22)$$

with $b(X) := c \|X\| \max_{i=0, \ldots, N} \|Q(\tilde{g}_i)\|$. Since the left-hand side of (22) is bounded below by a constant (see Lemma 2.1), the claim 2 holds true and also does the condition in (H_2).

III. Concluding remarks

This paper provides sufficient conditions to guarantee the existence of an optimal stationary policy for the long-run average cost control problem, i.e., that of minimizing (2) subject to (1). The derived results apply for discrete-time stochastic systems, with linear state space and possibly nonlinear control structure. The main condition relies on a minimum second moment matrix state, see Assumption 2.1 in connection.
References


