State-feedback Stabilization for High-order Stochastic Nonlinear Systems without Strict Triangular Conditions

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Abstract—For a class of high-order stochastic nonlinear systems which are neither necessarily feedback linearizable nor affine in the control variables without the strict triangular conditions, this paper studies the problem of state-feedback stabilization for the first time. The main contribution of this paper is the development of a systematic design approach for stochastic nonlinear systems in the absence of triangular conditions. This methodology enables us to remove the strict triangular conditions which have been the common assumption for global stabilization, and leads to a new result combining and generalizing the previous work. Finally, an illustrative example is provided to demonstrate the effectiveness of the proposed control design methodology.

I. INTRODUCTION

Let us consider a class of multi-input high-order stochastic nonlinear systems as follows

\[ dz = f_0(z, x_1)dt + h_0(z, x, u, \eta_1)\eta_1^{p_0+2} dt + g_0(z, x_1)dw, \]
\[ dx_i = (x_{i+1} + f_i(z, x_i))dt + h_i(z, x, u, \eta_1)\eta_1^{p_0+2} dt + g_i(z, x_i)dw, i = 1, \cdots, r - 1, \]
\[ dx_r = (u^{+2} + f_r(z, x_r))dt + h_r(z, x, u, \eta_1)\eta_1^{p_0+2} dt + g_r(z, x_r)dw, \]
\[ d\eta_s = \eta_{s+1}dt + \phi_s(z, x, u, \eta_s)dw, s = 1, \cdots, q - 1, \]
\[ d\eta_q = vdt + \phi_q(z, x, u, \eta_q)dw, \]

(1)

where \( z = (z_1, \cdots, z_m)^T \in R^m, x = x_r = (x_1, \cdots, x_r)^T \in R^r, \eta = \eta_1 = (\eta_1, \cdots, \eta_q)^T \in R^q \) are the measurable states of the three subsystems and \( u \in R, v \in R \) are the control inputs, respectively, \( x_i = (x_1, \cdots, x_i)^T, \eta_r = (\eta_1, \cdots, \eta_q)^T, i = 1, \cdots, r, s = 1, \cdots, q, \omega \) is an \( r \)-dimensional standard Wiener process defined on a probability space \((\Omega, F, P)\) with \( \Omega \) being a sample space, \( F \) being a \( \sigma \)-field, and \( P \) being a probability measure. \( p_i \geq 1, i = 0, 1, \cdots, r, r \) are odd integers. The functions \( f_i, h_i, g_i(i = 0, 1, \cdots, r) \) and \( \phi_s(s = 1, \cdots, q) \) are smooth with the properties \( f_i(0, 0) = 0, g_i(0, 0) = 0, \phi_s(z, x, u, 0) = 0 \).

When \( p_i = 1 \) and \( z = 0, \eta_s = 0, i = 0, 1, \cdots, r, s = 1, \cdots, q, \) for this class of stochastic nonlinear systems, the design of a global stabilization controller has achieved remarkable development. Ref. [1] and [2] respectively presented the basic stability theory of stochastic control systems, which have now been widely applied and laid the mathematical foundation for the design and analysis of stochastic nonlinear controller. According to the difference of selected Lyapunov functions, the existing literature on controller design can be mainly divided into two types. One type is based on quadratic Lyapunov functions which are multiplied by different weighting functions, see, e.g., [3]-[5]. Another essential improvement belongs to Krstić and Deng. By introducing the quartic Lyapunov function, Ref. [6]-[7] presented asymptotical stabilization control under the assumption that the nonlinearities equal to zero at the equilibrium point of the open-loop system. Subsequently, for several classes of stochastic nonlinear systems with unmodeled dynamics and uncertain nonlinear functions, by combining Krstić and Deng’s method with stochastic small-gain theorem [8], and with dynamic signal and changing supply function [9]-[10], different adaptive output-feedback control schemes were studied.

However, similar to its deterministic counterpart in [11]-[16] and the other related papers in [17]-[18], in the case of \( p_i > 1, i = 0, 1, \cdots, r, \) some interesting features of (1) are that the Jacobian linearization of the system is neither controllable nor feedback linearizable, so the existing design tools are hardly applicable to (1). Recently, for (1) with \( \eta_s = 0, s = 1, \cdots, q, \) Ref. [19] addressed state-feedback stabilization for high-order stochastic nonlinear systems with stochastic inverse dynamics for the first time. Very recently, Ref. [20] and [21] investigated the problem of the inverse optimal stabilization and output-feedback control for high-order stochastic nonlinear systems, respectively. It should be pointed out that all the mentioned results require the systems satisfy the strict triangular conditions. For system (1) without triangular conditions restriction, to the best of our knowledge, there is no published result.

In this paper, motivated by the results in [6]-[7], [19] and [22], we study the problem of stabilization for system (1) by relaxing the triangular conditions restriction, which combines and generalizes the previous work in [6] and [19]. Under some moderate assumptions, smooth state-feedback controllers are designed, which ensures that the closed-loop system has an almost surely unique solution on \([0, \infty)\), the equilibrium at the origin of the closed-loop system is globally asymptotically stable (GAS) in probability, and the states can be regulated to the origin almost surely.

The remainder of the paper is organized as follows. Section II provides some preliminary results. Controller

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design and stability analysis are given in Section III. A simulation example is provided to demonstrate the state-feedback controllers in Section IV. Concluding remarks are collected in Section V.

Notations: Throughout this paper, the notations are standard. \( R_+ \) denotes the set of all nonnegative real numbers, \( R^n \) denotes the real \( n \)-dimensional space. For a given vector or matrix \( X, X^T \) denotes its transpose, \( \text{Tr} \{ X \} \) denotes its trace when \( X \) is square, and \( | X | \) is the Euclidean norm of a vector \( X \). \( C^1 \) denotes the set of all functions with continuous \( \theta \)th partial derivative. \( K \) denotes the set of all functions: \( R_+ \rightarrow R_+ \), which are continuous, strictly increasing and vanishing at zero; \( \mathcal{K}_\infty \) denotes the set of all functions which are of class \( K \) and unbounded; \( \mathcal{KL} \) denotes the set of all functions \( \beta(s, t) : R_+ \times R_+ \rightarrow R_+ \), which are of \( K \) for each fixed \( t \), and decrease to zero as \( t \rightarrow \infty \) for each fixed \( s \).

II. PRELIMINARY RESULTS

Consider the following stochastic nonlinear system
\[
dx = f(x)dt + g^T(x)dw, \quad \forall \ x_0 \in R^n, \tag{2}
\]
where \( x \in R^n \) is the state of the system, \( \omega \) is an \( r \)-dimensional standard Wiener process defined on a probability space \( (\Omega, \mathcal{F}, P) \), \( x_0 \) is the initial state. The Borel measurable functions \( f : R^n \rightarrow R^n \) and \( g : R^n \rightarrow R^{n \times r} \) are locally Lipschitz in \( x \in R^n \).

The following definitions and lemmas will be used throughout the paper.

**Definition 1** [7]: For any given \( V(x) \in C^2 \), associated with stochastic system (2), the differential operator \( \mathcal{L} \) is defined as
\[
\mathcal{L}V(x) = \frac{\partial V}{\partial x} f(x) + \frac{1}{2} \text{Tr} \{ g(x) \frac{\partial^2 V}{\partial x^2} g^T(x) \}.
\]

**Definition 2** [7]: For the stochastic system (2) with \( f(0) = 0 \) and \( g(0) = 0 \), the equilibrium \( x(t) = 0 \) of (2) is called to be globally asymptotically stable (GAS) in probability if for any \( \varepsilon > 0 \), there exists a class \( K \) Lyapunov function \( \beta(\cdot, \cdot) \) such that \( P \{ | x(t) | < \beta(|x_0|, t) \} \geq 1 - \varepsilon, \forall t \geq 0, x_0 \in R^n \setminus \{0\} \).

**Lemma 1** [7]: Consider the stochastic system (2). If there exist a \( C^2 \) function \( V(x) \), class \( K \) functions \( \delta_1 \) and \( \delta_2 \), constants \( c_1 > 0 \) and \( c_2 \geq 0 \), and a nonnegative function \( W(x) \) such that
\[
\delta_1(|x|) \leq V(x) \leq \delta_2(|x|), \quad \mathcal{L}V \leq -c_1 W(x) + c_2,
\]
then

a) For (2), there exists an almost surely unique solution on \( [0, \infty) \);

b) When \( c_2 = 0 \), \( f(0) = 0 \), \( g(0) = 0 \), and \( W(x) = \delta_3(|x|) \) is continuous, then the equilibrium \( x = 0 \) is GAS in probability and \( P \{ \lim_{t \to \infty} | x(t) | = 0 \} = 1 \), where \( \delta_3(\cdot) \) is a class \( K \) function.

**Lemma 2** [13]: Let \( x, y \) be real variables, then for any positive integers \( m, n \) and positive real number \( a \), one has
\[
ax^m y^n \leq b|x|^m |y|^n + \frac{n}{m+n} \left( \frac{m+n}{m} \right)^{\frac{n}{m}} a^{\frac{m+n}{n}},
\]
where \( b > 0 \) is any real number.

**Remark 1**: Compared with deterministic systems, the main technical obstacle in the Lyapunov design for stochastic systems is that Itô differentiation involves not only the gradient but also the higher order Hessian term, in which the second-order differential \( \frac{\partial^2 V}{\partial x^2} \) makes the controller design much more difficult than that of the deterministic case.

III. CONTROLLER DESIGN AND STABILITY ANALYSIS

The objective of this paper is to design, under appropriate conditions, smooth state-feedback controllers for system (1), such that the closed-loop system is GAS in probability at the origin.

The following assumptions are made on system (1).

**Assumption 1**: There are nonnegative smooth functions \( f_{01}(z, x_1) \), \( f_{j0}(z, x_1) \) \( (j = 1, \cdots, m) \), \( f_{i1}(z, x_i) \) and \( g_{i1}(z, x_i) \), \( i = 1, \cdots, r \), such that
\[
\begin{align*}
|f_{01}(z, x_1)| & \leq (|z|^p + |x_1|^p) f_{01}(z, x_1), \\
|f_{j0}(z, x_1)| & \leq (|z|^p + |x_1|^p) f_{01}(z, x_1), \\
|f_{i1}(z, x_i)| & \leq (|z|^p + |x_1|^p + \cdots + |x_i|^p) f_{i1}(z, x_i), \\
|g_{i1}(z, x_i)| & \leq (|z|^p + |x_1|^p + \cdots + |x_i|^p) g_{i1}(z, x_i),
\end{align*}
\]
where \( g_{j0}(z, x_1) \) represents the \( j \)-th row of \( g_0(z, x_1) \), \( j = 1, \cdots, m \).

**Assumption 2**: Suppose that there are a smooth Lyapunov function \( V_0(z) \) which is positive definite and proper, and a smooth function \( \alpha(z) \) with \( \alpha(0) = 0 \) such that
\[
\begin{align*}
\mathcal{L}V_0(z) = & \frac{\partial V_0}{\partial z} f_0(z, x_1) + \frac{1}{2} \text{Tr} \{ g_0^T(z, x_1) \frac{\partial^2 V_0}{\partial z^2} g_0(z, x_1) \} \\
\leq & -(|z|^p + |x_1|) \bar{g}(z, x_1),
\end{align*}
\]
where \( \bar{g}(\cdot) \) is continuous and \( 0 \leq \bar{g}(z, x_1) \leq (|z|^p + |x_1|^p) g_1(z, x_1) \) for a continuous function \( g(z, x_1) \geq 0 \).

**Assumption 3**: \( p_0 \geq p_1 \geq \cdots \geq p_r \geq 1 \) are odd integers.

**Remark 2**: When \( \eta_0 = 0, s = 1, \cdots, q \), system (1) reduces to the norm form of high-order stochastic nonlinear system whose stabilization problem has been discussed in [19]-[21]. It should be emphasized that Assumption 2 is imposed on the \( z \)-subsystem with the term \( h_0(z, x, u, \eta_1) \eta_1^{p_0+2} = 0 \). As demonstrated in [19], Assumptions 1-3 play the essential roles in the stabilization of system (1). More details can be found in the following design procedure (see Step 1).

**Remark 3**: When \( z = 0, x_i = 0, i = 1, \cdots, r \), system (1) reduces to Eq.(3.1) and (3.2) considered in [6]. Meanwhile, as demonstrated in Remark 2, the system investigated in [19] is also the special case of (1). Therefore, the results in this paper combine and generalize the previous work in [6] and [19].

**Remark 4**: Consider the following simple stochastic nonlinear system:
\[
\begin{align*}
dx_1 & = (x_1^2 + x_1 x_2 \eta_1^6) dt + \frac{1}{3} x_1^3 dw, \\
dx_2 & = (u + x_1 x_2 \eta_1^6) dt + x_2 dw, \\
d\eta_1 & = v dt + u^2 x_1 \eta_1 dw. \tag{3}
\end{align*}
\]
Obviously, the drift terms in (3) do not satisfy the triangular conditions commonly used in [6] and [19]-[21]. Therefore, from (1) and Assumptions 1-3, one can easily obtain that system (3) can be stabilized by the design methodology in this paper.

Therefore, from this point of view, this paper leads to a new result combining and generalizing the previous work.

The design of state-feedback controller for system (1) is divided into two steps. In Step I, one only considers the first two subsystems without the terms \( h_i(z, x, u, \eta_j) \eta_i^{p_0+2} \), \( i = 0, 1, \cdots, r \). By taking the similar design procedure as that in [19], a smooth controller \( u \) is constructed. Then in Step II, by combining the obtained results in Step I and using backstepping technique in [6], a smooth stabilizing state-feedback controller \( v \) is designed for system (1).

**Step I:** Consider

\[
\begin{align*}
\dot{z} &= f_0(z, x_1)dt + g_0(z, x_1)d\omega, \\
\dot{x}_i &= (x_i^{p_1} + f_i(z, x_i))dt + g_i(z, x_i)d\omega, \\
\dot{x}_r &= (u^{p_r} + f_r(z, x_r))dt + g_r(z, x_r)d\omega.
\end{align*}
\]

In this step, we construct a smooth state feedback controller \( u = u(z, x) \) that globally stabilizes system (4).

Since the design procedure is almost the same as that of [19], for simplicity, we summarize the controller as follows:

\[
\begin{align*}
x_1^*(z) &= \alpha(z), \\
x_2^*(z, x_1) &= -\xi_1 \alpha_1(z, x_1), \\
x_i^*(z, x_{i-1}) &= -\xi_{i-1} \alpha_{i-1}(z, x_{i-1}), \quad i = 2, \cdots, r, \\
u(z, x_r) &= -\xi_r \alpha_r(z, x_r),
\end{align*}
\]

where \( \alpha_i(z, x_i) > 0, i = 1, \cdots, r \), are smooth functions. It follows that

\[
\mathcal{L}V(z, \xi_1, \cdots, \xi_r) \leq -2(|z|^{p_0+3} + |\xi_1|^{p_0+3} + \cdots + |\xi_r|^{p_0+3}),
\]

where \( V(z, \xi_1, \cdots, \xi_r) = (r + 2) V_0 + \sum_{i=1}^r \xi_i^{p_i+1}, \xi_i = x_i - x_i^*(z, x_{i-1}), i = 1, \cdots, r \).

**Step II:** In this step, using the controller (5), a smooth state feedback controller \( v = v(z, x, \eta) \) that globally stabilizes the system (1) is constructed.

To do so, first apply the controller (5) to the entire system (1) to obtain a closed-loop system of the following form:

\[
\begin{align*}
d\Xi &= F(\Xi)dt + H(\Xi, \eta_1) \eta_1^{p_0+2} dt + G(\Xi)d\omega, \\
d\eta_s &= \eta_{s+1} dt + \hat{\phi}_s(\Xi, \eta_s)d\omega, \quad s = 1, \cdots, q - 1, \\
d\eta_q &= vd\omega + \hat{\phi}_q(\Xi, \eta_q)d\omega,
\end{align*}
\]

where \( \Xi = (z^T, x^T)^T = (z_1, \cdots, z_m, x_1, \cdots, x_r)^T \) and

\[
\begin{align*}
F(\Xi) &= \begin{bmatrix} f_0(z, x_1) \\
                        x_2^{p_1} + f_1(z, x_1) \\
                        \vdots \\
                        u^{p_r} + f_r(z, x_r) \end{bmatrix} \\
H(\Xi, \eta_1) &= \begin{bmatrix} h_0(z, x, u, \eta_1) \\
                        h_1(z, x, u, \eta_1) \\
                        \vdots \\
                        h_r(z, x, u, \eta_1) \end{bmatrix} \\
G(\Xi) &= \begin{bmatrix} g_1(z, x_1) \\
                        \vdots \\
                        g_r(z, x_r) \end{bmatrix},
\end{align*}
\]

\[
\hat{\phi}_s(\Xi, \eta_s) = \phi_s(z, x, u(z, x), \eta_s),
\]

where \( s = 1, \cdots, q, g_j(0, z, x_1) \) represents the \( j \)-th row of \( g_0(z, x_1), j = 1, \cdots, m \).

**Remark 5:** By assumption, the function \( \hat{\phi}_s(\Xi, 0) = \phi_s(z, x, u(z, x), 0) = 0 \). Using the Taylor expansion formula with integration remainder, there exist smooth functions \( \varphi_{sk}(\Xi, \eta_s), s = 1, \cdots, q, k = 1, \cdots, s \), such that

\[
\hat{\phi}_s(\Xi, \eta_s) = \sum_{k=1}^s \eta_k \varphi_{sk}(\Xi, \eta_s).
\]

Introduce the following coordinate transformation:

\[
\xi_i = \eta_i - \beta_{i-1}(\Xi, \eta_{i-1}), \quad i = 1, \cdots, q,
\]

where \( \beta_i(\Xi, \eta_i), i = 0, 1, \cdots, q - 1 \), are virtual smooth controllers to be designed later and \( \beta_0 = 0 \). Then, by Itô’s differentiation rule, for \( i = 1, \cdots, q \), one has

\[
\begin{align*}
d\xi_i &= d(\eta_i - \beta_{i-1}(\Xi, \eta_{i-1})) \\
       &= (\eta_{i+1} + F_i(\Xi, \eta_i))dt + G_i(\Xi, \eta_i)d\omega,
\end{align*}
\]

where \( \eta_{q+1} = v \), and

\[
\begin{align*}
F_i(\Xi, \eta_i) &= -\sum_{i=1}^{i-1} \frac{\partial \beta_{i-1}}{\partial \eta_i} \eta_{i+1} - \frac{\partial \beta_{i-1}}{\partial z} f_0(z, x_1) \\
&\quad + h_0(z, x, u, \eta_1) \eta_1^{p_0+2} - \sum_{s=1}^r \frac{\partial \beta_{i-1}}{\partial \eta_s} (x_s^{p_s} + f_s(z, x_s)) \\
&\quad + h_s(z, x, u, \eta_1) \eta_1^{p_0+2} - \frac{1}{2} \sum_{m=1}^{i-1} \frac{\partial^2 \beta_{i-1}}{\partial \eta_m \partial \eta_i} \phi_m(\Xi, \eta_m) \\
&\quad - \frac{1}{2} \sum_{s,l=1}^m \frac{\partial^2 \beta_{i-1}}{\partial x_s \partial x_l} g_{s,l}(0, z, x_1) \phi_T(\Xi, \eta_s) \\
&\quad + \frac{1}{2} \sum_{s,l=1}^m \frac{\partial^2 \beta_{i-1}}{\partial z_s \partial \eta_l} g_{s,l}(0, z, x_1) \phi_T(\Xi, \eta_s)
\end{align*}
\]
\[ -\frac{1}{2} \sum_{s=1}^{i-1} \sum_{l=1}^{r} \frac{\partial^2 \beta_i}{\partial x_l \partial \eta_s} g_l(z, \bar{x}_l) \phi_T(\Xi, \bar{\eta}_s) \]
\[ -\frac{1}{2} \sum_{s=1}^{m} \sum_{l=1}^{r} \frac{\partial^2 \beta_i}{\partial x_l \partial \eta_s} g_l(z, \bar{x}_l) g_0^T(z, x_1), \]
\[ G_i(\Xi, \bar{\eta}_i) = \phi_i(\Xi, \bar{\eta}_i) - \sum_{l=1}^{i-1} \frac{\partial \beta_i}{\partial \eta_l} \phi_l(\Xi, \bar{\eta}_l) \]
\[ - \frac{\partial \beta_i}{\partial z} g_0(z, x_1) - \sum_{s=1}^{r} \frac{\partial \beta_i}{\partial x_s} g_s(z, \bar{x}_s). \]
\[ \text{(11)} \]

We employ a Lyapunov function of the form
\[ U(\Xi, \eta) = \hat{V}(\Xi) + \sum_{i=1}^{q} \frac{1}{p_0 + 3} \zeta_i^{p_0 + 3}, \]
\[ \text{(12)} \]
and set out to select the function \( \beta_{i-1}(\Xi, \bar{\eta}_{i-1}) \) to make \( \mathcal{L}U(\Xi, \eta) \) negative definite, where \( \hat{V}(\Xi) = V(z, \xi_1, \cdots, \xi_r) \).

Along the solutions of (7), using (6) and \( F_1(\Xi) = 0 \), we have
\[ \mathcal{L}U(\Xi, \eta) = \frac{\partial \hat{V}}{\partial \Xi} F(\Xi) + \frac{1}{2} \text{Tr}\{G^T(\Xi) \frac{\partial^2 \hat{V}}{\partial \Xi^2} G(\Xi)\} \]
\[ + \frac{\partial \hat{V}}{\partial \Xi} H(\Xi, \eta_1) \eta_1^{p_0+2} + \zeta_i^{p_0+2} \eta_2 + \zeta_i^{p_0+2} (v) \]
\[ + F_q(\Xi, \bar{\eta}_q) + \sum_{i=2}^{q-1} \zeta_i^{p_0+2} (\eta_{i-1} + F_i(\Xi, \bar{\eta}_i)) \]
\[ + \frac{p_0 + 2}{2} \sum_{i=1}^{q} c_i^{p_0+2} G_i^T(\Xi, \bar{\eta}_i) G_i(\Xi, \bar{\eta}_i) \]
\[ \leq -2 |z|^{p_0+3} + |\xi_1|^{p_0+3} + \cdots + |\xi_r|^{p_0+3} \]
\[ + \zeta_i^{p_0+2} (\beta_1(\Xi, \eta_1) + \frac{\partial \hat{V}}{\partial \Xi} H(\Xi, \eta_1)) \]
\[ + \zeta_i^{p_0+2} (v + F_q(\Xi, \bar{\eta}_q)) + \sum_{i=1}^{q-1} \zeta_i^{p_0+2} \zeta_{i+1} \]
\[ + \sum_{i=2}^{q-1} \zeta_i^{p_0+2} (\beta_i(\Xi, \bar{\eta}_i) + F_i(\Xi, \bar{\eta}_i)) \]
\[ + \frac{p_0 + 2}{2} \sum_{i=1}^{q} c_i^{p_0+2} G_i^T(\Xi, \bar{\eta}_i) G_i(\Xi, \bar{\eta}_i). \]
\[ \text{(13)} \]

By using Lemma 2, one gets
\[ \sum_{i=1}^{q-1} \zeta_i^{p_0+2} \zeta_{i+1} \leq \frac{p_0 + 2}{p_0 + 3} \sum_{i=1}^{q-1} \zeta_i^{p_0+3} \]
\[ + \frac{1}{p_0 + 3} \sum_{i=2}^{q} \zeta_i^{p_0+3}. \]
\[ \text{(14)} \]

By the definition of \( G_i(\Xi, \bar{\eta}_i) \) in (11), one can obtain
\[ \frac{p_0 + 2}{2} \sum_{i=1}^{q} c_i^{p_0+2} G_i^T(\Xi, \bar{\eta}_i) G_i(\Xi, \bar{\eta}_i) \]
\[ \leq (p_0 + 2) \sum_{i=1}^{q} \zeta_i^{p_0+1} |G_i(\Xi, \bar{\eta}_i)|^2 \]
\[ \leq (p_0 + 2) \sum_{i=1}^{q} \zeta_i^{p_0+1} \left| \frac{\partial \beta_i}{\partial z} g_0(z, x_1) \right| \]
\[ + (p_0 + 2) \sum_{i=1}^{q} \zeta_i^{p_0+1} \left| \frac{\partial \beta_i}{\partial \eta} g_0(z, x_1) \right|^2 \]
\[ + \sum_{s=1}^{r} \left| \frac{\partial \beta_i}{\partial x_s} g_s(z, \bar{x}_s) \right|^2 \]
\[ \leq |z|^{p_0+3} + |\xi_1|^{p_0+3} + \cdots + |\xi_r|^{p_0+3} \]
\[ \text{(15)} \]

Using (8), similar to Eq.(3.15)-(3.20) in [6], one can prove that
\[ \langle p_0 + 2 \sum_{i=1}^{q} \zeta_i^{p_0+1} \left| \frac{\partial \beta_i}{\partial z} g_0(z, x_1) \right| \]
\[ + (p_0 + 2) \sum_{i=1}^{q} \zeta_i^{p_0+1} \left| \frac{\partial \beta_i}{\partial \eta} g_0(z, x_1) \right|^2 \]
\[ + \sum_{s=1}^{r} \left| \frac{\partial \beta_i}{\partial x_s} g_s(z, \bar{x}_s) \right|^2 \]
\[ \leq \sum_{i=1}^{q} \zeta_i^{p_0+3} \psi_i(\Xi, \bar{\eta}_i), \]
\[ \text{(16)} \]
where \( \psi_i(\Xi, \bar{\eta}_i) \) is a nonnegative smooth function.

By using Assumptions 1 and 3, Eq.(5), and \( \xi_i = x_i - x_i^* \), one can get
\[ \langle p_0 + 2 \sum_{i=1}^{q} \zeta_i^{p_0+1} \left| \frac{\partial \beta_i}{\partial z} g_0(z, x_1) \right| \]
\[ + (p_0 + 2) \sum_{i=1}^{q} \zeta_i^{p_0+1} \left| \frac{\partial \beta_i}{\partial \eta} g_0(z, x_1) \right|^2 \]
\[ + \sum_{s=1}^{r} \left| \frac{\partial \beta_i}{\partial x_s} g_s(z, \bar{x}_s) \right|^2 \]
\[ \leq \sum_{i=1}^{q} \zeta_i^{p_0+1} (|z|^{p_0+3} + |\xi_1|^{p_0+3} + \cdots + |\xi_r|^{p_0+3}) \]
\[ \leq \sum_{i=1}^{q} \zeta_i^{p_0+1} (|z|^{p_0+3} + |\xi_1|^{p_0+3} + \cdots + |\xi_r|^{p_0+3}) \]
\[ \leq \sum_{i=1}^{q} \zeta_i^{p_0+1} (|z|^{2p_r} + |\xi_1|^{2p_r} + \cdots + |\xi_r|^{2p_r}) \]
\[ \leq \sum_{i=1}^{q} \zeta_i^{p_0+1} (|z|^{2p_r} + |\xi_1|^{2p_r} + \cdots + |\xi_r|^{2p_r}) \]
\[ \leq |z|^{p_0+3} + |\xi_1|^{p_0+3} + \cdots + |\xi_r|^{p_0+3}, \]
\[ \text{(17)} \]
where \( h_{i1}(\Xi), h_{i2}(\Xi), h_{i3}(\Xi) \) are nonnegative smooth functions.

By Lemma 2, for \( s = 1, \cdots, r \), one obtains
\[ \zeta_i^{p_0+1} |z|^{2p_r} h_{i3}(\Xi) = \zeta_i^{p_0+1} |z|^{2p_r} h_{i4}(\Xi) \]
\[ \leq \frac{1}{q} |z|^{p_0+3} + \zeta_i^{p_0+3} h_{i5}(\Xi), \]
\[ \zeta_i^{p_0+1} |\xi_s|^{2p_r} h_{i3}(\Xi) \leq \frac{1}{q} |\xi_s|^{p_0+3} + \zeta_i^{p_0+3} h_{i6}(\Xi), \]
\[ \text{(18)} \]
where \( h_{i4}(\Xi) = |z|^{2p_r-2} h_{i3}(\Xi), h_{i5}(\Xi) \) and \( h_{i6}(\Xi)(s = 1, \cdots, r) \) are nonnegative smooth functions.

Substituting (18) into (17), one can get
\[ \langle p_0 + 2 \sum_{i=1}^{q} \zeta_i^{p_0+1} \left| \frac{\partial \beta_i}{\partial z} g_0(z, x_1) \right| \]
\[ + \sum_{s=1}^{r} \left| \frac{\partial \beta_i}{\partial x_s} g_s(z, \bar{x}_s) \right|^2 \]
\[ \leq |z|^{p_0+3} + |\xi_1|^{p_0+3} + \cdots + |\xi_r|^{p_0+3} \]
Substituting (14)-(16) and (19) into (13), a direct calculation leads to

\[
LU(\Xi, \eta) \leq -(|z|^{p_0+3} + |\xi_1|^{p_0+3} + \ldots + |\xi_r|^{p_0+3})
\]

\[
+ \zeta_0^{p_0+3}(h_{15}(\Xi) + h_{11}(\Xi) + \ldots + h_{ir}(\Xi)).
\]

Choosing

\[
\beta_1(\Xi, \eta_i) = -\frac{p_0}{p_0 + 3} \zeta_1 - \frac{\partial V}{\partial \Xi} H(\Xi, \eta_1) - \zeta_1 \psi_1(\Xi, \eta_1)
\]

\[
- \zeta_1 (h_{15}(\Xi) + h_{11}(\Xi) + \ldots + h_{ir}(\Xi))
\]

\[
\beta_i(\Xi, \eta_i) = -2 \zeta_1 - F_i(\Xi, \eta_i) - \zeta_1 \psi_1(\Xi, \eta_i)
\]

\[
- \zeta_1 (h_{15}(\Xi) + h_{11}(\Xi) + \ldots + h_{ir}(\Xi))
\]

\[
v = -\frac{p_0 + 4}{p_0 + 3} \zeta_1 - \frac{\partial V}{\partial \Xi} F_i(\Xi, \eta_i) - \zeta_1 \psi_q(\Xi, \eta_q)
\]

\[
- \zeta_1 (h_{q5}(\Xi) + h_{q1}(\Xi) + \ldots + h_{qr}(\Xi)).
\]

Substituting (21) into (20) yields

\[
LU(\Xi, \eta) \leq -(|z|^{p_0+3} + |\xi_1|^{p_0+3} + \ldots + |\xi_r|^{p_0+3})
\]

\[
+ |\zeta_1|^{p_0+3} + \ldots + |\zeta_r|^{p_0+3}).
\]

**Remark 6:** In the Lyapunov function (12), \(\zeta_i^{p_0+3}\) is adopted, which is different from \(\zeta_i^4\) in the Eq.(3.4) in [6]. If we used \(\zeta_i^4\), from (18) we know that the nonlinear terms \(|z|^4\) and \(|\xi|^4\) will be brought about, which cannot be damped out by \(-|z|^{p_0+3}\) and \(-|\xi|^{p_0+3}\) in (13). However, with \(\zeta_i^{p_0+3}\) in (12), we can deal with these nonlinear terms easily, which can be found from (17)-(20).

Based on (22), we immediately obtain the main results in this paper.

**Theorem 1.** If Assumptions 1-3 hold for system (1), under the state-feedback controllers (5) and (21), then

1) The closed-loop system has an almost surely unique solution on \([0, \infty)\); 
2) The equilibrium at the origin of the closed-loop system is GAS in probability; 
3) The states can be regulated to the origin almost surely, more precisely,

\[
P \left\{ \lim_{t \to \infty} \left( |z| + \sum_{i=1}^{r} |x_i| + \sum_{s=1}^{q} |\eta_q| \right) = 0 \right\} = 1.
\]

Proof: By \(V(z, \xi_1, \ldots, \xi_n) = (r + 2)V_0 + \sum_{i=1}^{r} \frac{\xi_i^{p_i+4}}{p_i + 4}\), (12) and (22), using Lemma 1, obviously, the closed-loop system (1), (5) and (21) has an almost surely unique solution on \([0, \infty)\), the equilibrium at the origin of the closed-loop system is GAS in probability, and the states can be regulated to the origin almost surely.

**IV. A SIMULATION EXAMPLE**

Consider the following system:

\[
dz = -z^3(3 + \sin^2 x_1) dt + \eta_1^3 dt + z^2 \sin x_1 d\omega,
\]

\[
dx_1 = u^3 dt + u\eta_1^3 dt + \frac{1}{3} x_1^2 \sin z d\omega,
\]

\[
d\eta_1 = \eta_2 dt + u x_1 \eta_1 d\omega,
\]

\[
d\eta_2 = v dt + u^2 x_1 \eta_2 d\omega.
\]

(24)

It is easy to obtain that

\[
| - z^3(3 + \sin^2 x_1) | \leq 4 |z|^3, |z^2 \sin x_1 | \leq |z|^3 + |x_1|^3,
\]

\[
\frac{1}{3} x_1^2 \sin z \leq |z|^3 + |x_1|^3.
\]

which satisfies Assumption 1. Choosing \(V_0 = \frac{1}{2} z^4\), we have \(LV_0 \leq -z^6\) and \(p_0 = 3\), then Assumption 2 is satisfied. Obviously, Assumption 3 is satisfied with \(p_0 = p_1 = 3\).

Here, without detailed arguments, we only state the final results as follows:

\[
u(x_1) = -\left( \frac{13}{6} \right)^{\frac{1}{2}} x_1,
\]

\[v(z, x_1, \eta_1, \eta_2) = \left( -6 z^2 - \frac{5}{2} \left( \frac{13}{6} \right)^{\frac{3}{2}} x_1^4 \right) (-z^3(3 + \sin^2 x_1) + \eta_1^3) \]

\[-4 \left( \frac{13}{6} \right)^{\frac{1}{2}} x_1^4 - 10 \left( \frac{13}{6} \right)^{\frac{1}{2}} x_1^3 z \]

\[-\left( \frac{13}{6} \right)^{\frac{1}{2}} - 5 \left( \frac{13}{6} \right)^{\frac{1}{2}} x_1 \eta_1 \]

\[-5 \left( \frac{13}{6} \right)^{\frac{1}{2}} x_1^4 z \]

\[-\left( \frac{13}{6} \right)^{\frac{1}{2}} - 5 \left( \frac{13}{6} \right)^{\frac{1}{2}} x_1 \eta_1 \]

\[\eta_2 + 2 z^3
\]

\[-\left( \frac{13}{6} \right)^{\frac{1}{2}} x_1^4 + 2 \left( \frac{13}{6} \right)^{\frac{1}{2}} x_1^4 z + \frac{11}{6} \eta_1.\]

(25)

In the practical simulation, one chooses the initial values \(z(0) = -0.6, x_1(0) = 0.1, \eta_1(0) = -1\) and \(\eta_2(0) = 4\). Fig.1 gives the responses of (24) and (26), from which, one can see that the states are regulated to zero asymptotically. To show more clearly the initial response of the system, Fig.2 gives the simulation during the first 1 second with same initial values as in Fig.1. From figures, the efficiency of the controllers are demonstrated.

**Remark 7:** From (24), one can obtain that the drift terms do not satisfy the triangular conditions commonly used in the previous work. Therefore, all the previously design methods are inapplicable to achieve global stabilization of (24). However, using the design methodology in this paper,
one can easily stabilize (24), which can be found from Fig.1 and Fig.2.

V. CONCLUSIONS

For high-order stochastic nonlinear system (1) in the absence of triangular conditions, this paper investigates the problem of state-feedback stabilization for the first time. The novelty of this paper lies in combining and generalizing the previous work in [6] and [19] by relaxing the triangular conditions. Under appropriate assumptions, smooth state-feedback controllers are designed to achieve the desired control objectives.

There are some related problems to be investigated, e.g., how to design state-feedback inverse optimal controller for system (1)? How to generalize the results in this paper to more general systems is an important problem deserving consideration.