\( L_1 \) Adaptive Controller for a Class of Systems with Unknown Nonlinearities

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Abstract—This paper presents an extension of the \( L_1 \) adaptive controller to a class of general nonlinear uncertain systems, nonaffine in control. The control signal interacts with system states and time-varying uncertainties in an unknown nonlinear way. The adaptive controller ensures uniformly bounded transient for system’s both input and output signals simultaneously. The performance bounds can be systematically improved by increasing the adaptation rate. Simulation results verify the theoretical findings.

I. INTRODUCTION

The prior results of \( L_1 \) adaptive control theory were limited to systems affine in control \cite{1}–\cite{4}. In this paper, we consider a class of more general uncertain systems, where the dependence upon the control signal is nonlinear and time-varying. Input nonlinearities like dead-zone, backlash, and hysteresis have been discussed in \cite{5}–\cite{8}. In this paper a more general class of input nonlinearities is considered. We prove that subject to a set of mild assumptions the system can be transformed into an equivalent linear system with time-varying unknown parameters and disturbances. We extend the methodology from \cite{9} to accommodate unknown time-varying high-frequency gain. The \( L_\infty \) norm bounds for the error signals between the closed-loop adaptive system and the closed-loop reference system can be systematically reduced by increasing the adaptation rate.

The paper is organized as follows. Section III gives the problem formulation. In Section V, the \( L_1 \) adaptive control architecture is presented. Stability and uniform performance bounds are presented in Section VI. In Section VII, simulation results are presented, while Section VIII concludes the paper.

II. PRELIMINARIES

Consider a single-input single-output (SISO) linear time-varying (LTV) system:

\[
\begin{align*}
\dot{x}(t) &= A(t)x(t) + B(t)x_i(t), \quad x(0) = 0, \\
x_{out}(t) &= C(t)x(t) + D(t)x_i(t),
\end{align*}
\]

where \( A(t) \in \mathbb{R}^{n \times n} \), \( B(t) \in \mathbb{R}^{n \times 1} \), \( C(t) \in \mathbb{R}^{1 \times n} \), \( D(t) \in \mathbb{R} \). Let \( \mathcal{M} \) define the input-output map of this system \( x_{out} = \mathcal{M}\{x_i\} \), where \( x_i, x_{out} \) are the system input and system output respectively.

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\[ \text{Lemma 1:} \quad [10] \text{Suppose the system state equation (1) is uniformly exponentially stable, and there exist finite positive constants } b, c, d > 0 \text{ such that for all } t \]

\[ \|B(t)\| \leq b, \quad \|C(t)\| \leq c, \quad \|D(t)\| \leq d. \]

Then the state equation is also uniformly BIBO stable. Further, if the input \( x_i(t) \in \mathcal{L}_\infty \), then \( \|x_{out}\|_{\mathcal{L}_\infty} \leq \|\mathcal{M}\|_{\mathcal{L}_\infty} \|x_i\|_{\mathcal{L}_\infty}. \]

\[ \text{Definition 1:} \quad \text{The cascaded system of } \mathcal{M} \text{ and } \mathcal{G} \text{ is defined as } \mathcal{M}\mathcal{G}, \text{ i.e.} \]

\[ (\mathcal{M}\mathcal{G})\{x_i\} = \mathcal{M}\{\mathcal{G}\{x_i\}\} . \]

III. PROBLEM FORMULATION

Consider the following system dynamics:

\[
\begin{align*}
\dot{x}(t) &= A_m x(t) + b f(x(t), u(t), t), \\
y(t) &= c^T x(t), \quad x(0) = x_0,
\end{align*}
\]

where \( x(t) \in \mathbb{R}^n \) is the system state vector (measurable), \( u(t) \in \mathbb{R} \) is the control signal, \( y(t) \in \mathbb{R} \) is the regulated output, \( b, c \in \mathbb{R}^n \) are known constant vectors, \( A_m \) is a known \( n \times n \) Hurwitz matrix, and \( f : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) is an unknown nonlinear function.

\[ \text{Assumption 1:} \quad [\text{Semiglobal Lipschitz condition on } x] \text{ For any } \delta > 0, \text{ there exist } L_\delta > 0 \text{ and } B > 0 \text{ such that} \]

\[ |f(x, u, t) - f(\bar{x}, u, t)| \leq L_\delta ||x - \bar{x}||_\infty, \quad |f(0, u, t)| \leq B, \quad \text{for all } ||x||_\infty \leq \delta \text{ and } ||\bar{x}||_\infty \leq \delta \text{ uniformly in } u \text{ and } t. \]

\[ \text{Assumption 2:} \quad [\text{For any } \delta > 0, \text{ there exist } d_{f_{\delta}}(\delta) > 0, \text{ and } d_{f_{\delta}}(\delta) > 0 \text{ such that for any } u \text{ and} \]

\[ ||x||_\infty \leq \delta \text{ the partial derivatives of } f(x, u, t) \text{ with respect to } x \text{ and } t \text{ are piece-wise continuous and bounded} \]

\[ \left| \frac{\partial f(x, u, t)}{\partial x} \right| \leq d_{f_{\delta}}(\delta), \quad \left| \frac{\partial f(x, u, t)}{\partial t} \right| \leq d_{f_{\delta}}(\delta). \]

\[ \text{Assumption 3:} \quad \text{[Known sign of control effectiveness]} \text{ There exist upper and lower bounds } \omega_u > \omega_l > 0 \text{ such that} \]

\[ \omega_l \leq \frac{\partial f(x, u, t)}{\partial u} \leq \omega_u \text{ uniformly in } t \text{ and } x \text{ for all } t \geq 0. \]

The control objective is to design a full-state feedback adaptive controller to ensure that \( y(t) \) tracks a given bounded reference signal \( r(t) \) with quantifiable transient and steady-state performance.

IV. DEFINITIONS AND EQUIVALENT LTV SYSTEM

Let \( k \) be a positive design parameter. We define two SISO LTV systems with input \( x_i \) and output \( x_{out} \):

\[
\begin{align*}
C : \quad \dot{x}_{out}(t) &= -k \omega(t)x_{out}(t) + k \omega(t)x_i(t), \\
\bar{C} : \quad \left\{ \begin{array}{l}
\dot{x}_{out1}(t) = -k \omega(t)x_{out1}(t) + k \omega(t)x_i(t), \\
x_{out}(t) = x_i(t) - x_{out1}(t),
\end{array} \right.
\end{align*}
\]
where $\omega(t)$ is any time-varying signal subject to
\begin{equation}
\omega_t \leq \omega(t) \leq \omega_u, \quad |\hat{\omega}(t)| < \omega_d.
\end{equation}

Choose constant $k$ to ensure that there exists $\rho_r$ such that
\begin{equation}
\frac{\|G\|_{L_1} \sigma_b + \|H \mathcal{C}_{L_1} \| k_g r}{1 - L_{\rho} \|G\|_{L_1}} < \rho_r,
\end{equation}
where $\sigma_b$ is the upper bound of $f(0,0,t)$, $k_g = -1/(c^T A_m^{-1} b)$, $r_0$ is the signal with its Laplace transformation of $\xi(t) = x_0$. $\mathcal{H}$ is the map of $H(s) = (sI - A_m)^{-1} b$, $\tilde{G} = \mathcal{H} \mathcal{C}$ is the cascaded system of the LTI system $\mathcal{H}$ and the LTV system $\mathcal{C}$, which was defined in (5), and
\begin{equation}
\rho = \rho_r + \beta,
\end{equation}
with $\beta$ being an arbitrary positive constant. It follows from Lemma 1 that $\|G\|_{L_1}$ and $\|H \mathcal{C}_{L_1}\|$ are finite. One can prove that $\|\tilde{G}\|_{L_1}$ can be arbitrarily small, when $k \to \infty$, and hence (7) can always be satisfied.

**Lemma 2:** The cascaded system $\mathcal{C} \mathcal{F}$ is BIBO stable with finite $L_1$ norm, where $\mathcal{C}$ is defined in (4) and $\mathcal{F}$ is $\frac{z^b}{b}$, $b > 0$.

If $\|x\|_\infty \leq \rho$ and $u \geq 0$, it follows from Assumptions 1 and 3 that
\begin{equation}
f(x, u, t) = f(x, 0, t) + \int_{\mu=0}^{u} \frac{\partial f(x(0), \mu, 0)}{\partial \mu} d\mu 
\geq -L_{\rho} \|x\|_\infty - \sigma_b + \epsilon + \omega_t u,
\end{equation}
where $\sigma_b = B + \epsilon$ and $\epsilon$ is an arbitrarily positive constant. Repeating the above derivation, we could obtain the entire bounds for $f(x, u, t)$ when $\|x\|_\infty \leq \rho$:
\begin{align}
\omega_u u - L_{\rho} \|x\|_\infty - \sigma_b + \epsilon &\leq f(x, u, t), \\
\omega_u u + L_{\rho} \|x\|_\infty + \sigma_b - \epsilon &\leq f(x, u, t), \\
\omega_u u - L_{\rho} \|x\|_\infty - \sigma_b + \epsilon &\leq f(x, u, t), \\
\omega_u u + L_{\rho} \|x\|_\infty + \sigma_b - \epsilon &\leq f(x, u, t),
\end{align}
(10) (11)

We further define $\rho_u$, as:
\begin{equation}
\rho_u = ||C||_{L_1} (\rho_r L_{\rho} + \sigma_b + k_g r \|x\|_\infty) / \omega_t.
\end{equation}
(12)
From Lemma 1, we note that $\|C\|_{L_1}$ is finite.

Define
\begin{equation}
\gamma_1 = \frac{\|H \mathcal{C} \|_{1/2} \mathcal{C}_{\rho} \|_{L_1} \gamma_0 + \beta_1 < \beta},
\end{equation}
where $\beta$ was introduced in (8), $\gamma_0$ and $\beta_1$ are any positive constant. There always exist $\gamma_0$ and $\beta_1$ satisfy (13), since they can be arbitrarily small. We note that $\mathcal{C}$ is still a linear low-pass system although it is time-varying. Note that $\|H \mathcal{C} \|_{1/2} \mathcal{C}_{\rho} \|_{L_1} \leq \|H \mathcal{C}_{L_1} \| \mathcal{C} \mathcal{F}_{L_1} \| 1/2 \mathcal{C}_{\rho} \|_{L_1}$. Since $\|\mathcal{C} \mathcal{F}_{L_1}$ is finite in Lemma 2, and $\mathcal{C}_{\rho} \|_{L_1}$ and $\mathcal{C} \mathcal{F}$ are stable and proper, we can conclude that the $\|H \mathcal{C} \|_{1/2} \mathcal{C}_{\rho} \|_{L_1}$ is finite. Similarly, we have finite $\|C\|_{1/2} \mathcal{C}_{\rho} \|_{L_1}$. Further, let
\begin{align}
\rho_u = \rho_u + \gamma_2, \\
\gamma_2 = \left(\frac{L_{\rho}}{\omega'}\right) ||C||_{L_1} \gamma_1 + \left(\frac{1}{\omega'}\right) ||C\|_{1/2} \mathcal{C}_{\rho} \|_{L_1} \gamma_0.
\end{align}

It follows from Lemma 4 in [1] that there exists $\epsilon_o \in \mathbb{R}^n$ such that
\begin{equation}
\epsilon_o^T H(s) = N_n(s)/N_d(s),
\end{equation}
where the order of $N_d(s)$ is one more than the order of $N_n(s)$, and both $N_n(s)$ and $N_d(s)$ are stable polynomials.

In this section we demonstrate that the nonlinear system in (3) can be transformed into an equivalent linear system with unknown time-varying parameters.

**Lemma 3:** (i) If $\|x_t\|_\infty \leq \rho$, there exist $u(t)$, $\omega(t)$, $\theta(t)$ and $\sigma(t)$ over $[0, t]$ such that
\begin{align}
\omega_t &< \omega(t) < \omega_u, \\
|\theta(t)| &< \theta_b, \\
|\sigma(t)| &< \sigma_b,
\end{align}
(17) (18) (19)
\begin{equation}
f(x(t), u(t), \tau) = \omega(t) u(t) + \theta(t) |x(t)|_\infty + \sigma(t).
\end{equation}
(20)

(ii) If in addition, $\dot{\omega}(t)$ and $\dot{\theta}(t)$ are bounded, then $u(t)$, $\omega(t)$, $\theta(t)$ are differentiable with finite derivatives.

**V. $L_1$ Adaptive Controller**

Consider the following state predictor (or passive identifier) for generation of the adaptive laws:
\begin{align}
\dot{\hat{x}}(t) &= A_m \dot{\hat{x}}(t) + b \left(\dot{\omega}(t) u(t) + \dot{\theta}(t) |x(t)|_\infty + \dot{\sigma}(t)\right), \\
\hat{y}(t) &= c^T \hat{x}(t), \quad \hat{x}(0) = x_0.
\end{align}
(21)

The adaptive estimates $\hat{\omega}(t)$, $\hat{\theta}(t)$, $\hat{\sigma}(t)$ are defined as:
\begin{align}
\dot{\tilde{\theta}}(t) &= \Gamma \text{Proj} \left[\dot{\theta}(t), -|x(t)|_\infty \hat{x}^T(t) P b\right], \quad \tilde{\theta}(0) = \tilde{\theta}_0, \\
\dot{\tilde{\sigma}}(t) &= \Gamma \text{Proj} \left[\dot{\sigma}(t), -\hat{x}^T(t) P b\right], \quad \tilde{\sigma}(0) = \tilde{\sigma}_0, \\
\tilde{\omega}(t) &= \Gamma \text{Proj} \left[\hat{\omega}(t), -\hat{x}^T(t) P b u(t)\right], \quad \tilde{\omega}(0) = \tilde{\omega}_0,
\end{align}
(22) (23)
where $\hat{x}(t) = \dot{\hat{x}}(t) - x(t)$, $\Gamma \in \mathbb{R}^+$ is the adaption gain, $P$ is the solution of the algebraic equation $A_m^T P + P A_m = -Q$, $Q > 0$, and the projection operator ensures that the adaptive estimates $\hat{\omega}(t)$, $\hat{\theta}(t)$, $\hat{\sigma}(t)$ remain inside the compact sets $[\omega_l, \omega_u]$, $[\theta_b, \theta_b]$, $[\sigma_b, \sigma_b]$, respectively, with $\theta_b$, defined as follows
\begin{equation}
\theta_b = L_{\rho},
\end{equation}
(23)
The control signal is generated through gain feedback of the following system:
\begin{align}
\dot{\chi}(t) &= \dot{\omega}(t) u(t) + \dot{\theta}(t) |x(t)|_\infty + \dot{\sigma}(t) - k g r(t), \\
u(t) &= -k \chi(t),
\end{align}
(24)

where $k \in \mathbb{R}^+$ is arbitrary positive constant.

The complete $L_1$ adaptive controller consists of (21), (22) and (24) subject to the $L_1$-norm upper bound in (7).
VI. ANALYSIS OF $L_1$ ADAPTIVE CONTROLLER

A. Closed-loop Reference System

Given any $\omega(\tau)$, $\theta(\tau)$ and $\sigma(\tau)$ satisfying (17), (18) and (19) for all $\tau \in [0, t]$, we design a closed-loop reference system:

$$
\dot{x}_{\text{ref}}(\tau) = A_m x_{\text{ref}}(\tau) + b(\omega(\tau)u_{\text{ref}}(\tau) + \theta(\tau)\|x_{\text{ref}}(\tau)\|_{\infty} + \sigma(\tau) ),
$$

(25)

$$
\dot{\chi}_{\text{ref}}(\tau) = \omega(\tau) u_{\text{ref}}(\tau),
$$

(26)

$$
u_{\text{ref}}(\tau) = -k_\chi \chi_{\text{ref}}(\tau),
$$

(27)

$$
y_{\text{ref}}(\tau) = c^T x_{\text{ref}}(\tau) , \quad x_{\text{ref}}(0) = x_0,
$$

(28)

with its property summarized in the next Lemma.

Lemma 4: For the closed-loop reference system in (25)-(28) subject to the $L_1$-norm upper bound in (7), if $\|x_0\|_{\infty} \leq \rho_r$, then

$$
\|x_{\text{ref}}\|_{L_\infty} \leq \rho_r,
$$

(29)

$$
\|u_{\text{ref}}\|_{L_\infty} \leq \rho_u,
$$

(30)

where $\rho_r$ is introduced in (7) and $\rho_u$, is defined in (12).

Proof of Lemma 4 is similar to Lemma 1 in [4] and omitted.

B. Tracking error signal

Consider

$$
\|x_t\|_{L_\infty} \leq \rho,
$$

(31)

$$
\|u_t\|_{L_\infty} \leq \rho_u.
$$

(32)

If (31) holds, Lemma 3 (i) implies that the system in (3) can be rewritten over $\tau \in [0, t]$: as:

$$
\dot{x}(\tau) = A_m x(\tau) + b(\omega(\tau)u(\tau) + \theta(\tau)\|x(\tau)\|_{\infty} + \sigma(\tau) ),
$$

(33)

where $\omega(\tau)$, $\theta(\tau)$, $\sigma(\tau)$ are unknown time-varying signals subject to the bounds (17), (18) and (19) for all $\tau \in [0, t]$.

If further (32) holds, it follows from (23) and (33) that $\dot{x}(\tau)$ and $u(\tau)$ are bounded. Hence, Lemma 3 (ii) and (31)-(32) imply that the derivatives of $\omega(\tau)$, $\theta(\tau)$, $\sigma(\tau)$ are bounded:

$$
\|\dot{\omega}(\tau)\| \leq d_\omega(\rho, \rho_u) < \infty,
$$

(34)

$$
\|\dot{\theta}(\tau)\| \leq d_\theta(\rho, \rho_u) < \infty,
$$

(35)

$$
\|\dot{\sigma}(\tau)\| \leq d_\sigma(\rho, \rho_u) < \infty.
$$

(36)

Lemma 5: For the system in (3) and the $L_1$ adaptive controller in (21), (22) and (24), for any $t$ such that (31)-(32) holds, we have

$$
\|\ddot{x}_t\|_{L_\infty} \leq \sqrt{\frac{\theta_m(\rho, \rho_u)}{\lambda_{\min}(P) \Delta}} ,
$$

(37)

where

$$
\theta_m(\rho, \rho_u) \triangleq 4\rho^2 + 4\rho_u^2 + (\omega_u - \omega_l)^2 + 4\frac{\lambda_{\max}(P)}{\lambda_{\min}(Q)} \Delta ,
$$

$$
\Delta = \left( \frac{1}{2} (\omega_u - \omega_l) d_\omega(\rho, \rho_u) + \theta_0 d_\theta(\rho, \rho_u) + \sigma_0 d_\sigma(\rho, \rho_u) \right).
$$

Proof of Lemma 5 is similar to Lemma 3 in [4] and omitted. The only difference is that we have time-varying $\omega(t)$ instead of constant values. Hence, $d_\omega(\rho, \rho_u)$ also show up in the definition of upper-bound of $\dot{x}_t$.

C. Transient and Steady-State Performance

Theorem 1: Consider the reference system in (25)-(28) and the closed-loop $L_1$ adaptive controller in (21), (22), (24) subject to (7). If $\|x_0\|_{\infty} \leq \rho_r$, and the adaptive gain is chosen to verify the lower bound:

$$
\Gamma > \frac{\theta_m(\rho, \rho_u)}{\lambda_{\min}(P) \gamma_0^2},
$$

(38)

we have:

$$
\|x\|_{L_\infty} < \rho,
$$

(39)

$$
\|u\|_{L_\infty} < \rho_u,
$$

(40)

$$
\|\dot{x}\|_{L_\infty} < \gamma_1,
$$

(41)

$$
\|x - x_{\text{ref}}\|_{L_\infty} < \gamma_2,
$$

(42)

$$
\|u - u_{\text{ref}}\|_{L_\infty} < \gamma_2,
$$

(43)

where $\gamma_1$ and $\gamma_2$ are defined in (13) and (15).

Proof. The proof will be done by contradiction. Assume that (39)-(40) are not true. Then, since $\|x(0)\|_{\infty} < \rho$, $\|u(0)\| = 0 < \rho_u$ and $x(\tau)$, $u(\tau)$ are continuous, there exists $t \geq 0$ such that

$$
\|x(\tau)\|_{L_\infty} = \rho, \text{ or }\]

$$
\|u(\tau)\|_{L_\infty} = \rho_u.
$$

(44)

(45)

while

$$
\|x_t\|_{L_\infty} \leq \rho, \quad \|u_t\|_{L_\infty} \leq \rho_u.
$$

(46)

In what follows, we prove (46) ensures

$$
\|x - x_{\text{ref}}\|_{L_\infty} < \gamma_1,
$$

(47)

$$
\|u - u_{\text{ref}}\|_{L_\infty} < \gamma_2.
$$

(48)

The proof of (47)-(48) will be done by contradiction, too. Assume that (47)-(48) are not true. Then, since $\|x(0) - x_{\text{ref}}(0)\|_{\infty} = 0 \leq \gamma_1$, $\|u(0) - u_{\text{ref}}(0)\|_{\infty} = 0 \leq \gamma_2$, and $x(\tau)$, $x_{\text{ref}}(\tau)$, $u(\tau)$, $u_{\text{ref}}(\tau)$ are continuous, there exists $t \in [0, t]$ such that

$$
\|x(t) - x_{\text{ref}}(t)\|_{\infty} = \gamma_1, \text{ or }\]

$$
\|u(t) - u_{\text{ref}}(t)\|_{\infty} = \gamma_2.
$$

(49)

while

$$
\|x - x_{\text{ref}}\|_{L_\infty} \leq \gamma_1, \quad \|u - u_{\text{ref}}\|_{L_\infty} \leq \gamma_2.
$$

(50)

We note that (46) implies $\|x_t\|_{L_\infty} \leq \rho$, $\|u_t\|_{L_\infty} \leq \rho_u$. Hence, it follows from Lemma 3 that there exist $\omega(t)$, $\theta(t)$ and $\sigma(t)$ with bounded derivatives such that the system in (3) is equivalent to LTV system in (33) over $[0, t]$, which will be used in the following analysis. It follows from (38) and Lemma 5 that

$$
\|\ddot{x}_t\|_{L_\infty} \leq \gamma_0.
$$

(51)
\[ \dot{r}(\tau) = \omega(\tau)u(\tau) + \tilde{\theta}(\tau)\|x(\tau)\|_{\infty} + \sigma(\tau), \quad r_1(\tau) = \theta(\tau)\|x(\tau)\|_{\infty} + \sigma(\tau). \]

It follows from (24) that \( \dot{\chi}(t) = \omega(t)u(t) + r_1(t) - k_2r(t) + \dot{r}(t). \) Consequently
\[ \begin{aligned}
\dot{\chi}(t) + k_2\omega(t)\chi(t) &= r_1(t) - k_2r(t) + \dot{r}(t), \\
u(t) &= -k_3\chi(t). \end{aligned} \]

Using the definition of \( C \) from (4), we can write
\[ \omega(t)u(t) = -C\{r_1(t) - k_2r(t) + \dot{r}(t)\}, \]
and the system in (3) consequently takes the form:
\[ x(t) = (H\tilde{C})\{r_1(t)\} + (HC)\{k_2r(t) - \dot{r}(t)\} + x_{in}(t), \]
where \( x_{in}(t) \) is the signal with its Laplasea transformation \( sI - A_m)\)^{-1}x_0. Using the same \( \omega(t), \theta(t) \) and \( \sigma(t) \) as in (33) and apply the reference control algorithm as in (25)-(27). We have
\[ x_{ref}(t) = (HC)\{r_{ref1}(t)\} + (H\tilde{C})\{r_2(t)\} + x_{in}(t). \]

Let \( e(\tau) = x(\tau) - x_{ref}(\tau) \). It follows from (54) and (55) that
\[ e(\tau) = (H\tilde{C})\{r_2(\tau)\} - (HC)\{\dot{r}(\tau)\}, \]
over \([0, t]\) where
\[ r_2(\tau) = \theta(\tau)(\|x(\tau)\|_{\infty} - \|x_{ref}(\tau)\|_{\infty}). \]

We have \( \tilde{x}(s) = H(s)\tilde{r}(s) \), which leads to (56) that
\[ e(\tau) = (H\tilde{C})\{r_2(\tau)\} - (HC)\{\dot{r}(\tau)\}, \]
over \([0, t]\) where
\[ r_2(\tau) = \theta(\tau)(\|x(\tau)\|_{\infty} - \|x_{ref}(\tau)\|_{\infty}). \]

We have the following upper bound:
\[ \|e_1\|_{L_\infty} \leq \|G\|_{L_1}r_2(\tau)\|_\infty + \|HC\frac{1}{C_0}H^T\|_{L_1}\|\tilde{x}(\tau)\|_{L_\infty}. \]

Since \( \|x(\tau)\|_{\infty} - \|x_{ref}(\tau)\|_{\infty} \leq \|x(\tau) - x_{ref}(\tau)\|_{\infty} \leq \|e_1\|_{L_\infty} \) for any \( \tau \in [0, t] \), it follows from (23), (102) and (57) that
\[ \|r_2\|_{L_\infty} \leq L_\rho\|e_1\|_{L_\infty}. \]
From (58) we have \( \|e_1\|_{L_\infty} \leq \|G\|_{L_1}L_\rho\|e_1\|_{L_\infty} + \|HC\frac{1}{C_0}H^T\|_{L_1}\|\tilde{x}(\tau)\|_{L_\infty} \). Eq. (51) and the \( L_1 \)-norm upper bound from (7) lead to the following upper bound \( \|e_1\|_{L_\infty} \leq \|HC\frac{1}{C_0}H^T\|_{L_1}\|\tilde{x}(\tau)\|_{L_\infty} \gamma_0 \), which along with (13) leads to
\[ \|e_1\|_{L_\infty} \leq \gamma_1 - \beta_1 \leq \gamma_1. \]

We notice that from (53) and (55) one can derive \( \omega(t)u(t) - u_{ref}(t) = -C\{r_2(t)\} - C\tilde{r}(t) \). Since
\[ C\{\tilde{r}(t)\} = (C\frac{1}{C_0}H^T)\{\tilde{r}(t)\} = (C\frac{1}{C_0}H^T)\{\tilde{x}(\tau)\}, \]
It follows from (51), (59), and the definition of \( \gamma_2 \) in (15) that
\[ \|u - u_{ref}(t)\|_{L_\infty} \leq (L_\rho/\omega)(C_{\tilde{C}}\|\chi(1) - \beta_1\|_{L_1} + (1/\omega)) \|C\frac{1}{C_0}H^T\|_{L_1}\|\gamma_0 \leq \gamma_2. \]

We note that the upper bounds in (60) and (61) contradict the equality in (49), which proves (47)-(48).

Since \( \omega(t), \theta(t) \) and \( \sigma(t) \) are bounded, it follows from (7) and Lemma 4 that
\[ \|x_{ref1}\|_{L_\infty} \leq \rho_r, \quad \|u_{ref}(t)\|_{L_\infty} \leq \rho_u. \]

Combining (47)-(48) and (62), we have
\[ \|x_i\|_{L_\infty} \leq \rho, \quad \|u_i\|_{L_\infty} \leq \rho_u, \]
which contradict the equality in (45). Hence, (39)-(40) are proved.

Since (46) holds for any \( \bar{\ell} \), (42)-(43) follow from (47)-(48) immediately. The upper bound in (41) follows from (51) directly.

We note that the reference system is only for analysis purpose and not implementable since it uses unknown information \( \omega(t), \theta(t) \) and \( \sigma(t) \). If adaptive gain \( \Gamma \) is large enough, \( \gamma_0 \) can be arbitrarily small, as well as \( \gamma_1 \) and \( \gamma_2 \). This implies that both the control signal and system response of the adaptive and reference system could be arbitrarily close.

VII. SIMULATIONS

Consider the dynamics of a single-link robot arm with nonlinearity rotating on a vertical plane:
\[ I\ddot{q}(t) + F(q(t), \dot{q}(t), u(t), t) = 0, \]
where \( q(t) \) and \( \dot{q}(t) \) are the measured angular position and velocity, respectively, \( u(t) \) is the input torque, \( I \) is the unknown moment of inertia, \( F(q(t), \dot{q}(t), u(t), t) \) is an unknown nonlinear function that lumps the forces, torques and inputs due to gravity, friction, disturbance and other external sources. Let \( x = [x_1 x_2]^T = [q q^T]. \) The system in (64) can be presented in the canonical form:
\[ \dot{x}(t) = A_m x(t) + b f(x(t), u(t), t), \quad y(t) = c^T x(t), \]
where \( b = [0 1]^T, \quad c = [1 0]^T. \) Let \( A_m = \begin{bmatrix} 0 & -1 \\ 1 & -1.4 \end{bmatrix}, \)
\( f(x(t), u(t), t) = \begin{bmatrix} 1 & 1.4 \end{bmatrix} x(t) - \begin{bmatrix} 2 & 0 \end{bmatrix} \sin(u(t)) + \sin(u(t)) \sin(0.5t), \)
so that the compact sets can be conservatively chosen as \( \omega_1 = 0.5, \omega_u = 0, \theta_0 = 10, \sigma_0 = 10. \) The control objective is to design \( u(t) \) to achieve tracking of a bounded reference input \( y_{des}(t) \) by \( q(t) \). The definition of \( y_{des}(t) \) is as follows,
\[ \dot{x}_{des}(t) = A_m x_{des}(t) + k_2 r(t), \]
\[ y_{des}(t) = c^T x_{des}(t), \]
where \( \|r\|_{L_\infty} \leq 1. \)

In the implementation of the \( L_1 \) adaptive controller, we set \( Q = 2I, \) and hence \( P = \begin{bmatrix} 1.4143 & 0.5000 \\ 0.5000 & 0.71430 \end{bmatrix}, \)
\( k = 20, \) the adaptive gain \( \Gamma = 10000. \)

The simulation results of the \( L_1 \) adaptive controller are shown in Figures 1(a)-1(b) for reference input \( y_{des}(t) \) with \( r = \cos(0.5t). \) We further change the nonlinearity function between \( u \) and \( x_1: F(x_2(t), x_1(t), t) = \begin{bmatrix} 2+0.2 \sin(u(t)) + \cos(u(t)) \sin(2x_1(t)) + x_2^2(t) + x_1^2(t) + \sin(5t) \end{bmatrix}. \) The simulation results are shown in 2(a)-2(b). We note that the \( L_1 \) adaptive controller guarantees smooth and uniform transient
performance in the presence of different unknown nonlinearities without requiring any retuning. We also notice that $x_1(t), \dot{x}_1(t)$, and reference input $y_{des}(t)$ are almost the same in Figs. 1(a) and 2(a).

VIII. CONCLUSION

A novel $L_1$ adaptive controller architecture is presented that has guaranteed transient response in addition to stable tracking for general uncertain systems in the presence of unknown state, input and time-dependent nonlinearities. The control signal and the system response approximate the same signals of a closed-loop reference system, which can be designed to achieve desired specifications.

REFERENCES


with the initial values being bounded, the determinant of $A_0$ is:

$$
\det(A_0) = |u(\tau)(\omega_b - \omega_0(\tau)) + g_b - g(\tau)|. \tag{76}
$$

For any $\tau \in [0, \bar{t}]$, where $\bar{t}$ is an arbitrary constant or infinite, it follows from (76) that $\det(A_0(\tau)) \neq 0$ over $[0, \bar{t}]$. Then with conditions (17) and (69) So

$$
\frac{df(x(\tau), u(\tau), \tau)}{d\tau} \frac{d(\omega(\tau)u(\tau) + g(\tau))}{d\tau} \tag{77}
$$

for any $\tau \in [0, \bar{t}]$.

In what follows, we prove (17) and (69) by contradiction. If (17) and (69) are not true, since $\omega(\tau)$ and $g(\tau)$ are continuous, it follows from initial value that there exists $\bar{t} \in [0, \bar{t}]$ such that at least one of the following cases happen

1. $\lim_{\tau \to \bar{t}} \omega(\tau) = \omega_u$,
2. $\lim_{\tau \to \bar{t}} g(\tau) = g_l$,
3. $\lim_{\tau \to \bar{t}} g(\tau) = g_b$,
4. $\lim_{\tau \to \bar{t}} g(\tau) = -g_b$.

while

$$\omega_l < \omega(\tau) < \omega_u, \ |g(\tau)| < g_b, \ \forall \tau \in [0, \bar{t}). \tag{83}$$

1. Let's consider the case

$$\lim_{\tau \to \bar{t}} \omega(\tau) = \omega_u, \tag{84}$$

first. Since $u(t)$ is continuous, there are only three situations:

$$u(\bar{t}) > 0, \tag{85}$$

$$u(\bar{t}) < 0, \tag{86}$$

$$u(\bar{t}) = 0. \tag{87}$$

If (85) is true, there exists $t_0 < \bar{t}$ such that

$$u(\tau) > 0, \tau \in [t_0, \bar{t}). \tag{88}$$

It follows from (83) that

$$\omega_l < \omega(t_0) < \omega_u, \ |g(t_0)| < g_b, \tag{89}$$

$$\omega_l < \omega(\tau) < \omega_u, \ |g(\tau)| < g_b, \ \forall \tau \in [t_0, \bar{t}). \tag{90}$$

Using the initial condition from (71)-(73), we can integrate to obtain

$$f(x(\tau), u(\tau), \tau) = \omega(\tau)u(\tau) + g(\tau), \forall \tau \in [0, \bar{t}), \tag{91}$$

$$\lim_{\tau \to \bar{t}} \text{sign}(g(\tau)) \ln(g_b - |g(\tau)|) \tag{92}$$

and hence

$$\lim_{\tau \to \bar{t}} \text{sign}(\omega_0(\tau)) \ln(\omega_b - |\omega_0(\tau)|) = \infty. \tag{93}$$

Since it is obvious that $\text{sign}(g(t_0)) \ln(g_b - |g(t_0)|)$ and $\text{sign}(\omega_0(t_0)) \ln(\omega_b - |\omega_0(t_0)|)$ are bounded, it follows from (92) that $\lim_{\tau \to \bar{t}} \text{sign}(g(\tau)) \ln(g_b - |g(\tau)|) = \infty$, and hence

$$\lim_{\tau \to \bar{t}} g(\tau) = g_b. \tag{95}$$

It follows from (91) that

$$\lim_{\tau \to \bar{t}} f(x(\tau), u(\tau), \tau) = \lim_{\tau \to \bar{t}} (\omega_u u(\bar{t}) + g_b), \tag{96}$$

which follows from (94) and (95) implies

$$\lim_{\tau \to \bar{t}} f(x(\tau), u(\tau), \tau) = \omega_u u(\bar{t}) + g_b. \tag{97}$$

It follows from (68) that

$$f(x(t), u(t), \tau) \geq \omega_u u(t) - \sigma_b + \frac{\epsilon}{2}, \tag{98}$$

which contradicts (97), and therefore (79) is not true. If (86) is true, using the similar methodology, we could derive

$$\lim_{\tau \to \bar{t}} f(x(\tau), u(\tau), \tau) = -\omega_u u(\bar{t}) - g_b. \tag{99}$$

It follows from (91) that

$$\lim_{\tau \to \bar{t}} f(x(\tau), u(\tau), \tau) = g(\tau), \tag{100}$$

and hence

$$\lim_{\tau \to \bar{t}} ||f(x(\tau), u(\tau), \tau)||_\infty \leq g_b. \tag{101}$$

We note that (101) contradicts (68) and (79) is not true.

2. Following the same steps as above, one can derive contradicting arguments to (80), (81) and (82) respectively.

Since (79)-(82) are not true, then the relationships in (17), (69) hold. Eq. (70) follows from (17), (69) and (91) directly.

Using derivation similar to Lemma 2 in [4], we can prove that there exist $\theta(\tau)$ and $\sigma(\tau)$ over $\tau \in [0, \bar{t}]$ such that

$$|\theta(\tau)| < \theta_b, \tag{102}$$

$$|\sigma(\tau)| < \sigma_b, \tag{103}$$

$$g(\tau) = \theta(\tau)||x(\tau)||_\infty + \sigma(\tau). \tag{104}$$

In the above analysis, we note that $g(t)$ is continuous and $|g(t)|$ never approaches $g_b$. $\omega(\tau)$ also never approaches its bounds. It follows from (76) that the determinant of $A_n$ is finite. Since $u(t)$ is bounded, we conclude from (74) that $\omega(\tau)$ and $g(\tau)$ are bounded. Since $\dot{x}$ is bounded, in the light of Assumption 2, it follows similarly that $\frac{dg(x(\tau), \tau)}{d\tau}$ and $\frac{d||x(\tau)||_\infty}{d\tau}$ are bounded, although the derivative $\frac{d||x(\tau)||_\infty}{d\tau}$ may not be continuous. Hence, from Lemma 2 in [4], we conclude that $\theta(\tau)$ and $\delta(\tau)$ are bounded. This concludes the proof.

□