Coherent Control of Linear Quantum Systems: A Differential Evolution Approach

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Abstract—In this paper, we propose a new method to construct an optimal coherent quantum controller, which is required to be physically realizable. This method is based on an evolutionary optimization method, namely the Differential Evolution approach. The aim is to provide a straightforward algorithm to deal with both nonlinear and nonconvex constraints arising in the quantum controller design. The solution to our problem involves the solutions of a complex algebraic Riccati equation and a Lyapunov equation. The efficacy of the proposed method is demonstrated through a case study on an entanglement control problem for an ideal quantum network comprising two cascaded optical parametric amplifiers.

I. INTRODUCTION

The development of computational methods to solve quantum control problems is important if we want to apply the existing quantum control theory to real quantum systems. This is a challenging research topic because we often have to deal with difficult nonconvex optimization problems when designing a quantum controller. This issue has been pointed out in [1] where the authors propose to use a rank constrained LMI approach (see [2]) to synthesize a physically realizable coherent quantum LQG controller. Indeed, this approach is applied only to solve a relaxed feasibility version of the original quantum control problem. Therefore, the original quantum control problem, which is naturally a nonconvex nonlinear optimization problem, remains to be solved. Moreover, the efficacy of the rank constrained LMI method is strongly dependent on finding a suitable initial point to begin the numerical iteration. We might have spent great deal of effort to find such a point. In addition, when dealing with a high order quantum system, the approach of [1] will lead to a very complicated rank constrained LMI problem.

These facts have motivated us to develop a more straightforward and reliable algorithm to find a solution to the linear coherent quantum control problem. Thus, as our main contribution, we propose to use the population-based stochastic optimization method, namely Differential Evolution (DE) approach (see [3]), to solve a class of linear coherent quantum control problems which includes the quantum LQG control problem as described in [1]. As an evolutionary method, DE has three main elements: the mutation, recombination and selection operators. The success of DE methods is also dependent on the fitness test procedure which is formulated according to the control problem to be solved. Furthermore, this approach can be potentially applied to handle other classes of quantum control problem such as those in [4]. We also note that evolutionary algorithms have been successfully applied to solve other quantum control problems using, for example, Genetic Algorithms (GA) and Evolution Strategies (ES); e.g., see [5], [6]. However, the DE algorithm is used in this paper because it has more attractive properties and is likely to have better performance than GA and ES; e.g., see [3].

We demonstrate our DE-based approach through a case study on the problem of quantum network entanglement control. This problem has drawn a lot of attention in the quantum information and communication theory literature due to the fact that entanglement is a fundamental property required in quantum information processing (e.g., see [7]). Thus, its generation, preservation and restoration has been extensively studied and many techniques have been proposed to attain these goals; e.g., see [8], [9].

We consider a simple ideal quantum network consisting of two cascaded optical parametric amplifiers (OPAs) interacting through an optical field (see Fig. 1). The aim is to increase the entanglement level of the quantum network through the application of a dynamic coherent quantum controller (see Fig. 2) to replace the simple optical field connection. Thus, the quantum controller is not only required to satisfy the physical realizability condition; e.g., see [1], but also to drive the controlled quantum system such that an entanglement criterion is satisfied; e.g., see [10]. The entanglement level can be measured based on logarithmic negativity as defined in [11]. Moreover, our case study is motivated by [9] where the authors employ a homodyne detector to obtain information about the quantum network. The acquired measurement is then fed back to the network through a static gain controller to control the dynamic behaviour of the cavities being considered. It has been shown in [9] that not only is sudden-death disentanglement avoided, but also entanglement level is increased using this approach.

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However, it has also been reported in [9] that it is hard to significantly improve the entanglement without the use of such a classical feedback control scheme. Our example gives an indication of future real applications in that we have shown that it is possible to enhance the entanglement of a quantum network using a dynamic coherent quantum controller. Moreover, we also show that in our approach, we have some flexibility in choosing the objective function involved in the quantum controller design.

We use the following notation throughout this paper. If \( M = [m_{jk}] \) is an \( m \times n \) matrix, then \( M^*, M^T \) and \( M^\dagger \) denote the operation of taking the complex conjugate of each entry of \( M \) without transposition, the regular transpose of \( M \) without taking the complex conjugate of its entries, and the complex conjugate transpose of \( M \), respectively. That is, 
\[
M^* = [m_{jk}^*], \quad M^T = [m_{kj}] \quad \text{and} \quad M^\dagger = [m_{kj}^*] = (M^*)^T.
\]

II. PROBLEM STATEMENT

We consider a linear quantum system described by the following non-commutative stochastic dynamic model (see [1]):
\[
\begin{align*}
\dot{x}(t) &= Ax(t)dt + Bdu(t) + B_w dw(t); \\
\dot{y}(t) &= Cx(t)dt + D_w dw(t)
\end{align*}
\]
where \( x(t) \) is a vector of system variables, \( w(t) \) is a quantum Wiener process and \( u(t) \) is a control input. The dimensions of the vectors \( x(t), w(t) \) and \( u(t) \) are compatible with those of the plant coefficient matrices whose entries are real numbers. That is, \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times n_u}, B_w \in \mathbb{R}^{n \times n_w}, C \in \mathbb{R}^{n_u \times n} \) and \( D_w \in \mathbb{R}^{n_u \times n_w} \). The control input \( u(t) \) of the quantum system (1) is modelled as
\[
\dot{u}(t) = \beta_u(t)dt + \tilde{u}(t)
\]
where \( \beta_u(t) \) and \( \tilde{u}(t) \) denote signal and quantum noise components of \( u(t) \), respectively. Moreover, \( \beta_u(t) \) is considered as an adapted, self-adjoint process commuting with \( x(t) \) satisfying \( \beta_u(t)x(t)^T - (x(t)\beta_u(t)^T)^T = 0 \). The quantum noise component \( \tilde{u}(t) \) is independent of \( w(t) \). Moreover, it is also assumed that \( x(0)x(0)^T - (x(0)x(0)^T)^T = \Theta \), where \( \Theta \) is a real skew-symmetric commutation matrix which is defined as
\[
\Theta := \begin{cases} \text{diag}(J, J, \ldots, J) & \text{if canonical} \\ \text{diag}(0, J, \ldots, J) & \text{if degenerate canonical} \end{cases}
\]
(3)

Note that \( \text{diag}(\bullet) \) denotes the block diagonal matrix with \( \bullet \) on its diagonal; the zero matrix in (3) is an \( m \times m \) matrix with \( 0 < m \leq n \); and \( J \) is the real skew-symmetric matrix:
\[
J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.
\]

Furthermore, we wish to construct a dynamic coherent quantum controller, which is also assumed to be a non-commutative stochastic quantum system. Plant-controller coherency means that a closed loop quantum system is formed without direct measurement of the system variables as in the classical control case. A general dynamic quantum controller is written as
\[
\begin{align*}
\dot{x}_c(t) &= A_Kx_c(t)dt + \sum_{j=1}^{2} B_{Kj} dw_{Kj}(t) + B_{K3} dy(t); \\
\dot{u}(t) &= C_Kx_c(t)dt + dw_{K1}(t)
\end{align*}
\]
(5)
where \( x_c(t) \) is a vector of self-adjoint operators (controller variables) and each \( w_{Kj} \) (for \( j = 1, 2 \)) is a non-commutative quantum Wiener process which is also independent of \( w(t) \). The controller is assumed to be of \( n \)-th order, and hence \( A_K \in \mathbb{R}^{n \times n} \). Also, \( B_{K2} \) has the same dimension as \( A_K \), and \( B_{K1} \) has the same number of columns as the rows of \( C_K \). With initial condition \( x_c(0) = x_{c0} \), it is assumed that \( x_c(0)x_c(0)^T - (x_c(0)x_c(0)^T)^T = \Theta_K \), where \( \Theta_K \) is the controller commutation matrix.

Moreover, we assume that there is no initial coupling between the plant and the controller. That is, \( x(0)x_o(0)^T - (x(0)x_o(0)^T)^T = 0 \). Then, interconnecting the controller (5) with the open loop quantum system (1), we obtain a closed loop quantum system written as
\[
\dot{y}(t) = A\eta(t)dt + Bdw(t)
\]
(6)
where
\[
\eta(t) = \begin{bmatrix} x(t) \\ x_c(t) \end{bmatrix}, \quad \omega(t) = \begin{bmatrix} w(t) \\ w_{K1}(t) \\ w_{K2}(t) \end{bmatrix}, \quad A = \begin{bmatrix} A & BC_K \\ B_{K3}C & A_K \end{bmatrix},
\]
\[
B = \begin{bmatrix} B_w & B & 0 \\ B_{K3}D_w & B_{K1} & B_{K2} \end{bmatrix}.
\]
(7)

Here, \( \beta_u(t) \equiv C_Kx_c(t) \) and \( \tilde{u}(t) \equiv w_{K1}(t) \). Note that the matrix \( A \) is required to be Hurwitz.

The realization of the coherent quantum controller (5) is subject to a physical realizability condition (see Theorem 3.4 in [4] or Theorem 1 in [1]). That is,

**Definition 1:** Let \( \Theta_K \) be a given real skew-symmetric commutation matrix. Then the coherent quantum controller (5) is physically realizable if and only if its coefficient matrices: \( A_K, B_{K1}, B_{K2}, B_{K3} \) and \( C_K \) are such that
\[
A_K\Theta_K + \Theta_KA_K^T + \sum_{j=1}^{3} B_{Kj}\Gamma_j B_{Kj}^T = 0;
\]
(8)
where
\[
B_{K1} = \Theta_KC_K^T \text{diag}_{n_s/2} \Gamma_j
\]
(9)
where \( \Gamma_j := \text{diag}_{n_j/2}(J) \) for \( j = 1, 2, 3 \); \( n_j \) is the dimension of \( w_{Kj} \) with \( w_{K3}(t) \equiv y(t) \). Note that Definition 1 can be derived from Theorem 3.4 in [4] with the following notation:
\[
\begin{align*}
A &= A_K; \\
B &= \begin{bmatrix} B_{K1} & B_{K2} & B_{K3} \end{bmatrix}; \\
C &= C_K; \\
D &= \begin{bmatrix} I_{n_u \times n_u} & 0 \end{bmatrix}; \\
w &= \begin{bmatrix} w_{K1}^T & w_{K2}^T & y^T \end{bmatrix}^T.
\end{align*}
\]
(10)

**Remark 1:** To construct a coherent quantum controller, we could require that \( \Theta_K \) is a canonical commutation matrix which implies that each of its diagonal blocks is equal to \( J \) as in (3). However, in our approach, we allow \( \Theta_K \) to be a general skew-symmetric matrix. A suitable state space...
transformation can then be applied to the controller (5) to obtain $\Theta_K$ in canonical or degenerate canonical form; see also [1]. Moreover, the equality condition (8) is a nonconvex nonlinear constraint which poses difficulties if we are going to solve the quantum controller synthesis problem using a regular optimization method. It has also been reported in [1] that an analytical solution to this problem has not yet been developed.

III. COHERENT QUANTUM CONTROLLER SYNTHESIS

An algorithm to solve the coherent quantum LQG control problem using the rank constrained LMI approach (see [2]) has been proposed in [1]. However, this approach tends to become very complicated if it is used to solve higher dimension quantum control problems. In addition, as with many nonconvex problem solvers, the efficacy of the rank constrained LMI approach is strongly dependent on an initial point from which the numerical iteration is started.

These facts have motivated us to propose another method to solve the coherent quantum controller synthesis problem using a population-based stochastic optimization method, namely Differential Evolution (DE) (see [3]). As with other evolutionary methods, the DE algorithm has three essential elements: the mutation, recombination and selection operators, which are applied along with a fitness test procedure which is tailored to the problem to be solved.

Thus, we need to reformulate our controller design problem as a constrained nonlinear optimization problem in order to fit into the DE framework. That is, we want to find an optimal $(K_+,\mathcal{L}_+)$ to solve

$$\min_{K,\mathcal{L}} f(K,\mathcal{L})$$

subject to

$$g_k(K,\mathcal{L}) = 0; \quad h_l(K,\mathcal{L}) \leq 0$$

for $k = 1, 2, \ldots, a$ and $l = 1, 2, \ldots, b$, where $a$ and $b$ are the total number of equality and inequality constraints, respectively. Note that $f(K,\mathcal{L})$ is the objective function to be minimized; $K = (A_K, B_{K1}, B_{K2}, B_{K3}, C_K)$; and $\mathcal{L}$ denotes any other variables that may arise.

Since the physical realizability condition (8) is an essential property of the controller and the closed loop system is required to be asymptotically stable, then we choose

$$g_1(K,\Theta_K) = A_K \Theta_K + \Theta_K A_K^T + \sum_{j=1}^3 B_{Kj} \Gamma_j B_{Kj}^T = 0; \quad h_1(K) = \tilde{\lambda}_r(A) < 0.$$  \hspace{1cm} (13)

These conditions must always be satisfied as part of the equality and inequality constraints. Here, $\tilde{\lambda}_r(A)$ denotes the maximum of the real parts of the eigenvalues of $A$.

Remark 2: Applying the DE approach, we are quite flexible in defining the objective function $f(K,\mathcal{L})$ according to the goal of the control problem to be solved. In particular, if a coherent quantum LQG control problem is to be solved (see [1]), we need to define

$$z(t) := C_z x(t) + D_z \beta_u(t)$$  \hspace{1cm} (15)

where $C_z \in \mathbb{R}^{n \times n}$ and $D_z \in \mathbb{R}^{n \times n_u}$ are such that the objective function can be represented as

$$f(K, t_f) = \int_0^{t_f} (z^T(t)z(t)) \, dt$$  \hspace{1cm} (16)

where $t_f$ is the final time and $\langle \cdot \rangle$ denotes quantum expectation (e.g., see [4]). For infinite horizon case ($t_f \uparrow \infty$), (16) is equivalent to

$$f(K, P) = \text{Tr} \left( C P C^T \right)$$  \hspace{1cm} (17)

where $C = [C_z D_z C_K]$ and $\text{Tr}(\cdot)$ denote the trace. Also, the other equality and inequality constraints are

$$g_2(K, P) = AP + PA^T + BB^T = 0; \quad h_2(P) = -P < 0.$$  \hspace{1cm} (18)

For a second order quantum system, we only need to solve a quadratic equation to obtain a solution $\Theta_K$ to (13). However, for a higher order quantum system, this is generally not the case. Hence, we transform (13) into a complex algebraic Riccati equation (ARE) in order to obtain a solution $\Theta_K$ to (13). We first substitute (9) into (13) and then multiply by $i = \sqrt{-1}$. This leads to the Riccati equation

$$A_K \Sigma_K + \Sigma_K A_K^T + \Sigma_K C_K^T \bar{\Gamma}_1 C_K \Sigma_K + B_{K1} \bar{\Gamma}_2 B_{K1}^T + B_{K2} \bar{\Gamma}_3 B_{K3}^T = 0.$$  \hspace{1cm} (20)

Moreover, referring to Theorem 13.1 and Theorem 13.3 in [12], we are assured that there exists a Hermitian solution $\Sigma_K$ to (20) although it is not a unique solution. However, we need to confirm that $\Sigma_K$ is a purely imaginary solution as defined in (21). A sufficient condition for this to hold is given in the following theorem:

**Theorem 1:** Suppose that the complex ARE (20) has a Hermitian solution $\Sigma_K = \Phi_K + i\Pi_K$ such that

$$\lambda_k(\check{A}_K + \lambda_l^*(\check{A}_K) \neq 0, \quad \forall k, l = 1, 2, \ldots, N$$  \hspace{1cm} (23)

where $\check{A}_K := A_K - \Pi_K C_K^T \Gamma_1 C_K$. Then, $\Sigma_K$ is indeed an imaginary solution, that is $\Sigma_K = i\Pi_K$, which satisfies the physical realizability condition (8).

**Proof:** Suppose that a Hermitian matrix $\Sigma_K = \Phi_K + i\Pi_K$ satisfies (20), where $\Phi_K$ is a real symmetric matrix and
ΠK is a real skew-symmetric matrix. Then, substituting ΣK into (20), we obtain
\[(A_K - ΠK C'_K Γ_1 C_K) Φ_K + Φ_K (A_K - ΠK C'_K Γ_1 C_K)' + i(A_K Π_K + Π_K A'_K + Φ_K C'_K Γ_1 C_K Φ_K) - ΠK C'_K Γ_1 C_K Π_K + B_{K2} Π'_2 B_{K2} + B_{K3} Π'_3 B_{K3} = 0.\] (24)
The left-hand side of (24) is equal to zero if and only if its real and imaginary parts are equal to zero. That is,
\[(A_K - ΠK C'_K Γ_1 C_K) Φ_K + Φ_K (A_K - ΠK C'_K Γ_1 C_K)' = 0;\] (25)
\[A_K Π_K + Π_K A'_K + Φ_K C'_K Γ_1 C_K Φ_K - ΠK C'_K Γ_1 C_K Π_K + B_{K2} Π'_2 B_{K2} + B_{K3} Π'_3 B_{K3} = 0.\] (26)
Moreover, it follows from Lemma 2.7 in [12] that if condition (23) is satisfied, then Φ_K must equal to zero in order that (25) holds. Therefore, ΣK is indeed an imaginary solution. That is ΣK = iΠK. Furthermore, the imaginary part (26) of (20) will lead to the satisfaction of the real physical realizability condition (8). That is, Θ_K = Π_K.

Note that here, \(\lambda_k(\hat{A}_K)\) denotes the k-th eigenvalue of \(\hat{A}_K\) and \(\lambda'_k(\hat{A}_K)\) denotes the complex conjugate of the l-th eigenvalue of \(\hat{A}_K\). Then, the solution ΣK to (20) is computed using the technique presented in Chapter 13 of [12].

**IV. CASE STUDY: ENTANGLEMENT CONTROL**

The quantum controller design method developed in Section III is now applied to solve a quantum entanglement control problem of two cascaded optical parametric amplifiers (OPAs) (see [13]) interacting through an optical field (see Fig. 1) or a coherent quantum controller (see Fig. 2). This particular application is motivated by [9] where the OPAs are referred to as damped optical cavities. The entanglement control mechanism used in [9] is applied to avoid finite-time entanglement sudden-death as well as to enhance entanglement level. It is attained through direct measurement feedback from a homodyne detector to control the dynamic behaviour of both OPAs using a static gain controller. This is in contrast to our approach where we apply a simple coherent quantum controller without a feedback loop to achieve enhanced entanglement (see Fig. 2).

The dynamic model of the first OPA is described in terms of the following complex linear quantum stochastic differential equation (e.g., see [14]):
\[d\hat{a}_1(t) = \chi_1 \hat{a}_1(t) dt - \kappa_1 \hat{a}_1(t) dt - i\Delta_1 \hat{a}_1(t) dt + \sqrt{2\kappa_1} d\hat{w}_1(t);\]
\[d\hat{w}_1(t) = -\sqrt{2\kappa_1} \hat{a}_1(t) dt + d\hat{w}_1(t);\] (27)
and that of the second OPA is
\[d\hat{a}_2(t) = \chi_2 \hat{a}_2(t) dt - \kappa_2 \hat{a}_2(t) dt - i\Delta_2 \hat{a}_2(t) dt + \sqrt{2\kappa_2} d\hat{w}_2(t);\] (28)
where, for each OPA, \(\hat{a}\) is an annihilation operator with \(\hat{a}^*\) as its corresponding creation operator; \(\hat{w}\) and \(\hat{u}\) are the input signals; \(\hat{y}\) is the output signal; \(\chi\) is a complex coupling constant (\(\alpha + i\beta\)); \(\Delta\) is a detuning parameter; and \(\kappa\) is the loss rate of the OPA. Then, we define expressions for the amplitude and phase quadratures in terms of each corresponding annihilation and creation operators, and input and output signals as in Table I:

**TABLE I**

<table>
<thead>
<tr>
<th>Amplitude</th>
<th>Phase</th>
</tr>
</thead>
<tbody>
<tr>
<td>(q_1 = \frac{1}{\sqrt{2}}(a_1 + a_1^*))</td>
<td>(p_1 = \frac{1}{\sqrt{2}}(a_1 - a_1^*))</td>
</tr>
<tr>
<td>(q_2 = \frac{1}{\sqrt{2}}(a_2 + a_2^*))</td>
<td>(p_2 = \frac{1}{\sqrt{2}}(a_2 - a_2^*))</td>
</tr>
<tr>
<td>(u_{11} = w_1 + w_1^*)</td>
<td>(u_{12} = -i(w_1 - w_1^*))</td>
</tr>
<tr>
<td>(y_{11} = y_1 + y_1^*)</td>
<td>(y_{12} = -i(y_1 - y_1^*))</td>
</tr>
<tr>
<td>(u_{21} = u_2 + u_2^*)</td>
<td>(u_{22} = -i(u_2 - u_2^*))</td>
</tr>
</tbody>
</table>


Thus, based on the expressions in Table I, the dynamic equations (27) and (28) can be written in the form of state equations (1) with the system matrices:

\[A = \begin{bmatrix} -\alpha_1 - \kappa_1 & -\beta_1 + \Delta_1 & 0 & 0 \\ -\beta_1 - \Delta_1 & \alpha_1 - \kappa_1 & 0 & 0 \\ 0 & 0 & -\beta_2 - \Delta_2 & \alpha_2 - \kappa_2 \\ 0 & 0 & \alpha_2 - \kappa_2 & -\beta_2 - \Delta_2 \end{bmatrix};\]
\[B = \begin{bmatrix} 0 & 0 & \sqrt{\kappa_1} & 0 \\ 0 & 0 & 0 & \sqrt{\kappa_1} \\ \sqrt{\kappa_2} & 0 & 0 & 0 \\ 0 & \sqrt{\kappa_2} & 0 & 0 \end{bmatrix};\]
\[C = \begin{bmatrix} -2\sqrt{\kappa_1} & 0 & 0 & 0 \\ 0 & -2\sqrt{\kappa_1} & 0 & 0 \end{bmatrix};\]
\(D_w = I_{2 \times 2}\) (29)

and the state vector \(x\), control input vector \(u\), noise vector \(w\) and output vector \(y\) are defined as follows:
\[x := \begin{bmatrix} q_1 \\ p_1 \\ q_2 \\ p_2 \end{bmatrix}; u := \begin{bmatrix} u_{11} \\ u_{21} \end{bmatrix}; w := \begin{bmatrix} u_{12} \\ u_{11} \end{bmatrix}; y := \begin{bmatrix} y_{11} \\ y_{12} \end{bmatrix}.\] (30)

If the output \(y\) of the first OPA becomes the control input \(u\) to the second OPA as depicted in Fig. 1, we obtain
\[dx(t) = Ax(t)dt + Bdw(t);\] (31)
where
\[A = \begin{bmatrix} -\alpha_1 - \kappa_1 & -\beta_1 + \Delta_1 & 0 & 0 \\ -\beta_1 - \Delta_1 & \alpha_1 - \kappa_1 & 0 & 0 \\ -2\sqrt{\kappa_1 \kappa_2} & 0 & -\alpha_2 - \kappa_2 & -\beta_2 + \Delta_2 \\ 0 & -2\sqrt{\kappa_1 \kappa_2} & -\beta_2 - \Delta_2 & \alpha_2 - \kappa_2 \end{bmatrix};\]
\[B = \begin{bmatrix} \sqrt{\kappa_1} & 0 \\ 0 & \sqrt{\kappa_1} \\ \sqrt{\kappa_2} & 0 \\ 0 & \sqrt{\kappa_2} \end{bmatrix}.\] (32)

The entanglement of the cascaded system (31) can be evaluated in terms of covariance matrix \(V\) as the state vector \(x\) is Gaussian (e.g., see [10]). The covariance matrix \(V\) is the solution to the following Lyapunov equation (see [9])
\[AV + VA^T + BB^T = 0.\] (33)
Note that we need to choose a suitable set of parameter values such that $\mathcal{A}$ is Hurwitz and (33) has a unique positive definite solution. Then, according to [10], the entanglement criterion can be defined as follows:

**Definition 2:** The quantum system (31), (32) is said to be entangled if and only if there exists a complex vector $d_c = d_r + i d_i$ such that

$$\mathcal{N} := d_c^* \mathcal{M} d_c < 0 \quad (34)$$

where

$$\mathcal{M} := V + \frac{i}{2} \Omega; \quad \mathcal{M}^T = \mathcal{M}; \quad \Omega := \begin{bmatrix} J & 0 \\ 0 & -J \end{bmatrix} \quad (35)$$

As all eigenvalues of $\mathcal{M}$ are real numbers, the condition (34) is equivalent to the condition that at least one of the eigenvalues of $\mathcal{M}$ is a negative real number. Then, $d_c$ can be taken as the corresponding eigenvector of $\mathcal{M}$. Furthermore, the Gaussian entanglement level can be measured using the logarithmic negativity (see [11]):

$$\mathcal{E} := \max\{0, -\ln(2\nu)\} \quad (36)$$

where

$$\nu := \frac{1}{\sqrt{2}} \sqrt{\Psi - \sqrt{\Psi^2 - 4 \det(V)}}; \quad V := \begin{bmatrix} V_1 & V_2 \\ V_2^T & V_3 \end{bmatrix}; \quad \Psi := \det(V_1) + \det(V_2) - 2 \det(V_3) \quad (37)$$

The quantity defined in (36) tells us that the quantum system is entangled if and only if $\mathcal{E} > 0$. Otherwise, $\mathcal{E} = 0$.

To consider the same example as in [9], the parameter values of the system (29) are chosen as follows:

$$\alpha = 0; \quad \beta = -0.4; \quad \Delta_1 = \Delta_2 = 0.6; \quad \kappa_1 = \kappa_2 = 1. \quad (38)$$

With this set of parameters, the system (31) directly connected through an optical field is entangled with

$$\mathcal{N} = -0.0892; \quad \mathcal{E} = 0.2256$$

and a corresponding complex vector $d_c$ is (34) is

$$[-0.3084+0.4196 \quad 0.5410-0.2104 \quad 0.0827-0.2942 \quad -0.5463].$$

It is possible to increase this entanglement level by applying a coherent quantum controller (5) such that the controlled quantum system has a configuration as shown in Fig. 2. This controller can be designed using the optimization method proposed in Section III with the objective function

$$f(K, \Sigma_K, P) = g(M) \quad (39)$$

where $g(M)$ is the smallest eigenvalue of the corresponding matrix $M$, and the constraints are (22), (14), (18), (19) and

$$g_3(K) = \text{Re}(\Sigma_K) = 0 \quad (40)$$

where $\text{Re}(\Sigma_K)$ is the real part of $\Sigma_K$. Note that, instead of (33), the Lyapunov equation corresponding to the controlled quantum system is

$$\bar{A}P + P\bar{A}^T + \bar{B}\bar{B}^T = 0 \quad (41)$$

where the matrices $\bar{A}$ and $\bar{B}$ correspond to the controlled quantum system as in (7); $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}$ and $V = P_{11}$.

The parameters used in the DE numerical iteration to compute the optimal dynamic quantum controller are given in Table II. We used MATLAB R2008a to run our DE-based algorithm and applied a revised DE mutation-recombination operator as described in [15]. As DE is a population-based evolutionary method, all elements of the controller matrices are randomly generated. Thus, we need to evaluate the fitness of each candidate solution (or individual) with respect to a set of constraints involved in the quantum controller design problem. Since we use a penalty-based fitness test, this routine acquires information about how many constraints have been violated by a particular individual and the violation cost incurred, or the value of the objective function if there are no violations. Moreover, in order to have an efficient fitness test, we assume that a constraint violation in a lower level implies the one(s) in the higher level. Thus, the fitness test procedure is constituted in the following steps:

1. Evaluate the stability of the controlled quantum system $\bar{A}$ by referring to constraint (14);
2. Compute the solution of ARE (22);
3. Evaluate constraint (40) to check if the solution to (22) is imaginary Hermitian;
4. Compute the solution of (18) and ensure that (19) is satisfied;
5. Calculate the value of the objective function (39).

Applying the DE algorithm, the optimum quantum controller matrices are obtained as follows:

$$A_K = \begin{bmatrix} -907.6167 & -344.0940 & -86.2280 & -97.4048 \\ 236.9661 & -963.4073 & -377.8802 & -78.7616 \\ -145.2900 & 219.1872 & -904.0325 & 392.0398 \end{bmatrix}; \quad B_{K1} = \begin{bmatrix} -14.4690 & -81.5469 \\ -4.7526 & 28.2512 \\ -5.7654 & 0.6262 \\ -66.3207 & -12.5105 \end{bmatrix}; \quad B_{K3} = \begin{bmatrix} 29.3096 & -60.5369 \\ -33.2270 & 28.8097 \\ 18.6475 & 4.8260 \end{bmatrix};$$


$$\Theta_K = \begin{bmatrix} 0.6382 & -0.0264 & 1.7524 & 0.6076 & 0.0264 & 0.1545 & 4.6528 \end{bmatrix}. \quad (42)$$

Alternatively, if we assume that $B_{K2} = 0$, we obtain the following quantum controller matrices by applying the DE
The assumption that $B_{K2} = 0$ is based on the experience of [1] where the optimal $B_{K2}$ has very small entries. Therefore, it has an insignificant effect on the performance of the controlled system. In our example, we find that the entanglement level of the controlled quantum system obtained by applying controllers (42) and (43) are the same. That is, in both cases we obtain

\[ N = -0.1237; \quad E = 0.2944. \]

The corresponding complex vectors $d_c$ were

\[
\begin{bmatrix}
-0.0699+0.4613 & -0.4932+0.1997 & -0.0010+0.4652 & 0.5322 \\
-0.0694+0.4699 & -0.4921+0.2023 & -0.0017+0.4655 & 0.5319 \\
\end{bmatrix}
\]

for the controllers (42) and (43) respectively. Thus, through this example, we find that the optimal $B_{K2}$ is not necessarily small or equal to zero. However, this example does not show that any improvement in the objective function can be obtained by using a nonzero $B_{K2}$. This is consistent with the example presented in [1].

Another interesting aspect of this example is that we get only an approximately 30% entanglement improvement in terms of logarithmic negativity through the application of the coherent quantum controller as opposed to the direct interconnection of the two OPAs. This result is not surprising because it is hard to drastically enhance the entanglement level of this type of quantum network as reported in [9]. However, our method has shown the potential to improve the entanglement level of a realistic quantum network using a dynamic coherent quantum controller.

**Remark 3:** Instead of (39), another possible objective function is

\[ f(K, \Sigma_K, P) = \ln(2\nu) \] (44)

subject to (22), (14), (18), (19) and (40). Moreover, we could also formulate our entanglement control problem as an infinite horizon quantum LQG control problem. In this case, we would fix $d_c$ and define

\[ C_z = [d_r \quad d_i \quad 0 \quad 0]^T; \quad D_z = [0_{2\times2} \quad \sqrt{\rho I_{2\times2}}]^T \] (45)

in (15) where $\rho > 0$ is a control weighting factor, and $d_r$ and $d_i$ are defined in terms of the complex vector $d_c$. The investigation of these alternative objective functions is a topic for future research.

**V. Conclusions**

We have presented a new method to solve a linear coherent quantum control problem based on a DE approach. The solution to this problem involves the solutions of both complex algebraic Riccati equations and Lypunov equations. As a case study, we consider an entanglement control problem for two cascaded OPAs. Applying a suitable coherent quantum controller, we show that the entanglement level can be increased. This result indicates that our method can have potential future applications to realistic quantum networks.

Interestingly, with or without the term $B_{K2}$ in the quantum controller, we obtain the same amount of entanglement in terms of logarithmic negativity. This fact motivates an investigation of the significance of the inclusion of $B_{K2}$ in the realization of the dynamic coherent quantum controller. Moreover, applying the DE method, we can have some flexibility to choose the objective function to be minimized. Thus, we have suggested two more alternative objective functions which can be investigated in future research.

**REFERENCES**


