H∞ Neural Network Adaptive Control

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Abstract—This paper introduces an $H_\infty$ Adaptive Control architecture using neural networks for systems whose uncertainty has an unknown structure. This architecture merges ideas from robust control theory such as $H_\infty$ control design, the Small Gain Theorem, and $L_1$ stability theory with Lyapunov stability theory and recent theoretical achievements in adaptive control to develop an adaptive architecture for a systems whose uncertainty satisfies a local lipschitz bound. The method permits a control designer to simplify the adaptive tuning process, band limit the adaptive control signal, and treat unmatched uncertainty in a single design framework. The design framework is similar to that used in robust control, but without sacrificing performance. All of this is accomplished while providing notions of transient performance bounds dependent on the characteristics of two linear systems and the adaptation gain.

I. INTRODUCTION

Model reference adaptive control (MRAC) has numerous advantages over modern linear model-based control design methods. Classical methods are limited by uncertainties and nonlinearity. Robust control design reduces the effect of uncertainty and nonlinearity at the expense of reduced performance. However, both of these methods offer the advantage of frequency limited control action. Adaptive control offers the possibility of achieving a much higher degree of robust performance. However, a major disadvantage of adaptive control is that it lacks an accepted means of quantifying the behavior of the control signal apriori. Hence, most adaptive control laws will require a more extensive verification and validation process due to the time varying and nonlinear manner in which its gains are adapted. This process can lead to unacceptable transients during adaptation, which can be made worse by actuator limitations[5] and can yield a transient response that exceeds the practical limits of the plant. From this prospective, it is highly desirable to limit the frequency content of an adaptive control signal. The $H_\infty$ Adaptive architecture presented in this paper offers this capability.

Nearly all existing adaptive control architectures require one to be able to directly cancel uncertainty via the control input. This condition, known as matched uncertainty, has been treated in detail. However, this assumption severely limits the class of uncertainty that one can compensate in a theoretically verifiable manner. The apparent inability to compensate for unmatched uncertainty comes from the fact that Lyapunov based arguments require one to be able to algebraically cancel the uncertainty. However, this neglects insight gained from frequency domain based methods. To ensure close tracking, one must only suppress the effect of uncertainty at low frequencies due to the fact that physical systems have a filtering effect on high frequency disturbances.

There have been a few attempts to solve adaptive control problems with unmatched uncertainty. The most well known of these is adaptive backstepping[10]. However, these designs require a system to be in strict-feedback form. A relatively new work by Hoagg, Santillo, and Bernstein [4] develops a novel discrete-time adaptive MIMO output feedback controller for stabilization, command following, and disturbance rejection in minimum-phase systems. This Markov-parameter-based adaptive control algorithm requires knowledge of only the open-loop systems relative degree and a bound on the first nonzero Markov parameter. This adaptive control method requires that the plant be linear and its Markov parameters must be identified before it is controlled.

$L_1$ adaptive control is a novel approach that guarantees robust performance[1, 2]. $L_1$ Adaptive control is similar in concept to the subject of this paper due to the fact that small-gain-theorem-like arguments are used to prove stability, though the methods are derived from very different perspectives and the proofs are different. The benefit of $L_1$ adaptive control is its ability to provide fast and robust adaptation that leads to desired transient performance for both the system’s input and output signals simultaneously. The $L_1$ framework achieves this by introducing a stable linear filter between the control input and the adaptive element, thereby band limiting the adaptive control signal. The analysis tools in this paper can be used to analyze the $L_1$ adaptive control architecture and retain similar stability results (though not the same). The $H_\infty$ adaptive architecture is a synthesis framework that allows use of linear control design methods to achieve potentially lower bandwidth control signals using well understood design tools. This allows the design process to become very intuitive. Adjustment of the linear control design allows one to trade between reference model tracking error and adaptive control effort in a manner that is optimal with respect to the $H_\infty$ norm.

This paper presents the $H_\infty$ Adaptive Control architecture
for systems with a Lipschitz bound on its nonlinearity. The synthesis approach seeks to merge ideas from robust control theory such as $H_\infty$ control design and the Small Gain Theorem, $L$ stability theory and Lyapunov stability from nonlinear control, and recent theoretical achievements in adaptive control. By introducing some additional structure of the system uncertainty, a bound on the lipschitz constant, frequency domain considerations can be introduced in the adaptive design. This fusion of frequency domain and linear time domain ideas allows one to derive adaptive control architectures that allow a control designer to simplify the adaptive tuning process, band limit the adaptive control signal, and treat unmatched uncertainty in a single design framework. It is similar to that used in robust control without sacrificing performance. All of this is accomplished while providing notions of transient performance bounds. The presented norm bounds on the transient performance allows a designer to ensure that the response stays within a desired error tolerance of the reference model by increasing the adaptation gain and decreasing the $H_\infty$ norm of two different linear systems. Since the system state is bounded to a ball with computable size, the analysis is valid for locally Lipschitz nonlinearities. Moreover, though the bounds can be conservative, bounds are computable using straightforward numerical procedures and $H_\infty$ optimal control techniques provide guidance as to how to suppress these bounds.

II. Mathematical Preliminaries

Proofs in this paper use input-output stability to show stability and bounded transients[3], [6]. The $H_\infty$ control design paradigm uses $H_\infty$ optimal control theory to formulate a control decision. Since signals in the adaptive system can only be guaranteed bounded, system responses can only be bounded using $L_1$ norms (since the $L_\infty$ norm induces the $L_1$ system norm). The next theorem provides a relationship between the $H_\infty$ norm and the $L_1$ norm of linear systems[11].

**Theorem 2.1:** Let $G(s)$ be a be a strictly proper transfer function with finite $H_\infty$ norm. Furthermore, suppose

$$G(s) = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}$$

is a balanced realization such that

$$A\Sigma + \Sigma A^* + BB^* = 0 \quad \text{and} \quad A^*\Sigma + \Sigma A + C^*C = 0$$

where

$$\Sigma = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n) \geq 0$$

with $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$. Then

$$\sigma_i \leq \|G\|_\infty \leq \|G\|_{L_1} \leq 2 \sum_{i=1}^{n} \sigma_i$$

Theorem 2.1 illustrates that for a linear system, with finite $H_\infty$ norm, $\|G(s)\|_{L_1}$ can be upper bounded by a finite computable constant. Moreover, as the $H_\infty$ norm approaches zero, the $L_1$ norm approaches zero. The above theorem relies on the existence of a balanced realization to upper bound the $L_1$ norm. The balanced realization always exists[11].

III. Stability Analysis for Lipschitz Nonlinearities

A. $H_\infty$ Adaptive Control Law for Lipschitz Nonlinearities

Suppose that the system dynamics can be expressed as

$$\dot{x} = Ax + Bu + Df(x)$$

where $A \in \mathbb{R}^{nxn}$, $B \in \mathbb{R}^{nxm}$, $D \in \mathbb{R}^{nxj}$, $x \in \mathbb{R}^n$ is the system state, $u \in \mathbb{R}^m$ is the system control input, and $f(x) : \mathbb{R}^n \mapsto \mathbb{R}^j$ satisfies the following Lipschitz property

$$\|f(x) - f(y)\|_\infty \leq L \|x - y\|_\infty$$

so that $K \in \mathbb{R}^+$ is an upper bound for $f(\cdot)$ at the origin.

**Remark 3.1:** Note that $D$ allows for a broad class of uncertainty. If $D = B$, the system model reduces to the system used in MRAC theory subject to a matched uncertainty condition.

Suppose that there exist a nominal control law that renders $A$ Hurwitz and provides the desired system tracking characteristics assuming that the system uncertainty, $f(x)$, is zero.

$$u_n = -K_x x + K_r r$$

It is desired to track the ideal system behavior ($f(x) = 0$) within bounded error. To define the desired behavior, the following closed loop reference model is defined

$$\dot{x}_m = A_m x + B_m r$$

where $A_m = A - BK_x$ and $B_m = BK_r$. In order to compensate for the system uncertainty and ensure the reference model is tracked with bounded error, we augment the nominal control law with an adaptive signal. The total control is defined as

$$u = u_n - u_{ad}$$

where $u_{ad}$ will be defined shortly. Applying this control action to the system dynamics, we rewrite the system dynamics as

$$\dot{x} = A_m x - Bu_{ad} + BK_r r + Df(x)$$

It is assumed that $f(x)$ is unknown but can be approximated to a sufficient degree of accuracy by a linear in parameters neural network over a compact set. We assume that the neural network approximates $f(x)$ as

$$f(x) = W^T \beta(x) + \epsilon(x), \quad x \in D_x$$

where $\beta(x) : \mathbb{R}^n \mapsto \mathbb{R}^s$ is a vector of known basis functions, $W \in \mathbb{R}^{sxj}$ is a set of unknown ideal weights, and $\exists \epsilon > 0$ s.t. $\|\epsilon(x)\| < \epsilon^* < \infty \quad \forall x \in D_x$. The ideal weights are assumed to exist in a known compact set and the domain of approximation $D_x$ will be defined later. With this assumption, the closed loop dynamics become

$$\dot{x} = A_m x - Bu_{ad} + BK_r r + D(W^T \beta(x) + \epsilon(x))$$
In order to learn how to dominate the system uncertainty, we employ a state emulator to separate the control realization from the adaptation process (previously referred to as a state predictor). Consider the following state emulator
\[ \dot{\hat{x}} = A_{m}\hat{x} - Bu_{ad} + BK_{r}r + D\hat{W}^{T}\beta(x) \] (10)
where \( \hat{W} \in \mathbb{R}^{n \times j} \) is a set of adaptive weights to be determined online. Defining the emulation error as \( \hat{e} = x - \hat{x} \), the weight estimation error as \( \hat{W} = W - \hat{W} \), the emulation error dynamics can then be expressed as
\[ \dot{\hat{e}} = A_{m}\hat{e} - D\hat{W}^{T}\beta(x) + De(x) \] (11)
Forming the error between the reference model and the emulator, we define the emulator tracking error as \( e = x_{m} - \hat{x} \). From this definition, the emulator tracking error dynamics can be expressed as
\[ \dot{\hat{e}} = A_{m}e + Bu_{ad} + Dw(t) \] (12)
where \( w(t) = -\hat{W}^{T}\beta(x) \). Suppose that a full information (including \( w(t) \)) control law is designed to suppress the disturbing effect that \( w(t) \) has on the error dynamics in equation (12). Let the form of this control law be defined by
\[ \begin{split} \dot{x}_{e} &= A_{e}x_{e} + B_{e}[w]^{T} \\ u_{ad} &= C_{e}x_{e} + D_{e}[w]^{T} \end{split} \] (13)
This allows for a broad class of linear control laws to be applied. The existence of the unknown disturbance in the feedback control law may be necessary in general to ensure a low gain control design that can sufficiently reject the disturbance \( w(t) \). Let the weight update law for the adaptive weights, \( \hat{W} \), be defined as
\[ \dot{\hat{W}} = \hat{\Gamma}Proj(\hat{W}, \beta(x)e^{T}PD) \] (14)
where \( A_{m}^{T}P + PA_{m} = -Q, Q = Q^{T} > 0, \hat{\Gamma} = \Gamma I \), and the projection bound on the projection operator is \( W_{max} \). If the control law in (13) is designed to minimize the \( H_{\infty} \) norm of the transfer function from \( w(t) \) to \( e(t) \), then this defines the complete \( H_{\infty} \) adaptive control law. The complete control law is illustrated graphically in figure 1.

**B. Stability Analysis and Guaranteed Tracking Bounds**

This section derives sufficient conditions for stability and transient bounds of the \( H_{\infty} \) adaptive control law. This is accomplished by defining an auxiliary set of reference dynamics and considering tracking performance relative to this system. The error bounds can be made smaller by reducing the \( H_{\infty} \) norm associated with the emulator tracking error dynamics (12) and by increasing the adaptation gain. Note that the adaptive control structure can also be used for a control design that suppresses the \( L_{1} \) norm of (12). However, \( H_{\infty} \) control theory provides a convenient framework for gaining design insight into the tradeoffs involved in the control design process. Moreover, using \( H_{\infty} \) control theory allows us to approach the design process in an efficient algorithmic way.

For convenience, each theorem assumes the \( H_{\infty} \) adaptive control law is defined as in equations (10)-(14) and each theorem may implicitly rely on the theorems derived before it. To begin our analysis, the following lemma shows that the emulator error, \( \hat{e} \), is bounded.

**Lemma 3.1:** The emulator error, \( \hat{e} \), is bounded. Moreover, \( \|\hat{e}\| \) has the upper bound
\[ \|\hat{e}\| \leq \sqrt{\frac{2}{\lambda_{min}(Q)\lambda_{min}(P)}\epsilon^{T}(PD) + \frac{4}{\Gamma_{c}}\hat{W}^{2}_{F,\max}} \] (15)
where \( \|\cdot\|_{F} \) represents the Frobenius norm.

**Proof:** Choosing the following Lyapunov function candidate
\[ V = \hat{e}^{T}PD\hat{e} + tr(\hat{W}^{T}\Gamma^{-1}\hat{W}) \] (16)
implies
\[ \dot{V} \leq -\hat{e}^{T}Q\hat{e} + 2\hat{e}^{T}PD\epsilon \] (17)
Hence\( \hat{e} \) is bounded and the adaptive weights are bounded due to the projection operator. Since
\[ V(t) \leq \frac{2}{\lambda_{min}(Q)\lambda_{min}(P)}\epsilon^{T}(PD) + \frac{4}{\Gamma_{c}}\hat{W}^{2}_{F,\max} \] (18)
we have the result. \[ \square \]

**Remark 3.2:** If the neural network approximation error, \( \epsilon(x) = 0 \), the emulator error can be made arbitrarily small by increasing the adaptation gain.

The previous lemma proved that the emulation error is guaranteed to stay bounded. However, at this point, one can not claim that \( x \) or \( \hat{x} \) is bounded. If it can be shown that \( x \) or \( \hat{x} \) is bounded, then for perfect parameterization (\( \epsilon = 0 \)) the emulation error can be shown to converge asymptotically. We now develop sufficient conditions for boundedness of all signals. To facilitate development, express the closed loop emulator tracking error dynamics as
\[ E(s) = G(s)W(s) \] (19)
where
\[
G(s) = \begin{bmatrix}
A_m & BC_c & D \\
B_{c1} & A_c & 0 \\
I & 0 & 0
\end{bmatrix}
\]
and \(B_c\) is partitioned such that \(B = [B_{c1} \ B_{c2}]\) so that \(B_{c1}\) corresponds to the control law input \(e\) and \(B_{c2}\) corresponds to the control law input \(w(t)\). Furthermore, define \(F(s)\) as the proper transfer function
\[
F(s) = G(s) \left( D^T D \right)^{-1} D^T (sI - A_m)
\]
where \(F(s)\) has the realization
\[
F(s) = \begin{bmatrix}
A_F & B_F \\
C_F & D_F
\end{bmatrix}
\]
For convenience, the following constants are defined
\[
\alpha_1 = \left( 2 \sum_{i=1}^{n} \sigma_i \right) \quad \text{and} \quad \alpha_2 = \left( 2 \sum_{i,j}^{n} \sigma_j \right) + \|D_F\|_1
\]
where \(\sigma_i\) are from a balanced realization of \(G(s)\) and \(\sigma_j\) are from a balanced realization of \(F(s) - D_F\).

To derive sufficient conditions and transient bounds, first, the ideal system response is defined. Consider the following auxiliary reference dynamics
\[
\dot{x}_a = A_m x_a - B u_{a_d} + D f(x_a) + B_m r
\]
Let the auxiliary system tracking error be defined as \(e_a = x_m - x_a\). Then the auxiliary system tracking dynamics is given by
\[
\dot{e}_a = A_m e_a + B u_{a_d} + D (-f(x_a))
\]

The auxiliary tracking dynamics control law is defined similarly to equation (13) as
\[
\dot{x}_{ca} = A_c x_{ca} + B_{c1} e_a + B_{c2} (-f(x_a))
\]
\[
u_{a_d} = C_c x_{ca}
\]

Remark 3.3: Equations (24), (25), and (26) define the ideal response from the system assuming that the uncertainty is completely known without an adaptive element. It represents the best one can hope to achieve using adaptive control since the uncertainty is assumed to be completely known. The next theorem gives sufficient conditions for the auxiliary reference system to be stable.

Theorem 3.1: If \(1 - \alpha_1 L > 0\) then the auxiliary reference system is bounded. Moreover, the auxiliary reference system (24) tracks the reference model in transient to within the following norm bounds
\[
(1 - \|G\|_{L_1} L)\|x_a\|_{\infty} \leq \|x_m\|_{\infty} + \|G\|_{L_1} K
\]
\[
\|x_m\|_{\infty} \leq (1 + \|G\|_{L_1} L)\|x_a\|_{\infty} + \|G\|_{L_1} K
\]
A more conservative but easy to compute bound is given by
\[
(1 - \alpha_1 L)\|x_a\|_{\infty} \leq \|x_m\|_{\infty} + \alpha_1 K r
\]
\[
\|x_m\|_{\infty} \leq (1 + \alpha_1 L)\|x_a\|_{\infty} + \alpha_1 K
\]
where
\[
G(s) \in \mathcal{RH}_\infty
\]

Proof: Given the definition of the auxiliary reference system in equation (25) and equation (26), we form the following expressions
\[
X_a(s) = X_m(s) + G(s)f(x_a)(s)
\]
\[
X_m(s) = X_a(s) - G(s)f(x_a)(s)
\]
Applying the triangle inequality and the Schwarz inequality, this leads to norm bounds given by
\[
\|x_m\|_{\infty} \leq \|x_a\|_{\infty} + \|G\|_{L_1} \|f(x_a)\|_{\infty}
\]
\[
\leq (1 + \|G\|_{L_1} L)\|x_a\|_{\infty} + \|G\|_{L_1} K
\]
and
\[
\|x_a\|_{\infty} \leq \|x_m\|_{\infty} + \|G\|_{L_1} (K + L\|x_a\|_{\infty})
\]
\[
\leq \frac{1}{1 - \|G\|_{L_1} L} (\|x_m\|_{\infty} + \|G\|_{L_1} K)
\]
Hence the auxiliary reference system is bounded and equation (27) holds. Equation (28) follows from Theorem 2.1.

In general, for unmatched uncertainty it may be difficult to make \(\|G(s)\|_{L_1}\) small enough to provide sufficient transient bounds. The next theorem provides a useful additional bound on performance if it is only of interest to suppress the emulator tracking error in certain linear combinations of the reference model state (for instance, one may only care to suppress the position error between the reference model and the auxiliary reference system).

Theorem 3.2: Let \(\tilde{C}\) be a matrix describing the outputs whose emulator tracking error should be small during system transient. Then assuming
\[
\alpha_3 L < 1
\]
where
\[
G(s) \in \mathcal{RH}_\infty \quad \text{and} \quad \alpha_3 = \left( 2 \sum_{i=1}^{n} \sigma_i \right)
\]
such that \(\sigma_i\) are from a balanced realization of \(\tilde{C}G(s)\). Then
\[
\|\tilde{C}(x_a - x_m)\|_{\infty} \leq \alpha_3 \left( \frac{K}{1 - \alpha_1 L} \right) L\|x_m\|_{\infty}
\]

Proof: We can express \(x_m\) as
\[
X_m(s) = X_{aug}(s) - G(s)f(x_a)(s)
\]
Hence
\[
\|\tilde{C}(x_a - x_m)\|_{\infty} = \|\tilde{C}G(s)f(x_a)\|_{\infty}
\]
\[
\leq \|\tilde{C}G(s)\|_{L_1} \|f(x_a)\|_{\infty}
\]
\[
\leq \|\tilde{C}G(s)\|_{L_1} (K + L\|x_a\|_{\infty})
\]
From the proof theorem 3.1, we have
\[
\|\tilde{C}(x_a - x_m)\|_{\infty} \leq \left( \frac{K + L\|x_m\|_{\infty} + \|G(s)\|_{L_1} K}{1 - \|G(s)\|_{L_1} L} \right)
\]
The result follows from Theorem 2.1 and some algebra. □

The next theorem characterizes the effect of using adaptation to estimate the uncertainty. The difference is the error between the ideal auxiliary reference system and the $H_\infty$ adaptive control law.

**Theorem 3.3:** The emulator tracks the auxiliary reference system within the following bound

$$\|x_a - \hat{x}\|_\infty \leq \frac{\alpha_2 + \alpha_1 L}{1 - \alpha_1 L} \|\hat{e}\|_\infty$$

(30)

Moreover, the system state tracks the auxiliary reference system with the following bound

$$\|x_a - x\|_\infty \leq \frac{1 + \alpha_2}{1 - \alpha_1 L} \|\hat{e}\|_\infty$$

(31)

If in addition, $\epsilon(x) = 0$, then $x_a - \hat{x} \to 0$ as $t \to \infty$ and $x_a - x \to 0$ as $t \to \infty$.

**Proof:** See Appendix.

**Remark 3.4:** This theorem can be made less conservative by replacing $\alpha_1$ and $\alpha_2$ with $\|G(s)\|_{\ell_1}$ and $\|F(s)\|_{\ell_1}$, respectively.

**Remark 3.5:** Note that as the adaptation gain $\Gamma$ is increased and $\epsilon(x) \to 0$, Theorem 3.3 implies that the error between the ideal response, $x_a$, and the system state, $x$, can be made arbitrarily small.

**Remark 3.6:** Since $x_a - \hat{x}$ is bounded, so are $x$ and $\hat{x}$.

From the previous two theorems one can arrive at an estimate for the domain of approximation for the neural network.

**Theorem 3.4:** The system state is bounded as

$$\|x\|_\infty \leq \frac{1 + \alpha_2}{1 - \alpha_1 L} \|\hat{e}\|_\infty + \frac{\alpha_1}{1 - \alpha_1 L} \left[ L \|x_m\|_\infty + K \right] + \|x_m\|_\infty$$

(32)

**Proof:** Since $x = x - x_a - e_a + x_m$, this is a straight forward result from Theorems 3.2 and 3.3.

**Remark 3.7:** From the previous theorems, we see that the transient bound between the auxiliary system and the state emulator is made small by decreasing the $H_\infty$ norm of $G(s)$. However, the stability analysis gives bounds on the transient performance but does not guarantee that the system converges to the reference model asymptotically for the case of perfect parameterization. If asymptotic convergence to a reference model step response is desired, it can be accomplished by augmenting the emulator tracking error dynamics with an integrator state to ensure that the tracking output of the auxiliary reference system converges to the output of the reference model tracking variables. For analysis purposes, it is important to be sure not to include the integrator states in the output of $G(s)$.

**IV. Example**

Consider the following idealized model of aircraft wing rock dynamics[9].

$$\dot{x} = Ax + B(\delta_a + f(x))$$
$$y = Cx$$

(33)

where

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C^T = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

(34)

where $\delta_a$ is the roll control moment, $f(x)$ is a set of unknown system nonlinearities, $x = [\phi \ \dot{\phi}]^T$, and $\phi$ is the aircraft roll angle. The system nonlinearities are defined as

$$f(x) = \begin{bmatrix} 0.1 \phi \ 2\dot{\phi} - 0.6|\phi|\dot{\phi} \ 0.1|\phi|\dot{\phi} \ 0.2\phi^3 \end{bmatrix}$$

(35)

This nonlinearity has a Lipschitz constant of 2. Suppose we desire the system to behave like a second order system with natural frequency $\omega_n = 5$ rad/s and damping ratio $\zeta = 0.707$. With this choice, the nominal control law, defined in (4), is given by $K_x = [\omega_n^2 \ 2\omega_n\zeta]$ and $K_r = \omega_n^2$. To design the $H_\infty$ based adaptive signal, the system emulator tracking dynamics is augmented with a integrator to enforce zero steady state error in the auxiliary reference system. Using the performance measure, $z$,

$$z = Q^2 e + R^2 u_ad$$

the following system realization is used for the $H_\infty$ control design.

$$\begin{bmatrix} A_{aug} & B_{aug} & B_{aug} \\ \tilde{Q}^2 & 0 & R^2 \\ I & 0 & 0 \end{bmatrix}$$

(36)

where

$$A_{aug} = \begin{bmatrix} 0 & C \\ 0 & A - BK_x \end{bmatrix}, \quad B_{aug} = \begin{bmatrix} 0 \\ B \end{bmatrix}$$

(37)

For comparison purposes, we will compare a standard projection based adaptive law[8] to the new architecture using radial basis functions to approximate $f(x)$. System actuation limitations are modeled as a first order filter with a 10 Hz bandwidth. The nominal control law is unstable. The weighting matrices in emulator tracking error dynamics were selected as $Q^2 = diag(0.05 \ 1.0 \ 0.5)$ and $R^2 = 0.02$. This yields an $H_\infty$ control law that is band limited to 8 Hz. From this, $\alpha_1 = 0.05$ and $\alpha_2 = 2.0536$. Using the same adaptation gain, the tracking performance is compared between the standard adaptive law and the $H_\infty$ adaptive law for a 10 degree square wave at a frequency of $0.1Hz$. The standard adaptive law control signal is highly oscillatory and is relatively large in amplitude. This causes the system to oscillate some around the reference model. For this example, there is no choice of adaptive gain that could be found that achieved acceptable tracking and a low frequency control signal simultaneously. The $H_\infty$ control law provides satisfactory tracking performance without the control oscillations present in the standard adaptive law.

**V. Conclusion**

A neural network $H_\infty$ adaptive control architecture was derived that allows a control designer to tune the reference model tracking characteristics via linear control design techniques, band limit the adaptive control signal, and treat unmatched uncertainty a single design framework for systems...
Proof of Theorem 3.3

Forming the error between the auxiliary system tracking error (25) and the emulator tracking error (12), \( \bar{e} = e - e_a \), we have the following error dynamics

\[
\dot{\bar{e}} = \dot{e} - \dot{e}_a = A_m \bar{e} + B \bar{u}_{ad} + D \left( -W^T \beta(x) + f(x_a) \right) \tag{38}
\]

where \( \bar{u}_{ad} = u_{ad} - u_{ad_a} \) and

\[
\dot{x}_c = A_c \bar{x}_c + B_c \left[ \left( -\hat{W}^T \beta(x) + f(x_a) \right) \right]^T
\]

\[
\bar{u}_{ad} = C_c \bar{x}_c
\]

Hence one can write \( \bar{e}(t) \) in the Laplace domain

\[
\bar{E}(s) = G(s) \left[ -\hat{W}^T \beta(x) + f(x_a) \right] (s) \tag{39}
\]

Since \( \hat{W} = W - W \) and that \( f(x) = W^T \beta(x) + \epsilon(x) \)

\[
\bar{E}(s) = G(s) \left[ -\hat{W}^T \beta(x) - f(x) + \epsilon(x) + f(x_a) \right] (s) \tag{40}
\]

In the Laplace domain, the emulator error dynamics (11) can be expressed as

\[
(sI - A_m) \dot{\bar{e}}(s) = -D \left( \hat{W}^T \beta(x) \right) (s) + D \epsilon(s) \tag{41}
\]

Since \( D \) has full column rank

\[
\left( \hat{W}^T \beta(x) \right) (s) = \epsilon(s) - (D^T D)^{-1} D^T (sI - A_m) \bar{E}(s) \tag{42}
\]

This allows one to write

\[
\bar{E}(s) = F(s) \bar{E}(s) + G(s) \left( f(x_a) - f(x) \right) (s) \tag{43}
\]

Using the facts that \( \bar{e} = x_a - \bar{x} \), \( x = \bar{x} + \hat{x} \), and the Lipschitz property of \( f(\cdot) \)

\[
\| \bar{E} \|_\infty \leq \| F(s) \|_{L_1} \| \bar{e} \|_\infty + \| G(s) \|_{L_1} L (\| \bar{e} \|_\infty + \| \bar{e} \|_\infty)
\]

This implies the result in (30) using Theorem 2.1. Noting that \( x_m - x - e_a = \bar{e} - \hat{e} \) and the triangle inequality, one can easily derive the expression given in (31) using Theorem 3.2. Now consider the case of perfect parameterization. For \( \epsilon(x) = 0 \), \( \bar{e} \in L_2 \cap L_\infty \). Since \( \bar{e} \in L_\infty \), the dynamics of \( \bar{e} \) imply that \( \bar{e} \in L_\infty \). Hence, from Barbalat’s Lemma, \( \bar{e}(t) \to 0 \) as \( t \to \infty \). One can use similar arguments to show that \( (x_a - x)(t) \to 0 \) as \( t \to \infty \).