Performance Limits Imposed by Semi-Active Damping Constraints

Philip S. Harvey and Henri P. Gavin

Abstract—Semi-active damping control represents a class of parametric control problems in which control decisions affect the value of positively-constrained damping parameters. Transfer functions from control to performance cannot be written in closed form and performance is typically assessed in terms of the peak values of transient responses. In this study optimal control trajectories are designed to satisfy the Hamilton-Jacobi-Bellman conditions. The performance associated with such control trajectories may be viewed as the performance limit of a vibrating system controlled by adjusting one or more damping parameters.

I. INTRODUCTION

Isolation systems reduce the impedance between oscillators at frequencies above a certain threshold. For situations in which the frequency range of the disturbance is known and in which the disturbance is persistent isolation systems can be designed to provide desired performance objectives in terms of transmissibilities and other responses.

Seismic isolation can therefore protect shock-sensitive objects from base excitations of a sufficiently-high frequency content. When the excitation frequency approaches that of the isolation system, however, the peak displacement demand can exceed the isolation system capacity, resulting in impact loads that could defeat the very purpose of the isolation system. Passive damping suppresses displacement responses in the range of resonant frequencies but increases the isolation system transmissibility at higher frequencies. There is therefore a potential benefit to controlling damping rates to mitigate the negative influence of damping on seismic isolation systems for situations in which the disturbance could produce resonance in the isolation system. One such application is in the protection of building contents, such as data center equipment, from earthquake hazards — the energy content of earthquake ground motions in the low frequency range (less than 1 Hz) can be significant and is not well characterized due to limitations in the sensors used to record earthquakes.

The equipment isolation system addressed in this study operates on the principle of a nonlinear rolling pendulum. The isolated components are supported by large ball-bearings (2 cm in diameter) that roll on dish-shaped surfaces. The profile of the dish determines the period of motion, independent of the mass, and may be tailored to meet performance objectives. In this study dishes with a quadratic profile and a conical profile are investigated. Frictional damping is modulated in the isolation system in order to minimize objective function that weights total response accelerations and control effort in order to improve the isolation system transmissibility at high frequencies while also providing damping to suppress resonant behavior.

Semi-active damping is a type of parametric control, and is therefore inherently nonlinear. Because the damping is implemented through controllable friction, because the dynamic system is nonlinear (for the conical profile), and because the disturbance is nonstationary and transient, it is difficult to prove optimality of feedback control algorithms. Asymptotic stability of these systems is unconditionally guaranteed through the physics of the implementation; there is no need for stability constraints in the controller synthesis. The response of an optimally-controlled nonlinear and semi-active system excited by transients may be determined without ever proposing a feedback control rule by solving a dynamic programming problem.

In this study optimal control trajectories are computed to satisfy a set of Euler Lagrange equations and are therefore independent of any feedback parameterization. These solutions provide performance targets for parameterized feedback controllers. It will be shown that for short period impulsive excitations the optimal control has a clear physical interpretation which can inform a parametrized control rule. Further, the generality of the dynamic programming approach admits non-quadratic performance functions, the solutions of which can be more effective in suppressing peak responses.

II. METHOD

A. Euler-Lagrange Equations

A control trajectory $u(t)$ is to be applied to a non-autonomous system

$$\dot{x}(t) = f(x(t), u(t); t); \quad x(t_0) = x_0, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m \quad (1)$$

in order to minimize an integral cost

$$J = \int_{t_0}^{t_f} L(x(t), u(t); t) \, dt \quad , \quad (2)$$

subject to the constraints of the system dynamics (1). This is accomplished by minimizing the first variation of an augmented cost, $\delta J_A$,

$$J_A = \int_{t_0}^{t_f} \{ L(x, u; t) + \lambda^T(t) [f(x, u; t) - \dot{x}] \} \, dt \quad . \quad (3)$$

Defining the Hamiltonian,

$$\mathcal{H}(x, u; t) = L(x, u; t) + \lambda^T(t) f(x, u; t) \quad \quad (4)$$

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P.S. Harvey is with Department of Civil and Environmental Engineering, Duke Univ., Durham, NC 27708-0287, USA Philip.Harvey@duke.edu

H.P. Gavin is with the Faculty of Department of Civil and Environmental Engineering, Duke Univ., Durham, NC 27708-0287, USA Henri.Gavin@duke.edu
the Euler-Lagrange equations provide the necessary conditions for optimality [10],
\[
\dot{\lambda}(t) = - \left( \frac{\partial H}{\partial x} \right)^T + \lambda \frac{\partial \mathcal{L}}{\partial u} = 0
\] (5)
where \( \lambda(t_f) = 0 \).

B. Gradient descent method

A numerical solution to the Euler-Lagrange equation is found via a gradient descent method in which the state and co-state equations are solved by iterative modification. Provided an initial guess for the control trajectory, \( u_0(t) \) the associated state trajectories are computed with a forward simulation of the system dynamics and the integral cost is accumulated. Expressions for the system dynamics and the Lagrangian \( \mathcal{L}(x, u) \) may be analyzed for the following time-dependent Jacobians which can be evaluated given the solution of the system dynamics associated with \( u_0(t) \).

\[
\mathcal{J}(t) = \left[ \begin{array}{c c}
\frac{\partial f}{\partial x} & \frac{\partial f}{\partial u} \\
\frac{\partial g}{\partial x} & \frac{\partial g}{\partial u}
\end{array} \right]_{1 \times n}
\] (7)
The Jacobians are used to solve the costate equation (5) in reverse time from a terminal costate of zero. The gradient of the Hamiltonian (6) is then computed from the costates and the previously computed Jacobians and the control trajectory is updated in the direction that reduces the Hamiltonian gradient.

\[
u_{k+1}(t) = u_k(t) + \kappa_k \frac{\partial H(x, u, t)}{\partial u}
\] (8)
where \( \kappa_k \) is the scalar gradient gain and is held at a constant, small value for each trial \( k \). An appropriate choice of \( \kappa_k \) balances convergence with numerical stability.

Finding \( u(t) \) to satisfy (1), (5) and (6) represents a high dimensional (dynamic) optimization. In some cases the convergence of the gradient descent method depends critically upon the initial guess \( u_0(t) \) and on the update gain \( \kappa_k \).

C. Application to equipment isolation system

In this study, three cases were studied:

I. Parabolic bowl with quadratic Lagrangian, \( \mathcal{L}(x, u) \)

II. Parabolic bowl with quartic Lagrangian, \( \mathcal{L}(x, u) \)

III. Cone-shaped bowl with quadratic Lagrangian, \( \mathcal{L}(x, u) \)

For the parabolic bowl. The state vector includes the position and velocity of the mass, \( r(t) \) and \( \dot{r}(t) \). The third state equation models time delay of duration \( T_e \) and enforces that the frictional force is modulated by a positive-valued variable, regardless of the sign of the control, \( u \). In (9) the bowl has curvature \( \alpha \) and in (10) the bowl has a slope of \( \alpha \). The friction force amplitude is nominally \( f_0 \) and the system is forced with base accelerations \( \ddot{w}(t) \). The quadratic Lagrangian \( \mathcal{L}(x, u) \) is given by

\[
\mathcal{L}(x(t), u(t)) = [\dot{r}(t) + \ddot{w}(t)]^2 + qu(t)^2 = a(t)^2 + qu(t)^2
\] (11)
where the Lagrangian is the sum of the square of the absolute acceleration and the square of the control multiplied by a weighting factor \( q \). In the case of a quartic Lagrangian, the absolute acceleration is raised to the fourth order, \( a(t)^4 \).

The Jacobians (7) are given as follows for Case I:

\[
\left[ \frac{\partial f}{\partial x} \right] = \begin{bmatrix} 0 & 1 & 0 \\ -\alpha & 0 & -\frac{1}{m} f_0 sgn \dot{r} \end{bmatrix} \frac{1}{T_e}
\] (12)
\[
\left[ \frac{\partial f}{\partial u} \right] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \frac{1}{T_e} H[u]
\] (13)
\[
\left\{ \frac{\partial \mathcal{L}}{\partial u} \right\} = 2qu
\] (14)
For Case II, the only change is in the Lagrangian and

\[
\left\{ \frac{\partial \mathcal{L}}{\partial x} \right\} = 4(g \alpha r + \frac{1}{m} f_0 sgn \dot{r})^3 \begin{bmatrix} \frac{1}{T_e} \\ \frac{1}{T_e} \\ \frac{1}{T_e} \end{bmatrix} \] (16)
For conic-shaped bowl, expressions for \( \left\{ \frac{\partial \mathcal{L}}{\partial x} \right\} \) and \( \left\{ \frac{\partial f}{\partial x} \right\} \) are given by (15) and (13), respectively, whereas \( \left\{ \frac{\partial \mathcal{L}}{\partial u} \right\} \) and \( \left\{ \frac{\partial f}{\partial u} \right\} \) are given by

\[
\left[ \frac{\partial f}{\partial x} \right] = \begin{bmatrix} 0 \\ -\frac{2g \alpha}{r_0} \sech^2 \frac{r}{r_0} \frac{1}{r_0} \frac{1}{m} f_0 sgn \dot{r} \\ 0 \end{bmatrix}
\] (17)
and

\[
\left\{ \frac{\partial \mathcal{L}}{\partial x} \right\} = 2a(t) \begin{bmatrix} -\frac{2g \alpha}{r_0} \sech^2 \frac{r}{r_0} \frac{1}{r_0} \frac{1}{m} f_0 sgn \dot{r} \\ 0 \end{bmatrix}
\] (18)
where \( a(t) = -\frac{g \alpha}{r_0} \sech \frac{r}{r_0} \frac{1}{m} f_0 sgn \dot{r} \). Case III has more nonlinearity and it will be shown that gradient descent methodologies do not converge as successfully in this case.
D. Transient disturbance model

Because transient response to pulse-like excitation is of primary concern in this application, the base acceleration is obtained from the expression for base velocity which is essentially a Morlet wavelet.

\[ \dot{w}(t) = V_p \exp \left[ -\frac{\pi^2}{4} \left( \frac{t - 2N_cT_p}{N_cT_p} \right)^2 \right] \cos \left( \frac{2\pi}{T_p} \frac{t - 2N_cT_p}{T_p} \right) \tag{19} \]

where the pulse has a period of \( T_p \), a velocity amplitude of \( V_p \), and contains \( N_c \) cycles. An example transient excitation is shown in Figure 2. All transient simulations were carried out for a sufficiently long duration so as to capture the peak response.

E. Comparison to passive power-law damping

Semi-active damping systems are advantageous only in so far as they can out-perform passive damping systems. In order to assess these advantages, pulse response transmissibilities for the optimal semi-active systems are assessed in comparison to a set of passively damped systems in which the damping forces follow a power-law with velocity. The dynamics of the passive system are described by

\[ \ddot{v} + \beta |\dot{v}|^{n-1} \text{sgn} \dot{v} + g \alpha r = -\ddot{w} \, . \tag{20} \]

The exponent \( n \) in the passive systems in this study is 0.5 and the coefficient \( \beta \) ranges from 0.2 to 5.

III. RESULTS

Numerical values of the constants used in these simulations are shown in Table I. One objective of this study is to investigate the potential for parameterization of the optimal semi-active control actions in the form of a feedback control law. To this end, Figures 2, 4, 5, and 6 illustrate the relationships between measurable states and the control action. As discussed below, the clarity of the functional relationship between the states and the control actions depend upon the period of the excitation. Comparisons to passive power-law damping are made in figures plotting the ratio of the peak of the transient response to the peak of the transient excitation. These pulse response spectra show the dynamic amplification in terms of relative displacement, \( \max |\dot{r}|/\max |w| \), and total acceleration, \( \max |\ddot{r} + \ddot{w}|/\max |\ddot{w}| \).

A. Case I: Quadratic Lagrangian

To assess the ability of the gradient descent method to converge to similar optimal control trajectories, control trajectories originating from two initial guesses of \( u_0 = 0.0001 \) and \( u_0 = 1.0 \) are compared in Figure 3. Figures 3(a)-(c) correspond to semi-active devices with a slower response time (0.05s) and Figures 3(d)-(f) correspond to semi-active devices with a faster response time (0.01s). Figures 3(a) and (d) correspond to a short period pulse \( (T_p/T_n = 0.2) \); figures 3(b) and (e) correspond to a medium period pulse \( (T_p/T_n = 0.5) \); and figures 3(c) and (f) correspond to a long period pulse \( (T_p/T_n = 1.5) \). Two conclusions may be drawn from these results. First, for pulse periods shorter than the natural period the gradient descent method converges to similar optima from very different initial guesses. The same can not be said for longer period pulses. Second, for short period pulses, the optimal control appears to apply friction forces that are in phase with velocity (dissipative) but opposing the inertial forces (acceleration responses). This result shows that a semi-active control policy, termed pseudo-negative stiffness, is nearly optimal for short period pulse excitations. Such a result is promising for the goal of parameterizing these dynamic programming results into feedback control rules.

Figure 4 illustrates pulse response spectra of the relative displacement and the total acceleration for optimal semi-active damping and passive power-law damping. In terms of relative displacements, peak responses decrease monotonically with increasing levels of passive damping, \( \beta \), for all periods. In terms of total accelerations, however, it is seen that increasing passive damping also increases peak response accelerations for short periods \( (T_p < T_n/\sqrt{2}) \). Further, the optimized semi-active systems have lower transmissibility than any of the passively damped systems at short periods \( (T_p < T_n) \). Moreover, in terms of displacement capacity, the optimized semi-active systems show no resonant behavior. In fact, relative displacements are smallest for \( T_p \approx T_n \).

B. Case II: Quartic Lagrangian

In the control of transient responses, peak responses are of greatest concern. The dynamic programming context readily admits higher-order exponents in the state cost, as illustrated in (16). Control trajectories optimized to suppress peak responses apply greater effort at the extremes as shown
Fig. 3. Pseudonegative-stiffness control behavior for pulse excitation at $\omega_p/\omega_n = 5$ for (a) and (d), 2 for (b) and (e), and $2/3$ for (c) and (f); $T_v = 0.05$ for (a) - (c) and $T_v = 0.02$ for (d) - (f).
in Figure 5. Based upon the results of Cases I and II, a parameterized control rule for semi-active isolation using friction damping may be proposed as follows:

\[ u(r, \dot{r}) = (k_1 r + k_2 r^3)H(-r\dot{r}) \]

where \( k_1 \) and \( k_2 \) are feedback gains to be optimized. Note that because stability is guaranteed by the actions of the semi-active device, the optimization need not be constrained by a closed-loop stability condition.

C. **Case III: Cone-shaped Bowl**

The success observed in converging from constant initial guesses for \( u_0(t) \) to an optimized control trajectory for a quadratic bowl shape was not observed in the case of a conical bowl shape, as shown in Figure 6. The control behavior of this figure is not consistent with previous results and did not provide significant reductions in the objective function. Insight gained from solving Cases I and II was therefore applied to Case III in the form of an initial guess which was presumed to be close to optimal. This initial guess for a conical bowl was designed to provide a frictional force in opposition to the inertial force whenever possible (i.e., whenever \( r\dot{r} < 0 \)),

\[ u_0(t) = k_0 H(-r\dot{r}) \]

where \( k_0 = 0.001 \) was found to be a good initial guess. Figure 7 shows that the converged control is not substantially different than the initial guess in this case while Figure 8 shows that gradient based solution to the Euler-Lagrange equations reduced the value of the objective function by about twenty percent.
B. Conclusions

Numerical simulations presented in this paper support the following conclusions:

- Gradient descent methods are more effective in determining solutions to the Euler-Lagrange equations describing optimal semi-active control in linear isolation systems than in nonlinear isolation systems.
- When the objective is to suppress response accelerations of the isolated components, a semi-active friction control satisfying the Euler-Lagrange equations applies control forces that oppose inertial forces, thereby reducing acceleration at times in which accelerations, velocities, and displacements have the proper phase relationship.
- Parameterized feedback control methods may then be optimized using more tractable parametric optimization methods.
- Pulse response spectra of the optimally controlled isolation system exhibits no resonant effect in terms of displacement response and very little dynamic amplification in terms of acceleration response. This result illustrates that the goal of limiting response displacements without adversely affecting response accelerations has been met.

C. Future Work

Future work in this area will investigate other numerical methods for solving for optimal semi-active control trajectories, such as variation of extremals, quasilinearization, and second order gradient methods [3], [4], [8]. Higher dimensional systems for isolation of objects oscillating in the horizontal plane will be investigated. Dynamic frictional models will be implemented in order to model pre-slip compliance and stiction. And, importantly, optimal control performance computed via dynamic programming will be compared to the performance of parameterized feedback control laws and will be compared to passive damping methods.

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